A Construction Technique for Generalized Complex Orthogonal Designs and Applications to Wireless Communications

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Abstract

We introduce a construction technique for generalized complex linear processing orthogonal designs, which are $p \times n$ matrices X satisfying $X^H X = fI$, where f is a complex quadratic form, I is the identity matrix, and X has complex entries. These matrices generalize the familiar notions of orthogonal designs and generalized complex orthogonal designs. We explain the application of these matrices to space-time block coding for multiple-antenna wireless communications. In particular, we discuss the practical strengths of the space-time block codes constructed via our proposed technique.

1 Introduction

The theory of orthogonal designs dates back over a century [1, 2, 3]. Since Radon's classical result implying the set of dimensions for which real square orthogonal designs exist [3], several generalizations of real square orthogonal designs have followed, including generalized real orthogonal designs, complex orthogonal designs, generalized complex orthogonal designs, and generalized complex linear processing orthogonal designs. A thorough treatment of orthogonal designs is found in [4]. Tarokh, Jafarkhani, and Calderbank pioneered using generalized complex orthogonal designs to construct space-time block codes (STBCs), which are used to transmit data over wireless channels using multiple transmit antennas [5]. Their work extends Alamouti's scheme for wireless communications with two transmit antennas [6]. In this paper, we present a mathematically elegant technique of constructing generalized complex linear processing orthogonal designs, and we discuss the resulting designs' practical strengths.

In Section 2, we provide the necessary definitions and an introduction to space-time block coding. In Section 3, we present our construction technique for generalized complex linear processing orthogonal designs. In Section 4, we demonstrate how our construction technique can be used to generate generalized complex linear processing orthogonal designs with few or no zeros, and we discuss the trade-off between having few zeros and high rates. In Section 5, we discuss other implementations of this construction. In Section 6, we conclude the paper by reviewing the strengths of our construction technique.

2 Definitions and Background

2.1 Introduction to Orthogonal Designs

A *(real) orthogonal design* of order n and type $(s_1, s_2, \ldots, s_k)$ denoted $OD = OD(n; s_1, s_2, \ldots, s_k)$ in real variables $x_1, x_2, \ldots, x_k$, is a matrix A of order n with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_k\}$ satisfying

$$A^T A = \sum_{l=1}^{k} (s_l x_l^2) I_n,$$

where I_n is the identity matrix of order n. We note that over the appropriate algebraic structure

$$AA^T = A^T A = \sum_{l=1}^{k} (s_l x_l^2) I_n.$$

A *generalized real orthogonal design* of order n is a $p \times n$ matrix of k variables satisfying the same conditions. Generalized real orthogonal designs are also called rectangular real orthogonal designs. The rate of a generalized real orthogonal design is defined as $R = \frac{k}{p}$.

Geramita and Geramita [7] first defined a *complex orthogonal design* $COD = COD(n; s_1, s_2, \ldots, s_k)$ of type $(s_1, s_2, \ldots, s_k)$ in variables $x_1, x_2, \ldots, x_k$, as a matrix C of order n with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_k, \pm ix_1, \pm ix_2, \ldots, \pm ix_k,\}$ satisfying

$$C^H C = \sum_{l=1}^{k} (s_l x_l^2) I_n,$$

where I_n is the identity matrix of order n, H is the Hermitian conjugate (the transpose complex conjugate) and $i^2 = -1$. We note that over the appropriate algebraic structure

$$CC^H = C^H C = \sum_{l=1}^{k} (s_l x_l^2) I_n.$$

The type $(s_1, s_2, \ldots, s_k)$ of the design gives the number of times each variable occurs in each row (and column) and hence each s_ℓ is an integer. However, the defining equation

$$C^H C = \sum_{\ell=1}^{k} (s_\ell x_\ell^2) I_n,$$

can be simplified to

$$C^H C = \sum_{\ell=1}^{k} x_\ell^2 I_n,$$

by suitably normalizing the matrix (which does not impact the number of times each variable occurs in each row or column). The details of this normalization can be found in [5]. In this paper, we assume that the entries of a real or complex orthogonal design have been so normalized and abbreviate $(C)OD(n; s_1, s_2, \ldots, s_k)$ as $(C)OD(n;)$.

Alternate definitions of complex orthogonal designs C exist, allowing entries of complex variables $\{0, \pm z_1, \ldots, \pm z_k, \pm z_1^*, \ldots, \pm z_k^*\}$, where z^* denotes the complex conjugate of z and

$$C^H C = \sum_{l=1}^{n} |z_l|^2 I_n$$

[5, 8, 11]. In this paper, we use this latter definition of complex orthogonal designs.

A *generalized complex orthogonal design* of size n is a $p \times n$ matrix C with entries from $\{0, \pm z_1, \ldots, \pm z_k, \pm z_1^*, \ldots, \pm z_k^*\}$, or products of these complex indeterminants with the imaginary unit i, such that

$$C^H C = \sum_{l=1}^{k} |z_l|^2 I_n.$$

If the entries of C are allowed to be complex linear combinations of the complex variables $z_1, \ldots, z_k$ and their conjugates $z_1^*, \ldots, z_k^*$, then the design C is called a *generalized complex linear processing orthogonal design*. The rate of C is defined as $R = \frac{k}{p}$.

Henceforth, when no confusion should occur, we will sometimes use the term "orthogonal design" to refer to any of the above generalizations.

2.2 Introduction to Space-Time Block Coding

Tarokh, Jafarkhani, and Calderbank first used orthogonal designs and their generalizations as space-time block codes (STBCs) for wireless communications with multiple transmit antennas [5], building upon the work done by Alamouti for wireless communication with two antennas [6]. In space-time block coding, the matrix representation of the (generalized complex linear processing) orthogonal design serves as a transmission matrix. The entries of the $p \times n$ matrix denote the information symbols from an arbitrary real or complex signal constellation. The information symbols are to be sent over n transmit antennas, represented by the n columns. Hence, each antenna is responsible for sending the information symbols in one column. The number of rows, p, represents the number of channel uses. That is, each antenna must transmit p times. This number, p, is also called the decoding delay or memory length. The rate of the orthogonal design, $R = \frac{k}{p}$, is interpreted as the ratio of the number of transmitted information symbols to the decoding delay of these symbols at the receiver.

Two of the main problems of space-time block codes or orthogonal designs are as follows [8]:

1. Given n, find a $p \times n$ orthogonal design on k variables which maximizes the rate $R = \frac{k}{p}$.
2. Given n, find a $p \times n$ orthogonal design on k variables with maximal rate which minimizes p.

Hence, the goal is to include as many variables (or information symbols) in the fewest rows (or smallest decoding delay) possible.

The maximum rates of real and complex square orthogonal designs (and amicable orthogonal designs) have been known for many years; an account of this theory is given in [4]. Recently, Liang proved that for generalized complex orthogonal designs for $n = 2m - 1$ or $n = 2m$ antennas (*i.e.* with $n = 2m - 1$ or $n = 2m$ columns), the maximum rate is $R = \frac{m+1}{2m}$ [8]. In contrast, the maximum rate for generalized real orthogonal designs is $R = 1$. Additionally, Liang published algorithms for generating generalized complex and generalized real orthogonal designs of maximum rate [8]. It remains to solve Problem 2 above.

A third consideration for practical implementation is the number of zeros in a code: Compared to a code with fewer zeros, a code with more zeros results in a higher peak-to-mean power ratio for the transmit antennas to achieve the same bit error rates (BER). Equivalently, a code with fewer zeros provides a better BER with the same peak-to-mean power ratio per each transmit antenna. Having many zeros can also impede practical implementation since some transmit antennas must be turned off during transmission. Turning off transmit antennas during transmission is inconvenient, especially in high data rate wireless communication systems. Looking ahead to possible cryptographic applications of space-time block codes for hiding data, zeros in the transmission matrix could be a weakness. We conclude that it is important to look for construction techniques that yield high rate codes and/or codes with few or no zeros.

3 The Construction

In this section, we provide a vector-based version and an equivalent matrix-based version of our Construction Theorem for constructing generalized complex linear processing orthogonal design from (square or generalized) orthogonal designs (ODs) or (square or generalized) complex orthogonal designs (CODs). We are interested in using the resulting generalized complex linear processing orthogonal designs as space-time block codes (STBCs). We provide examples of codes constructed via our theorem and explain certain strengths of the construction, namely its simplicity and its ability to control the number of columns in the resulting STBCs.

Theorem 3.1 (Vector-Based Construction Theorem). *Let G be any $OD(n;)$ or $COD(n;)$ satisfying $G^H G = fI$. Let the set of column vectors of G be denoted by $Cols(G) = \{v_1, v_2, \ldots, v_n\}$. Build the column vector set of a STBC X as follows: Remove any pair of vectors $v_j \neq v_k$ from*

$Cols(G)$ and put $\frac{1}{\sqrt{2}}(v_j + iv_k)$ in $Cols(X)$. Form p, $1 \leq p \leq \frac{n}{2}$, such pairings, thus producing p vectors in $Cols(X)$. Now remove any number of remaining unpaired columns in $Cols(G)$ and put them (unaltered) in $Cols(X)$. The columns in $Cols(X)$ represent an $n \times q$ STBC, where q is any integer satisfying $1 \leq q < n$. X satisfies $X^H X = fI$.

Proof. For any $1 \leq j \leq n$, we can write $v_j = v_j^R + iv_j^C$, where v_j^C and v_j^R are real vectors. The off-diagonal entries of $X^H X$ are either of the form $\frac{1}{\sqrt{2}}(v_j + iv_k)^H \cdot \frac{1}{\sqrt{2}}(v_a + iv_b)$, $v_j^H \cdot \frac{1}{\sqrt{2}}(v_a + iv_b)$, $\frac{1}{\sqrt{2}}(v_j + iv_k)^H \cdot v_a$ or $v_j^H \cdot v_k$, where v_j, v_k, v_a, v_b are distinct vectors from $Cols(G)$. In the first case, we have:

$$\begin{aligned}
\frac{1}{\sqrt{2}}(v_j + iv_k)^H \cdot \frac{1}{\sqrt{2}}(v_a + iv_b) &= \frac{1}{2}(v_j^H - iv_k^H) \cdot (v_a + iv_b) \\
&= \frac{1}{2}(v_j^H \cdot (v_a + iv_b) - iv_k^H \cdot (v_a + iv_b)) \\
&= \frac{1}{2}(v_j^H \cdot v_a + iv_j^H \cdot v_b - iv_k^H \cdot v_a + v_k^H \cdot v_b) \\
&= 0
\end{aligned}$$

since $G^H G$ is a diagonal matrix. Similar computations show that off-diagonal entries of $X^H X$ of the form $v_j^H \cdot \frac{1}{\sqrt{2}}(v_a + iv_b)$, $\frac{1}{\sqrt{2}}(v_j + iv_k)^H \cdot v_a$, or $v_j^H \cdot v_k$ are also 0.

The diagonal entries of $X^H X$ are either of the form $v_j^H \cdot v_j$ or $\frac{1}{\sqrt{2}}(v_j + iv_k)^H \cdot \frac{1}{\sqrt{2}}(v_j + iv_k)$. In the first case, $v_j^H \cdot v_j = f$ since $G^H G = fI$. In the second case, we have

$$\begin{aligned}
\frac{1}{\sqrt{2}}(v_j + iv_k)^H \cdot \frac{1}{\sqrt{2}}(v_j + iv_k) &= \frac{1}{2}(v_j^H \cdot (v_j + iv_k) - iv_k^H \cdot (v_j + iv_k)) \\
&= \frac{1}{2}(v_j^H \cdot v_j + iv_j^H \cdot v_k - iv_k^H \cdot v_j + v_k^H \cdot v_k) \\
&= \frac{1}{2}(f + i0 - i0 + f) \\
&= f
\end{aligned}$$

since $G^H G = fI$.

We conclude that $X^H X = fI$. While X clearly contains the same number of rows as G, the number q of columns in X, $1 \leq q < n$, depends on the number of pairings formed during the column construction phase and on the number of unpaired vectors in $Cols(G)$ that are transferred into $Cols(X)$. ☐

Since many constructions of designs involve sub-matrices, we now provide a matrix version of this Construction Theorem.

Let

$$G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1}$$

be any $OD(n;)$ on the commuting variables $\{x_1, \cdots, x_k\}$, where A is $p \times q$, B is $p \times n-q$, C is $n-p \times q$ and D is $n-p \times n-q$. We write $A+iB$ where if $q \neq n-q$ the matrix A or B, whichever has the smaller number of columns has sufficient columns of all zeros concatenated to make $A+iB$ meaningful as a matrix (the extra columns can be any where). The same actions are taken to make $C+iD$ meaningful as a matrix.

Theorem 3.2. **[Matrix-Based Construction Theorem]** *Let G be any $OD(n;)$ or $COD(n;)$ partitioned as in (1) and satisfying $G^H G = fI_n$. Then*

$$X = \frac{1}{\sqrt{2}} \begin{bmatrix} A+iB \\ C+iD \end{bmatrix} \tag{2}$$

satisfies $X^H X = fI_q$. Hence X is a STBC which is $n \times q$ for any $[\frac{n}{2}] \leq q \leq n$.

Proof. Since $G^H G = fI_n$ we have

$$\begin{aligned} G^H G = & \begin{bmatrix} A^H & C^H \\ B^H & D^H \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ = & \begin{bmatrix} A^H A + C^H C & A^H B + C^H D \\ B^H A + D^H C & B^H B + D^H D \end{bmatrix} \\ = & fI_n. \end{aligned} \tag{3}$$

Hence

$$A^H A + C^H C = B^H B + D^H D = fI \quad \text{and} \quad A^H B + C^H D = B^H A + D^H C = 0. \tag{4}$$

We now consider

$$\begin{aligned} X^H X = & \frac{1}{2} \begin{bmatrix} A^H - iB^H \\ C^H - iD^H \end{bmatrix} \begin{bmatrix} A+iB \\ C+iD \end{bmatrix} \\ = & \frac{1}{2} \left[A^H A + iA^H B - iB^H A + B^H B + C^H C + iC^H D - iD^H C + D^H D \right] \\ = & \frac{1}{2}(2fI_p) \\ = & fI_p \end{aligned} \tag{5}$$

□

We note that other linear combinations (*i.e.* $sA+itB$ and $sC+itD$) are possible with slight modifications. The STBCs that result from the vector-based and matrix-based construction theorems are equivalent up to possible multiplication by i in certain columns. We therefore refer to the two theorems interchangeably as the "Construction Theorem." Two advantages of this construction are its mathematical elegance and the control it provides over the number of columns in the resulting codes. The construction is elegant because it provides a straightforward way to build generalized complex linear processing orthogonal designs with any number of columns by using readily available real or complex orthogonal designs. This is much simpler than the available construction techniques involving detailed algorithms [8] or amicable designs and weighing matrices [9], [4], [10]. Providing control over the number of columns in the STBC is of practical importance: We can accommodate varying requirements for the number of transmit antenna over complex constellations. We now provide some examples of codes built using this construction technique.

Let G satisfy $G^H G = (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2) I_8$, where G is given by:

$$G = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ -x_2 & x_1 & x_4 & -x_3 & x_6 & -x_5 & -x_8 & x_7 \\ -x_3 & -x_4 & x_1 & x_2 & x_7 & x_8 & -x_5 & -x_6 \\ -x_4 & x_3 & -x_2 & x_1 & x_8 & -x_7 & x_6 & -x_5 \\ -x_5 & -x_6 & -x_7 & -x_8 & x_1 & x_2 & x_3 & x_4 \\ -x_6 & x_5 & -x_8 & x_7 & -x_2 & x_1 & -x_4 & x_3 \\ -x_7 & x_8 & x_5 & -x_6 & -x_3 & x_4 & x_1 & -x_2 \\ -x_8 & -x_7 & x_6 & x_5 & -x_4 & -x_3 & x_2 & x_1 \end{bmatrix} \tag{6}$$

The construction can be used on G in many ways to control the number of columns in the resulting STBC. For example, the 8×4 STBC X_1 below is formed by pairing columns 1, 2, 3, 4 of G with columns 5, 6, 7, 8 of G, respectively:

$$X_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 + ix_5 & x_2 + ix_6 & x_3 + ix_7 & x_4 + ix_8 \\ -x_2 + ix_6 & x_1 - ix_5 & x_4 - ix_8 & -x_3 + ix_7 \\ -x_3 + ix_7 & -x_4 + ix_8 & x_1 - ix_5 & x_2 - ix_6 \\ -x_4 + ix_8 & x_3 - ix_7 & -x_2 + ix_6 & x_1 - ix_5 \\ -x_5 + ix_1 & -x_6 + ix_2 & -x_7 + ix_3 & -x_8 + ix_4 \\ -x_6 - ix_2 & x_5 + ix_1 & -x_8 - ix_4 & x_7 + ix_3 \\ -x_7 - ix_3 & x_8 + ix_4 & x_5 + ix_1 & -x_6 - ix_2 \\ -x_8 - ix_4 & -x_7 - ix_3 & x_6 + ix_2 & x_5 + ix_1 \end{bmatrix} \tag{7}$$

The 8×6 STBC X_2 below is formed by pairing columns 1 and 2 of G together, pairing columns 3 and 4 of G together, and using unpaired columns 5 through 8 of G:

$$X_2 = \begin{bmatrix} \frac{x_1+ix_2}{\sqrt{2}} & \frac{x_3+ix_4}{\sqrt{2}} & x_5 & x_6 & x_7 & x_8 \\ \frac{-x_2+ix_1}{\sqrt{2}} & \frac{x_4-ix_3}{\sqrt{2}} & x_6 & -x_5 & -x_8 & x_7 \\ \frac{-x_3-ix_4}{\sqrt{2}} & \frac{x_1+ix_2}{\sqrt{2}} & x_7 & x_8 & -x_5 & -x_6 \\ \frac{-x_4+ix_3}{\sqrt{2}} & \frac{-x_2+ix_1}{\sqrt{2}} & x_8 & -x_7 & x_6 & -x_5 \\ \frac{-x_5-ix_6}{\sqrt{2}} & \frac{-x_7-ix_8}{\sqrt{2}} & x_1 & x_2 & x_3 & x_4 \\ \frac{-x_6+ix_5}{\sqrt{2}} & \frac{-x_8+ix_7}{\sqrt{2}} & -x_2 & x_1 & -x_4 & x_3 \\ \frac{-x_7+ix_8}{\sqrt{2}} & \frac{x_5-ix_6}{\sqrt{2}} & -x_3 & x_4 & x_1 & -x_2 \\ \frac{-x_8-ix_7}{\sqrt{2}} & \frac{x_6+ix_5}{\sqrt{2}} & -x_4 & -x_3 & x_2 & x_1 \end{bmatrix} \tag{8}$$

4 Balancing Strengths and Weaknesses of the Construction

It is of practical importance to produce STBCs with few to no zero entries. A STBC with a relatively low number of zeros provides a better BER with the same peak-to-mean power ratio per each transmit antenna. This power savings is important for the practical implementation. Also, having relatively few or no zeros avoids the inconvenience of turning off transmit antennas during transmission, which is especially problematic in high data rate wireless communication systems. Additionally, we foresee cryptographic applications of STBCs in which zero entries might be a liability. Using our construction technique, it is straight-forward to build STBCs with few to no zeros.

While our construction technique provides STBCs with few to no zeros, we usually must compromise on the rate of the codes. It is known that designs of maximum rate 1 exist for every number of transmit antennas (*i.e.* every number of columns) using arbitrary real constellations, and that designs of maximum rate 1 exist for two transmit antennas (*i.e.* two columns) using arbitrary complex constellations [5, 8]. In contrast, there do not exist rate 1 designs for more than two transmit antennas using arbitrary complex constellations [8, 11]. In fact, we recall that Liang provided an algorithm for producing generalized complex orthogonal designs with $n = 2m - 1$ or $n = 2m$ columns that achieve the maximum rate of $R = \frac{m+1}{2m}$ [8]. However, these maximum rate codes have many zeros.

In order to produce codes with respectable rates using our construction technique, care must be taken. If we start with a rate 1 real OD, then there are no zeros in any column. When combining two columns each containing k distinct real variables, we necessarily reduce the number of variables from k distinct real variables to $\frac{k}{2}$ distinct complex variables. This is because the variables $x_j + ix_k$ and $x_k - ix_j$ are not distinct complex variables, despite x_j and x_k being distinct real variables. When using our construction, it seems that the smallest rate reduction (*i.e.* the smallest reduction of distinct variables) and therefore the highest rate codes result when G is a maximum rate *complex* orthogonal design, containing sizable submatrices of the form $x_j I$.

We now provide examples and discuss how our construction can be used to yield codes with few or no zeros, while balancing a need for respectable rates.

Remark 4.1. *Let G be any $OD(n; 1, 1, \dots, 1)$ or $COD(n; 1, 1, \dots, 1)$ satisfying $G^H G = f I_n$, where each variable occurs exactly once per row. It is possible to use the Construction Theorem to build a STBC X with fewer zero entries per column than G.*

Proof. We require that each variable occurs exactly once per row, which implies that each row (respectively, each column) has the same number, z, of zeros. Label the columns from left to right as $v_1, v_2, \dots, v_n$. Assuming $z > 0$, let a be the first row in which v_1 has a zero. Let v_k be the first vector having a nonzero entry in row a. Form $v_1 + iv_k$, which has a nonzero entry in row a. Since each variable occurs exactly once per row, there can be no cancellations. Therefore, $v_j + iv_k$ contains at most $z - 1$ zeros (and possibly far fewer). Remove columns v_1 and v_k from $Cols(X)$, entering $v_i + iv_k$ into $Cols(X)$. Continue making pairings in this way. Do not move any unpaired vectors from $Cols(G)$ into $Cols(X)$. The resulting $Cols(X)$ has at most $z - 1$ zeros per column. □

In fact, we next show examples to illustrate the following stronger remark:

Remark 4.2. *Let G be any $OD(n;)$ or $COD(n;)$ satisfying $G^H G = fI_n$. Then by suitable choice of the columns, it may be possible to ensure there are no zero entries in the STBC obtained via the Construction Theorem.*

Let G satisfy $G^H G = (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2)I_8$, where

$$G = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & 0 & 0 & 0 \\ -x_2 & x_1 & x_4 & -x_3 & 0 & -x_5 & 0 & 0 \\ -x_3 & -x_4 & x_1 & x_2 & 0 & 0 & -x_5 & 0 \\ -x_4 & x_3 & -x_2 & x_1 & 0 & 0 & 0 & -x_5 \\ -x_5 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_5 & 0 & 0 & -x_2 & x_1 & -x_4 & x_3 \\ 0 & 0 & x_5 & 0 & -x_3 & x_4 & x_1 & -x_2 \\ 0 & 0 & 0 & x_5 & -x_4 & -x_3 & x_2 & x_1 \end{bmatrix}. \tag{9}$$

The Construction Theorem can be used on G in many ways to control the number of columns in the final matrix and to control the number of zero entries in the final matrix. Therefore, the Construction Theorem can be used to obtain designs with a varying percentage of zeros. Furthermore, the Construction Theorem can be used to obtain designs with varied rates.

For example, X_1 below contains no zeros and has rate $\frac{1}{2}$, while X_2 below has more zero entries and a lower rate of $\frac{3}{8}$. By both criteria, X_1 is a better STBC. Since the initial code G has rate $\frac{5}{8}$, the rate of the resulting code X_1 has been reduced by $\frac{1}{8}^{th}$. However, the number of zeros has gone from a total of 24 to a total of 0. This illustrates the trade-off between the number of zeros and the rate of a code.

$$X_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 + ix_5 & x_2 & x_3 & x_4 \\ -x_2 & x_1 - ix_5 & x_4 & -x_3 \\ -x_3 & -x_4 & x_1 - ix_5 & x_2 \\ -x_4 & x_3 & -x_2 & x_1 - ix_5 \\ -x_5 + ix_1 & ix_2 & ix_3 & ix_4 \\ -ix_2 & x_5 + ix_1 & -ix_4 & ix_3 \\ -ix_3 & ix_4 & x_5 + ix_1 & -ix_2 \\ -ix_4 & -ix_3 & ix_2 & x_5 + ix_1 \end{bmatrix} \tag{10}$$

$$X_2 = \begin{bmatrix} \frac{x_1+ix_2}{\sqrt{2}} & \frac{x_3+ix_4}{\sqrt{2}} & x_5 & 0 & 0 & 0 \\ \frac{-x_2+ix_1}{\sqrt{2}} & \frac{x_4-ix_3}{\sqrt{2}} & 0 & -x_5 & 0 & 0 \\ \frac{-x_3-ix_4}{\sqrt{2}} & \frac{x_1+ix_2}{\sqrt{2}} & 0 & 0 & -x_5 & 0 \\ \frac{-x_4+ix_3}{\sqrt{2}} & \frac{-x_2+ix_1}{\sqrt{2}} & 0 & 0 & 0 & -x_5 \\ \frac{-x_5}{\sqrt{2}} & 0 & x_1 & x_2 & x_3 & x_4 \\ \frac{ix_5}{\sqrt{2}} & 0 & -x_2 & x_1 & -x_4 & x_3 \\ 0 & \frac{x_5}{\sqrt{2}} & -x_3 & x_4 & x_1 & -x_2 \\ 0 & \frac{ix_5}{\sqrt{2}} & -x_4 & -x_3 & x_2 & x_1 \end{bmatrix} \tag{11}$$

We also achieve a marked decrease in the number of zeros per column when starting with a COD constructed via the Adams-Lax-Phillips construction, which is summarized in [8]. In certain cases, the STBC constructed from a Adams-Lax-Phillips COD via our technique achieves the maximum rate. For example, the Adams-Lax-Phillips constructed COD $G(4;)$ shown in Equation (62) in [8] has one zero in each of its four columns, and it achieves the maximum rate of $\frac{3}{4}$ for square CODs with 4 columns. Pairings exist to construct a 4×3 STBC with no zeros. The rate remains $\frac{3}{4}$, which is the maximum rate for rectangular CODs with 3 columns. This example provides the minimum number of zeros (*i.e.* no zeros) and the maximum rate. While it is straight-forward to build STBCs with relatively few zeros from Adams-Lax-Phillips CODs, the resulting STBCs do not always achieve the maximum rate. For example, the Adams-Lax-Phillips constructed COD $G(16;)$ shown in Equation (48) in [8] has 11 zeros in each of its 16 columns of length 16. This square COD achieves the maximum rate of $\frac{5}{16}$ for square CODs with 16 columns. Pairings exist to construct a STBC with only 6 zeros in each of its 8 columns of length 16. The new code retains rate $\frac{5}{16}$, however the maximum rate for rectangular codes with 8 columns is $\frac{5}{8}$.

Using CODs constructed using Liang's algorithm [8] also results in STBCs with a marked decrease in the number of zeros per column. The 8×4 COD in Equation (98) in Liang has 2 zeros per column, and we note that no row has more than one zero. Therefore, pairings exist to produce an 8×2 STBC with no zeros. The 15×5 COD in Equation (100) in Liang has 5 zeros per column, and we note that every pair of columns shares one row in which both columns are zero. Therefore, pairings exist to produce a 15×4 STBC with only 1 zero per column. The 30×6 COD in Equation (101) in Liang has 10 zeros per column, and we note that every pair of columns shares two rows in which both columns are zero. Therefore, pairings exist to produce a 30×3 STBC with only 2 zeros per column.

When starting with using real orthogonal designs of the following form

x	y	z	0	a	0	0	$-b$	c	0	0	d	0	$-e$	f	0
y	$-x$	0	$-z$	0	$-a$	b	0	0	$-c$	$-d$	0	e	0	0	$-f$
z	0	$-x$	y	0	$-b$	$-a$	0	0	d	$-c$	0	e	0	0	$-f$
0	$-z$	y	x	b	0	0	a	$-d$	0	0	c	0	f	e	0
a	0	0	b	$-x$	y	z	0	0	$-e$	f	0	$-c$	0	0	$-d$
0	$-a$	$-b$	0	y	x	0	$-z$	e	0	0	$-f$	0	c	d	0
0	b	$-a$	0	z	0	x	y	$-f$	0	0	$-e$	0	$-d$	c	0
$-b$	0	0	a	0	$-z$	y	$-x$	0	f	e	0	d	0	0	$-c$
c	0	0	$-d$	0	e	$-f$	0	$-x$	y	z	0	a	0	0	$-b$
0	$-c$	d	0	$-e$	0	0	f	y	x	0	$-z$	0	$-a$	b	0
0	$-d$	$-c$	0	e	0	0	f	z	0	x	y	0	$-b$	$-a$	0
d	0	0	c	0	$-f$	$-e$	0	0	$-z$	y	$-x$	b	0	0	a
0	e	$-f$	0	$-c$	0	0	d	a	0	0	b	x	y	z	0
$-e$	0	0	f	0	c	d	0	0	$-a$	$-b$	0	y	$-x$	0	$-z$
f	0	0	e	0	d	c	0	0	b	$-a$	0	z	0	$-x$	y
0	$-f$	$-e$	0	$-d$	0	0	$-c$	$-b$	0	0	a	0	$-z$	y	x

(see the proof of Theorem 4.2 in [4]), we can pair columns 1 through 8, with columns 9 through 16, in order. This results in a rate $\frac{1}{2}$ code with no zeros. While this code has the desirable property of having no zeros, the rate is $\frac{1}{8}^{th}$ less than the maximum of $\frac{5}{8}$ achievable for CODs with 8 columns. Depending on the application, this reduction in rate may be acceptable in order to have no zeros.

We can also achieve a reduction in the number of zeros when the initial COD has repeated variables within rows. For example, there are a variety of ways to use the matrices displayed in [12] as initial CODs.

5 Other Constructions

In this section, we show that it is possible to concatenate different sized designs to build a STBC.

Remark 5.1. *Let G_j be any $OD(n_j;)$ or $COD(n_j;)$, $j = 1, 2, \ldots, k$, satisfying $G_j^H G_j = f_j I_{n_j}$. Then*

$$G = \begin{bmatrix} G_1 & 0 & \cdots & 0 \\ 0 & G_2 & & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \cdots & G_k \end{bmatrix}, \tag{12}$$

where 0 is the appropriately sized zero matrix is a STBC with $n = \sum_{j=1}^{k} n_j$ rows and columns. It can now be used as in the Construction Theorem to produce further codes as required.

For example, using G_1 and G_2 as given below:

$$G_1 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ -x_2 & x_1 & x_4 & -x_3 & x_6 & -x_5 & -x_8 & x_7 \\ -x_3 & -x_4 & x_1 & x_2 & x_7 & x_8 & -x_5 & -x_6 \\ -x_4 & x_3 & -x_2 & x_1 & x_8 & -x_7 & x_6 & -x_5 \\ -x_5 & -x_6 & -x_7 & -x_8 & x_1 & x_2 & x_3 & x_4 \\ -x_6 & x_5 & -x_8 & x_7 & -x_2 & x_1 & -x_4 & x_3 \\ -x_7 & x_8 & x_5 & -x_6 & -x_3 & x_4 & x_1 & -x_2 \\ -x_8 & -x_7 & x_6 & x_5 & -x_4 & -x_3 & x_2 & x_1 \end{bmatrix} \tag{13}$$

$$G_2 = \begin{bmatrix} x_9 & x_{10} & x_{11} & x_{12} \\ -x_{10} & x_9 & x_{12} & -x_{11} \\ -x_{11} & -x_{12} & x_9 & x_{10} \\ -x_{12} & x_{11} & -x_{10} & x_9 \end{bmatrix} \tag{14}$$

we can construct

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \tag{15}$$

and then apply the Construction Theorem to G to obtain a variety of STBCs. One such example is given by X below:

$$X = \begin{bmatrix} x_1 + ix_7 & x_2 + ix_8 & x_3 & x_4 & x_5 & x_6 \\ -x_2 - ix_8 & x_1 + ix_7 & x_4 & -x_3 & x_6 & -x_5 \\ -x_3 - ix_5 & -x_4 - ix_6 & x_1 & x_2 & x_7 & x_8 \\ -x_4 + ix_6 & x_3 - ix_5 & -x_2 & x_1 & x_8 & -x_7 \\ -x_5 + ix_3 & -x_6 + ix_4 & -x_7 & -x_8 & x_1 & x_2 \\ -x_6 - ix_4 & x_5 + ix_3 & -x_8 & x_7 & -x_2 & x_1 \\ -x_7 + ix_1 & x_8 - ix_2 & x_5 & -x_6 & -x_3 & x_4 \\ -x_8 + ix_2 & -x_7 + ix_1 & x_6 & x_5 & -x_4 & -x_3 \\ 0 & 0 & ix_9 & ix_{10} & ix_{11} & ix_{12} \\ 0 & 0 & -ix_{10} & ix_9 & ix_{12} & -ix_{11} \\ 0 & 0 & -ix_{11} & -ix_{12} & ix_9 & ix_{10} \\ 0 & 0 & -ix_{12} & ix_{11} & -ix_{10} & ix_9 \end{bmatrix}. \tag{16}$$

The novelty here is in concatenating matrices of different sizes to build the final matrix.

6 Conclusions

We explained a practical application of designs: Space-time block codes for multiple-antenna wireless communications. We presented an elegant technique for constructing space-time block codes. The construction technique is novel in that we use the inner structure of existing ODs or CODs in order to build a new STBC (without the need for amicable designs and without complicated algorithms).

The resulting space-time block codes have the following practical features: A varying number of columns, few to no zero entries, and respectable rates. By affording flexibility in the number of columns in the resulting STBC, we can accommodate varying requirements for the number of transmit antennas over complex constellations. By constructing STBCs with relatively few or no zeros, we require a relatively small peak-to-mean power ratio per each transmit antenna to achieve the same bit error rate, and we do not require any transmit antennas to be turned off during transmission.

References

[1] A. Hurwitz, Uber die Komposition der quadratischen Formen von beliebig vielen Variablem. *Nachr. Gesell. d. Wiss. Gottingen*, 1898, pp. 309–316.

[2] A. Hurwitz, Uber die Komposition der quadratischen Formen. *Math. An.*, Vol. 88, No. 5, 1923, pp.1–25.

[3] J. Radon, Lineare scharen orthogonaler matrizen. *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universitat.*, 1922, pp. 1–14.

[4] A. V. Geramita and J. Seberry, *Orthogonal designs: Quadratic forms and Hadamard matrices*, Lecture notes in pure and applied mathematics, vol. 43. Marcel Dekker, New York and Basel, 1979.

[5] Vahid Tarokh, Hamid Jafarkhani, and A. Robert Calderbank, Space-time block codes from orthogonal designs. *IEEE Trans. Inform. Theory*, Vol. 45, No. 5, 1999, pp.1456–1467.

[6] S. Alamouti A simple transmit diversity technique for wireless communications. *IEEE J. Select. Areas Commun.*, Vol. 16, Oct. 1998, pp.1451–1458.

[7] A. V. Geramita and J. M. Geramita, J. Combin. Theory. Series. A, Volume 25, 1978, no. 3, pp. 211–225.

[8] Xue-Bin Liang, Orthogonal designs with maximal rates. *IEEE Trans. Inform. Theory*, Vol. 49, No. 10, 2003, pp. 2468–2503. Special issue on space-time transmission, reception, coding and signal processing.

[9] G. Ganesan and P. Stoica, "Space-time diversity using orthogonal and amicable orthogonal designs," *Proc. IEEE International Conference on Acoustics, Speech, and Signal Processing*, vol. 5, pp. 2561-2564, June 2000.

[10] C. Yuen, Y. L. Guan and T. T. Tjhung, "Orthogonal space-time block code from amicable orthogonal design," *Proc. IEEE. Int. Conf. Acoustic, Speech and Signal Processing ICASSP 2004*, vol. 4, pp. 469-472, May 2004.

[11] Xue-Bin Liang and Xiang-Gen Xia, On the nonexistence of rate-one generalized complex orthogonal designs. *IEEE Trans. Inform. Theory*, Vol. 49, No. 11, 2003, pp. 2984–2989.

[12] Tran Le Chung; J. Seberry; Yejing Wang; B.J. Wysocki; T.A. Wysocki; Tianbing Xia; Ying Zhao. Two complex orthogonal space-time codes for eight transmit antennas. IEEE Electronic Letters, Vol. 40, No. 1, 2004, pp. 55–57.