Values of Minors of Some Infinite Families of Matrices Constructed from Supplementary Difference Sets and Their Application to the Growth Problem

C. Koukouvinos,* M. Mitrouli†‡ and Jennifer Seberry§

October 23, 2004

Abstract

We obtain explicit formulae for the values of the $2v-j$ minors, $j=0,1,2$ of $(1,-1)$ matrices of order $2v$, v odd, where the matrix is constructed using two circulant or type 1 incidence matrices of $2-\{v;k_1,k_2,\lambda\}$ sds. This allows us to obtain information on the growth problem for families of matrices with moderate growth. Some of our theoretical formulae imply growth close to the order $2v$ but experimentation has not yet supported this result. An open problem remains to establish whether the $(1,-1)$ CP incidence matrices of certain SBIBDs, can have growth greater than $2v$.

Key Words and Phrases: SBIBD, supplementary difference sets, Gaussian elimination, growth, complete pivoting.
AMS Subject Classification: 05B20, 15A15, 65F05, 65G05.
Short running title: Minors of matrices from difference sets and growth problem.

1 Introduction

A set of k residues $D=\{a_1,\ldots,a_k\}$ modulo v is called a (v,k,λ) difference set or cyclic difference set, if for every $d\not\equiv 0(mod v)$ there are exactly λ ordered pairs (a_i,a_j), $a_i,a_j\in D$ such that $a_i-a_j\equiv d(mod v)$.
For the purpose of this paper we will define two supplementary difference sets $2-\{v;k_1,k_2;\lambda\}$, abbreviated as sds, to be two circulant (or type 1) $v\times v$ matrices B_1 and B_2, with entries 0 or 1, which have exactly k_i entries $+1$ and $v-k_i$ entries 0, $i=1,2$ respectively, in each row and column and for which the inner product of any pair of rows of $[B_1\ \ B_2]$ is λ. The $(1,-1)$ incidence matrices of B_i, are obtained by letting $A_i=2B_i-J$, $i=1,2$, where J is the $v\times v$ matrix of all ones. Then we have for all methods below A is of order $2v$ is given by

*Department of Mathematics, National Technical University of Athens, Zografou 15773, Athens, Greece, e-mail: ckoukouv@math.ntua.gr
†Department of Mathematics, University of Athens, Panepistemiopolis 15784, Athens, Greece, e-mail: mmitroul@cc.uoa.gr
‡The work of this author was supported by a grant from the University of Athens.
§School of Information Technology and Computer Science, University of Wollongong, Wollongong, NSW, 2522, Australia, e-mail: jennie@uow.edu.au

$$A = \begin{bmatrix} A_1 & A_2 \\ A_2^T & -A_1^T \end{bmatrix} \tag{1}$$

$$A_1 A_1^T + A_2 A_2^T = 4(k_1 + k_2 - \lambda)I + 2(v - 2(k_1 + k_2 - \lambda))J = (2v - \Lambda)I + \Lambda\ J$$

where $\Lambda = 2v - 4(k_1 + k_2 - \lambda)$, and I is the identity matrix of order v.
We can write

$$AA^T = (2v + 2)I_{2v} - 2I_2 \times J_v.$$

It is easy to use the determinant simplification theorem [8] to see that for methods I and II, which follows, we have

$$det\ A = 2^v(2v - 1)(v - 1)^{v-1}.$$

In this paper we study the family $2-\{v; k_1, k_2; \lambda\}$ sds, and using $\lambda(v-1) = k_1(k_1-1)+k_2(k_2-1)$ we can conclude that

$$detA = 2^v(k_1 + k_2 - \lambda)^{v-1}(v^2 - 2v(k_1 + k_2) + 2k_1^2 + 2k_2^2).$$

Method I.
When $v = 4t + 1$ is a prime power, there exists a supplementary difference set with parameters $2 - \{v; \frac{v-1}{2}, \frac{v-1}{2}; \frac{v-3}{2}\} = 2 - \{4t + 1; 2t, 2t; 2t - 1\}$, see [14, 15].
Their incidence $(1, -1)$ matrices A_1 and A_2 satisfying the relation

$$A_1 A_1^T + A_2 A_2^T = (2v + 2)I - 2J.$$

Method II.
As described in Baumert [1, p.119], the best known n^{th} power residue difference sets are the quadratic residue sets of Paley [11].
When $v = 4t - 1$ is a prime power, the quadratic residues modulo v form a difference set with parameters $(v, k, \lambda) = (4t - 1, 2t - 1, t - 1)$. Thus, if we take this difference set twice we form a $2 - \{v; \frac{v-1}{2}, \frac{v-1}{2}; \frac{v-3}{2}\} = 2 - \{4t - 1; 2t - 1, 2t - 1; 2t - 1\}$ sds.
Their incidence $(1, -1)$ matrices are identical in this case, say $P = A_1 = A_2$, and then we have

$$2PP^T = (2v + 2)I - 2J.$$

We can write

$$AA^T = (2v + 2)I_{2v} - 2I_2 \times J_v.$$

It is easy to use the determinant simplification theorem [8] to see that for methods I and II we have

$$det\ A = 2^v(2v - 1)(v - 1)^{v-1}.$$

We also examine some other methods of constructing sds given in [18].
Method III.
When $q = 25+4b^2 \equiv 13(mod16)$ is a prime power, we consider the following $2-\{q; \frac{q-1}{4}, \frac{q-1}{2}; \frac{5q-17}{16}\}$ sds. In this case we have

$$A_1 A_1^T + A_2 A_2^T = (\frac{7q + 5}{4})I + (\frac{q - 5}{4})J.$$

As an example for $q = 29$ we obtain a $2 - \{29; 7, 14; 8\}$ sds, which gives

$$A_1 A_1^T + A_2 A_2^T = 52I + 6J.$$

Method IV.
When $q = 49 + 4b^2 \equiv 5(mod 16)$ is a prime power, we consider the following $2-\{q; \frac{q-1}{2}, \frac{3(q-1)}{4}; \frac{13q-33}{16}\}$ sds. In this case we have

$$A_1 A_1^T + A_2 A_2^T = (\frac{7q+17}{4})I + (\frac{q-17}{4})J.$$

As an example for $q = 53$ we obtain a $2 - \{53; 26, 39; 41\}$ sds, which gives

$$A_1 A_1^T + A_2 A_2^T = 97I + 9J.$$

Method V.
When $q = 1 + 4b^2 \equiv 5(mod 16)$ is a prime power, we consider the following $2-\{q; \frac{q-1}{2}, \frac{3(q-1)}{4}; \frac{13q-33}{16}\}$ sds. In this case we have

$$A_1 A_1^T + A_2 A_2^T = (\frac{7q+17}{4})I + (\frac{q-17}{4})J.$$

As an example for $q = 37$ we obtain a $2 - \{37; 18, 27; 28\}$ sds, which gives

$$A_1 A_1^T + A_2 A_2^T = 69I + 5J.$$

In the present paper we get values for the pivots [17] of $2 - \{v; k_1, k_2; \lambda\}$ sds, and $(1, -1)$ matrices of order $2v$ made from them. Calculations have given moderate values of growth for such matrices. An open problem concerning the possibility of finding $(1, -1)$ $2v \times 2v$ CP matrices having growth greater than $2v$ is posed.

Notation 1. Write A for a matrix of order n whose initial pivots are derived from matrices with CP structure. Write $A(j)$ for the absolute value of the determinant of the $j \times j$ principal submatrix in the upper lefthand corner of the matrix A and $A[j]$ for the absolute value of the determinant of the $(n-j) \times (n-j)$ principal submatrix in the bottom righthand corner of the matrix A. Throughout this paper when we have used i pivots we then find all possible values of the $A(n-i)$ minors. Hence, if any minor is CP it must have one of these values. The magnitude of the pivots appearing after the application of GE operations on a CP matrix W is given by

$$p_j = A(j)/A(j-1), \quad j = 1, 2, \dots, n, \quad A(0) = 1. \tag{2}$$

In particular for a CP A, constructed from the $\pm$ incidence matrices of 2-sds,

$$p_v = A(v)/A(v-1), \quad p_{v-1} = A(v-1)/A(v-2). \tag{3}$$

We use the notation M_j to denote the $j \times j$ minor of A.

2 Minors of Size $(2v-1)$

We denote by $\Delta(h,i,j,k,m)$ the following matrix of order $2v$:

$$\Delta(h,i,j,k,m) =$$

$$\left[\begin{array}{c|c|c|c} \overbrace{mI+(\Lambda-1)(J-I)}^{h} & \overbrace{(\Lambda+1)J}^{i} & \overbrace{-J}^{j} & \overbrace{J}^{k} \\ (\Lambda+1)J & mI+(\Lambda-1)(J-I) & J & -J \\ -J & J & mI+(\Lambda-1)(J-I) & -J \\ J & -J & -J & mI+(\Lambda-1)(J-I) \end{array}\right].$$

where $m = 2v = h+i+j+k$. Then by the Determinant Simplification Theorem [8]

$$\det\Delta(h,i,j,k,m) = (m-1)^{m-4}\left|\begin{array}{cccc} m+(h-1)(\Lambda-1) & (\Lambda+1)h & -h & h \\ (\Lambda+1)i & m+(i-1)(\Lambda-1) & i & -i \\ -j & j & m+(j-1)(\Lambda-1) & (\Lambda+1)j \\ k & -k & (\Lambda+1)k & m+(k-1)(\Lambda-1) \end{array}\right|$$

To find the $(2v-1)\times(2v-1)$ minors we remove the first row and column of A to get D. The matrix DD^T is obtained from $\Delta(h,i,j,k,m)$ by removing a row and the corresponding column and performing on it appropriate permutations of rows and columns. Removing a row is equivalent by taking $m-1$ rows and removing a column is equivalent by considering $h-1$ or $i-1$ or $j-1$ or $k-1$ columns. Thus det DD^T is det $\Delta(h-1,i,j,k,m-1)$ or det $\Delta(h,i-1,j,k,m-1)$ or det $\Delta(h,i,j-1,k,m-1)$ or det $\Delta(h,i,j,k-1,m-1)$.
In the sequel, we study the values of the $(2v-1)\times(2v-1)$ minors for the methods I-V mentioned in the Introduction.

Lemma 1 *The $(2v-1)\times(2v-1)$ minors of the matrix A given in (1), and constructed using Method II are:*

Proof. Here we use the $(1,-1)$ incidence matrices of the $2-\{4t-1; 2t-1, 2t-1; 2t-1\}$ sds.
PLEASE SPECIFY THESE VALUES
By the reasoning above, with $v = 4t-1$, $h =$, $i =$,$j =$, $k =$, $m =$, substituted into det $\Delta(h-1,i,j,k,m-1)$, det $\Delta(h,i,j-1,k,m-1)$, det $\Delta(h,i-1,j,k,m-1)$, and det $\Delta(h,i,j,k-1,m-1)$ we obtain the result.
Specifically the $(2v-1)\times(2v-1)$ minor is the square root of the determinant and are given by one of
1) det $\Delta(h-1,i,j,k,m-1) =$
2) det $\Delta(h,i-1,j,k,m-1) =$
3) det $\Delta(h,i,j-1,k,m-1) =$
4) det $\Delta(h,i,j,k-1,m-1) =$

□

3 Minors of size $(2v-2)$

We recall the partitioned matrix A of the design matrix, C, is composed from 2-$\{v;k_1,k_2;\lambda\}$ supplementary difference sets. Using the formula for the inner product of the rows of the $(1,-1)$ incidence matrix formed from these sds we see that the inner product is $2v-4(k_1+k_2-\lambda) = \Lambda$.

We now return to A with two rows and columns removed to find the generic matrix. In expanded form this gives for C, simplified by the Determinant Simplification Theorem [8], the matrix D given by

$$\begin{bmatrix} N & \Lambda u_2 & \Lambda u_3 & (\Lambda+2)u_4 & -2u_5 & 0 & 0 & 2u_8 \\ \Lambda u_1 & N & (\Lambda+2)u_3 & \Lambda u_4 & 0 & -2u_6 & 2u_7 & 0 \\ \Lambda u_1 & (\Lambda+2)u_2 & N & \Lambda u_4 & 0 & 2u_6 & -2u_7 & 0 \\ (\Lambda+2)u_1 & \Lambda u_2 & \Lambda u_3 & N & 2u_5 & 0 & 0 & -2u_8 \\ -2u_1 & 0 & 0 & 2u_4 & N & 2u_6 & 2u_7 & (\Lambda+2)u_8 \\ 0 & -2u_2 & 2u_3 & 0 & \Lambda u_5 & N & (\Lambda+2)u_7 & \Lambda u_8 \\ 0 & 2u_2 & -2u_3 & 0 & \Lambda u_5 & (\Lambda+2)u_6 & N & \Lambda u_8 \\ 2u_1 & 0 & 0 & -2u_4 & (\Lambda+2)u_5 & \Lambda u_6 & \Lambda u_7 & N \end{bmatrix}.$$

where $N = 2v - 2u_i + (u_i - 1)\Lambda$ on the diagonal.

This gives the determinant of A with two rows and columns removed, as $(2v-\Lambda)^{v-5}\sqrt{\det D}$. Diagramatically, we have used the matrix form

$$\begin{bmatrix} A_1 & A_2 \\ A_2^T & -A_1^T \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

For case I both rows and columns are removed from B_1; for case II one row is from B_1 and one from B_3 but both columns are from A_1; for case III one row is from B_1 and one from B_3 and one column is from B_1 and one column is from B_2.

$$
\begin{array}{cc c|l}
1 & 1 & & \\
\vdots & \vdots & \lambda_1 & \\
1 & 1 & & \\
1 & - & & v-2 \text{ rows} \\
\vdots & \vdots & k_1-\lambda_1 & \text{which have} \\
1 & - & & \text{inner} \\
- & 1 & & \text{product } \Delta \\
\vdots & \vdots & k_1-\lambda_1 & \text{with rows} \\
- & 1 & & \text{one and two} \\
- & - & & \\
\vdots & \vdots & v-2k_1+\lambda_1 & \\
- & - & & \\
\hline
1 & 1 & & \\
\vdots & \vdots & \lambda_2 & \\
1 & 1 & & \lambda=\lambda_1+\lambda_2 \\
1 & - & & v \text{ rows} \\
\vdots & \vdots & k_2-\lambda_2 & \text{which have} \\
1 & - & & \text{inner} \\
- & 1 & & \text{product } 0 \\
\vdots & \vdots & k_2-\lambda_2 & \text{with rows} \\
- & 1 & & \text{one and two} \\
- & - & & \\
\vdots & \vdots & v-2k_2+\lambda_2 & \\
- & - & &
\end{array}.
$$

Number of Rows of Each Type I and II

u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
λ_1-2	$k_1-\lambda_1$	$k_1-\lambda_1$	$v-2k_1+\lambda_1$	λ_2	$k_2-\lambda_2$	$k_2-\lambda_2$	$v-2k_2+\lambda_2$
λ_1-1	$k_1-\lambda_1-1$	$k_1-\lambda_1$	$v-2k_1+\lambda_1$	λ_2	$k_2-\lambda_2$	$k_2-\lambda_2$	$v-2k_2+\lambda_2$
λ_1-1	$k_1-\lambda_1$	$k_1-\lambda_1-2$	$v-2k_1+\lambda_1-1$	λ_2	$k_2-\lambda_2$	$k_2-\lambda_2$	$v-2k_2+\lambda_2$
λ_1-1	$k_1-\lambda_1$	$k_1-\lambda_1-2$	$v-2k_1+\lambda_1$	λ_2-1	$k_2-\lambda_2$	$k_2-\lambda_2$	$v-2k_2+\lambda_2$
λ_1-1	$k_1-\lambda_1$	$k_1-\lambda_1-2$	$v-2k_1+\lambda_1-1$	λ_2	$k_2-\lambda_2-1$	$k_2-\lambda_2$	$v-2k_2+\lambda_2$
λ_1-1	$k_1-\lambda_1-2$	$k_1-\lambda_1-2$	$v-2k_1+\lambda_1-1$	λ_2	$k_2-\lambda_1$	$k_2-\lambda_1$	$v-2k_2+\lambda_2$
λ_1	$k_1-\lambda_1-1$	$k_1-\lambda_1-1$	$v-2k_1+\lambda_1$	λ_2-1	$k_2-\lambda_2$	$k_2-\lambda_2$	$v-2k_2+\lambda_2$
λ_1	$k_1-\lambda_1-1$	$k_1-\lambda_1$	$v-2k_1+\lambda_1-1$	λ_2-1	$k_2-\lambda_2$	$k_2-\lambda_2$	$v-2k_2+\lambda_2$
λ_1	$k_1-\lambda_1-1$	$k_1-\lambda_1$	$v-2k_1+\lambda_1$	λ_2-1	$k_2-\lambda_2$	$k_2-\lambda_2$	$v-2k_2+\lambda_2$
λ_1	$k_1-\lambda_1-1$	$k_1-\lambda_1-1$	$v-2k_1+\lambda_1$	λ_2-1	$k_2-\lambda_2$	$k_2-\lambda_2$	$v-2k_2+\lambda_2$
λ_1	$k_1-\lambda_1-1$	$k_1-\lambda_1-1$	$v-2k_1+\lambda_1$	λ_2-1	$k_2-\lambda_2$	$k_2-\lambda_2$	$v-2k_2+\lambda_2$
λ_1	$k_1-\lambda_1-1$	$k_1-\lambda_1-1$	$v-2k_1+\lambda_1$	λ_2-1	$k_2-\lambda_2$	$k_2-\lambda_2$	$v-2k_2+\lambda_2$
λ_1	$k_1-\lambda_1-1$	$k_1-\lambda_1-1$	$v-2k_1+\lambda_1$	λ_2-1	$k_2-\lambda_2$	$k_2-\lambda_2$	$v-2k_2+\lambda_2$

Table 1

Case III

To help understand Case III we recall that in this case one column removed comes from the columns with k_1+k_2 ones per column and the other from the columns with $v-k_2+k_1$ ones per column in the original design. This means the generic form of these two columns is

1		1	
1		⋮	ρ
1		1	
1	k_1	−	
⋮		⋮	$k_1 - \rho$
1		−	
−		1	
⋮		⋮	$k_2 - \rho$
−	$v - k_1$	1	
−		−	
⋮		⋮	$v - k_1 - k_2 + \rho$
−		−	

1		1	
1		⋮	$k_2 - \rho$
1		1	
1	k_2	−	
⋮		⋮	ρ
1		−	
−		1	
⋮		⋮	$v - k_1 - k_2 + \rho$
−	$v - k_2$	1	
−		−	
⋮		⋮	$k_1 - \rho$
−		−	

Note they have inner product zero.

Number of Rows of Each Type III

u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 1$	$k_1 - \rho - 1$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$
$\rho - 2$	$k_1 - \rho$	$k_2 - \rho$	$v - k_1 - k_2 + \rho$	$k_2 - \rho$	ρ	$v - k_1 - k_2 + \rho$	$k_1 - \rho$

Table 2

4 Pivot structure for family of design matrices

Conjecture

Let A be an $2v \times 2v$ CP D-optimal design of KKSSS family which is constructed from $2-\{s^2+s+1; \frac{s(s-1)}{2}, \frac{s(s+1)}{2}; \frac{s(s-1)}{2}\}$ sds. Reduce A by GE. Set $\mathcal{P} = 2s^2+2s+1$.Then we conjecture

(i) $g(v, A) = \frac{s+1}{s}\mathcal{P}$, or $\frac{s}{s+1}\mathcal{P}$, or $\frac{s(s+1)}{s^2+s+1}\mathcal{P}$, or $\mathcal{P}$;

(ii) The last pivot is equal to $\frac{s+1}{s}\mathcal{P}$, or $\frac{s}{s+1}\mathcal{P}$, or $\frac{s(s+1)}{s^2+s+1}\mathcal{P}$, or $\mathcal{P}$;

(iii) The second last pivot can take the values given in Table 8.

(iv) Every pivot before the last has magnitude at most $2v$;

(v) The first four pivots are equal to $1, 2, 2, 4$;

(vi) The fifth pivot may be 2 or 3.

We prove (ii) and (iii) in this paper. (v) and (vi) were proved for Brouwer's SBIBD$(2s^2+2s+1, s^2, \frac{1}{2}s(s-1))$ in [7] and we also show they hold for the KKSSS family.

v	**growth**	**Pivot Pattern**
7	8	$(1,2,2,4,2,4,4,2,4,4,8,4,8,8)$
7	8	$(1,2,2,4,3,\frac{10}{3},3.2,4,4,4,4,4,8,8)$
22	12	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},4,3,6,6,2,4,4,8,6,\frac{20}{3},7,2,8,6,12,12)$

Table 8: Growth Factors and Pivots Patterns for small CP KKSSS designs

References

[1] L.D. Baumert, *Cyclic Difference Sets*, Lecture Notes in Mathematics, Vol. 182, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

[2] J. Day and B. Peterson, Growth in Gaussian elimination, *Amer. Math. Monthly*, 95 (1988), 489-513.

[3] H. Ehlich, Determinantenabschätzungen für binäre matrizen, *Math. Z.*, 83 (1964), 123-132.

[4] N. J. Higham, *Accuracy and Stability of Numerical Algorithms*, SIAM, Philadelphia, 1996.

[5] C. Koukouvinos, S. Kounias, and J. Seberry, Supplementary difference sets and optimal designs, *Discrete Math.*, 88 (1991), 49-58.

[6] C. Koukouvinos, M. Mitrouli and J. Seberry, Growth in Gaussian elimination for weighing matrices $W(n, n-1)$, *Linear Algebra and its Applications*, 306 (2000), 189-202.

[7] C. Koukouvinos, M. Mitrouli and J. Seberry, Values of minors of $(1, -1)$ incidence matrices of SBIBDs and their application to the growth problem, *Designs, Codes and Cryptography*, 23 (2001), 267-281.

[8] C. Koukouvinos, M. Mitrouli and J. Seberry, Values of minors of an infinite family of D-optimal designs and their application to the growth problem, *SIAM J. Matrix Anal. Appl.*, 23 (2001), 1-14.

[9] C. Koukouvinos, M. Mitrouli and J. Seberry, Values of minors of an infinite family of D-optimal designs and their application to the growth problem: II, (submitted).

[10] E. Lehmer, On residue difference sets, *Canad. J. Math.*, 5 (1953), 425-432.

[11] R.E.A.C. Paley, On orthogonal matrices, *J. Math. and Phys.*, 12 (1933), 311-320.

[12] J. Singer, A theorem in finite projective geometry and some applications to number theory, *Trans. Amer. Math. Soc.*, 43 (1938), 377-385.

[13] E. Spence, Skew-Hadamard matrices of the Goethals-Seidel type, *Canad. J. Math.*, 27 (1975), 555-560.

[14] J. Seberry Wallis, Some remarks on supplementary difference sets, *Colloquia Mathematica Societatis, Janos Bolyai*, No. 10, 1973, 1503-1526.

[15] J. Seberry Wallis and A.L. Whiteman, Some classes of Hadamard matrices with constant diagonal, *Bull. Austral. Math. Soc.*, 7 (1972), 233-249.

[16] A.L. Whiteman, A family of D-optimal designs, *Ars Combinatoria*, 30 (1990), 23-26.

[17] J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford University Press, London, 1988.

[18] M. Yamada, Supplementary difference sets and Jacobi sums, *Discrete Math.*, 103 (1992), 75-90.