Counting Techniques Specifying the Existence of Submatrices in Weighing Matrices

C. Kravvaritis[1,⋆], M. Mitrouli[1], and Jennifer Seberry[2]

[1] Department of Mathematics, University of Athens,
Panepistemiopolis 15784, Athens, Greece
{mmitroul, ckrav}@math.uoa.gr
[2] Centre for Computer Security Research, SITACS,
University of Wollongong, Wollongong, NSW, 2522, Australia
jennie@uow.edu.au

Abstract. Two algorithmic techniques for specifying the existence of a $k \times k$ submatrix with elements $0, \pm 1$ in a skew and symmetric conference matrix of order n are described. This specification is achieved using an appropriate computer algebra system.

Key Words and Phrases: Gaussian elimination, growth, complete pivoting, weighing matrices, symbolic computation.

AMS Subject Classification: 65F05, 65G05, 20B20.

1 Introduction

A $(0, 1, -1)$ matrix $W = W(n, k)$ of order n satisfying $WW^T = kI_n$ is called a *weighing matrix of order* n *and weight* k or simply a *weighing matrix*. A $W(n, n)$, $n \equiv 0 \pmod 4$, is called a Hadamard matrix of order n. A $W = W(n, k)$ for which $W^T = -W$, $n \equiv 0 \pmod 4$, is called a *skew–weighing matrix*. A $W = W(n, n-1)$ satisfying $W^T = W$, $n \equiv 2 \pmod 4$, is called a *symmetric conference matrix*. Conference matrices cannot exist unless $n-1$ is the sum of two squares: thus they cannot exist for orders $22, 34, 58, 70, 78, 94$. For more details and construction of weighing matrices the reader can consult the book of Geramita and Seberry [2]. Two matrices are said to be *Hadamard equivalent* or *H-equivalent* if one can be obtained from the other by a sequence of the operations: 1. Interchange any pairs of rows and/or columns; 2. Multiply any rows and/or columns through by -1. Two important properties of the weighing matrices, which follow directly from the definition, are: 1. Every row and column of a $W(n, k)$ contains exactly $n - k$ zeros 2. Every two distinct rows and columns of a $W(n, k)$ are orthogonal to each other, which means that their inner product is zero.

The usefulness and significance of studying properties of the weighing matrices lies in the fact that they have applications in several scientific areas. They are

⋆ This research was financially supported by IKY (National Foundation for Scholarships).

V.G. Ganzha, E.W. Mayr, and E.V. Vorozhtsov (Eds.): CASC 2005, LNCS 3718, pp. 294–305, 2005.

used in Coding Theory for producing error correcting codes with good properties regarding the minimum Hamming distance. They appear also in the Theory of Statistical Designs and in Cryptography. One of their most important applications is in Numerical Analysis, and in particular in the study of the problem of the growth factor [1], which appears in the technique of Gaussian Elimination (GE) for solving a system of the form $A \cdot \underline{x} = \underline{b}$, where $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is nonsingular. According to known theorems [5] the accuracy of the computed solution with GE, which means in fact the stability of GE, depends on the growth factor. So, is created the growth problem, which is actually the problem of determining the growth factor for various values of the order n.

Experiments that have been made in the past on the computer reveal that the weighing matrices have certain interesting properties regarding the structure of the pivots appearing after GE. We are interested in specifying the existence of submatrices with the maximum value of determinant, which are embedded inside a weighing matrix. Write $W(j)$ for the absolute value of the determinant of the $j \times j$ principal submatrix in the upper left corner of the matrix W. It can be proved that the magnitude of the pivots appearing after the application of GE operations on a CP (completely pivoted, no exchanges are needed during GE with complete pivoting) matrix W are given by

$$p_j = W(j)/W(j-1), \quad j = 1, 2, \ldots, n, \quad W(0) = 1. \tag{1}$$

It is obvious from the previous relationship that principal determinants (minors) of a matrix are strictly connected with the appearing pivots after GE, since the value of a pivot is the quotient of two minors. The purpose of this paper is to demonstrate two algorithmic techniques, which will prove the existence of specific submatrices embedded in every $W(n, n-1)$, for appropriate value of n. This will be done by showing that in every $W(n, n-1)$ the columns, which make up these matrices, can always be found. We have achieved our goal by applying the notion of symbolic manipulation on a Computer Algebra Package, such as Maple. By assigning all possible values to our variables we perform complete (exhaustive) searches for all the appearing cases. This is a technique that is used over and over in Cryptography to find impossibilities and possibilities.

In [3], the pivot structure of $W(n, n-1)$ was studied and the problem of specifying specific $k \times k$ $(0, 1, -1)$ matrices existing embedded in $W(n, n-1)$ was initially posed.

Lemma 1. *The possible absolute values of the determinants of all* $n \times n$ $(0, 1, -1)$ *matrices, where there is at most one zero in each row and column, is given in Table 1 for* $n = 2, 3, 4, 5$.

In [4] were proved the following lemmas 2,3 and 4:

Lemma 2. *Let* W *be a CP skew and symmetric matrix, of order* $n \geq 6$ *then if GE is performed on* W *the first two pivots are* 1 *and* 2.

Lemma 3. *Let* W *be a CP skew and symmetric conference matrix, of order* $n \geq 12$ *then if GE is performed on* W *the third pivot is* 2 *and the fourth pivot is* 3 *or* 4.

Table 1. Determinant Values for $n = 2, 3, 4, 5$

Order	*Maximum Determinant*	*Possible Determinant Values*
2×2	2	$0, 1, 2$
3×3	4	$0, 1, 2, 3, 4$
4×4	16	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16$
5×5	48	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18,$ $19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 32, 36, 40, 48$

2 Existence of Specific $(0, +, -)$ Submatrices in $W(n, n-1)$

Notation. Throughout this paper the elements of a $(0, 1, -1)$ matrix will be denoted by $(0, +, -)$. Let $\underline{y}^T_{\beta+1}$ the vectors containing the binary representation of each integer $\beta + 2^{k-1}$ for $\beta = 0, \ldots, 2^{k-1} - 1$. Replace all zero entries of $\underline{y}^T_{\beta+1}$ by -1 and define the $k \times 1$ vectors $\underline{u}_j = \underline{y}_{2^{k-1}-j+1}$, $j = 1, \ldots, 2^{k-1}$. We write U_k for all the $k \times (n - 2k + 1)$ matrices, in which $\underline{u}_j$ occurs x_j times. So

$$U_k = \begin{array}{ccccc} \overbrace{+\ldots+}^{x_1} & \overbrace{+\ldots+}^{x_2} & \ldots & \overbrace{+\ldots+}^{x_{2^{k-1}-1}} & \overbrace{+\ldots+}^{x_{2^{k-1}}} \\ +\ldots+ & +\ldots+ & \ldots & -\ldots- & -\ldots- \\ . & . & \ldots & . & . \\ . & . & \ldots & . & . \\ +\ldots+ & +\ldots+ & \ldots & +\ldots+ & -\ldots- \\ +\ldots+ & -\ldots- & \ldots & +\ldots+ & -\ldots- \end{array} = \begin{array}{ccccc} x_1 & x_2 & \ldots & x_{2^{k-1}-1} & x_{2^{k-1}} \\ + & + & \ldots & + & + \\ + & + & \ldots & - & - \\ \vdots & \vdots & & \vdots & \vdots \\ + & + & \ldots & - & - \\ + & - & \ldots & + & - \end{array}$$

where $x_1 + x_2 + \ldots + x_{2^{k-1}} = n - 2k + 1$.

Example 1.
$$U_3 = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \end{array}, \quad U_4 = \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & - & 1 & - & 1 & - & 1 & - \end{array}$$

2.1 An Algorithm Specifying the Existence of $k \times k$ $(0, +, -)$ Submatrices in a $W(n, n-1)$

The following algorithm specifies the existence of a $k \times k$ submatrix A in a $W(n, n-1)$, given that the upper left $(k-1) \times (k-1)$ submatrix B of A always exists in a $W(n, n-1)$.

Algorithm Exist 1
Step 1
Read the $k \times k$ matrix A and the $(k-1) \times (k-1)$ matrix B
Step 2
Create the matrix Z

$$Z = \left[\begin{array}{c|c|cccccc} & & 0 & + & + & \cdots & \cdots & + \\ & & y_{21} & 0 & y_{23} & \cdots & \cdots & y_{2k} \\ B & U_k & y_{31} & y_{32} & 0 & y_{34} & \cdots & y_{3k} \\ & & \vdots & \vdots & & \vdots & \ddots & \vdots \\ \hline + \; z_2 \cdots z_{k-1} & & y_{k1} & y_{k2} & y_{k3} & \cdots & y_{k,k-1} & 0 \end{array}\right], \text{ where } z_i,\ y_{ij} = \pm 1$$

If B contains columns with 0

they are excluded from the matrix $Z(:, n-k+1:n)$

Step 3

If A has r 0's

Demand that the r columns of $Z(:, n-k+1:n)$, in which the 0's are in the same position as in A, take the appropriate values y_{ij}:

they are identical with the r columns of A containing the 0's

Step 4

Procedure Solve

For all possible values of z_i,$i = 2, \ldots, k-1$

Form the system of $1 + \binom{k}{2}$ equations and 2^{k-1} variables which results from counting of columns and the inner products of every two distinct rows

Solve the system for all x_i

Find the minimum values for the x_i which correspond to the columns of A, given that the number of columns appearing in $Z(:, 1:k-1)$ is ≥ 1

Formulate (if necessary) conditions and/or restrictions for the order n or for some x_i:

the columns of A appear (the corresponding x_i are all ≥ 1)

End{of Procedure Solve}

Else

Do Procedure Solve

Complexity. Obviously, all the calculations of the algorithm are made in Step 4, where the system is solved. The system has $m = 1 + \binom{k}{2}$ equations with $v = 2^{k-1}$ variables, so it can be represented by an $m \times v$ matrix, with $v \geq m$. The solution of such a system requires about $f = m^2(v - \frac{m}{3})$ flops, if we use, for instance, QR factorization. Since the system is formed for all possible values of z_i, $i = 2, \ldots, k-1$, and z_i can be ± 1, we have $2^{k-1-2+1} = 2^{k-2}$ systems. Hence, we have totally $f \cdot 2^{k-2}$ flops.

Comments.

1. Clearly, any arbitrary $W(n, n-1)$ can be written always in the form of the matrix Z and z_i, y_{ij} can be ± 1.
2. In Procedure Solve the system of $1 + \binom{k}{2}$ equations and 2^{k-1} variables which results from the counting of all columns and the inner products of every two distinct rows, is formed only once. For every combination of all

possible values z_i,$i = 2, \ldots, k-1$, only the $k-1$ equations that result from the inner product between the $k-$th and the previous $k-1$ rows need to be changed every time.

3. Obviously, the system has exactly one solution only for k=3, otherwise it has infinite solutions, which are described by $2^{k-1} - 1 - \binom{k}{2}$ parameters.
4. If in the expression for the solution x_i appears n, we find the minimum value of n, for which we have $x_i \geq 1$. Otherwise, we either establish that always $x_i \geq 1$, or we apply conditions on the appearing parameters so that $x_i \geq 1$ holds.
5. If A contains some columns with 0's in its $(k-1) \times (k-1)$ upper left part, which is actually B, we exclude these columns from the submatrix $Z(:, n-k+1:n)$, and give to the corresponding z_i and y_{ij} appropriate values so that they are identical with the columns of A containing the 0's. If A contains a 0 in the k-th column or row, which means outside the $(k-1)\times(k-1)$ upper left sunmatrix B, then the corresponding column remains in $Z(:, n-k+1:n)$ and the variable z_i and y_{ij} in this column take appropriate values so that this column is identical with the one in A containing the 0 outside of B. After these subcases of Step 3 are examined, the matrix Z takes the desired form and the system is set up.
6. By saying "a submatrix A always exists in a $W(n, n-1)$" we mean actually that there exist always the columns of A in $W(n, n-1)$. Then, after a sequence of H-equivalent operations, A can appear on the upper left $k \times k$ block of the $W(n, n-1)$.
7. The Computer Package gives the ability to select the parameters among the variables before solving the system. In this way we take advantage of the appearance of the first $k-1$ columns of Z by assuming that the respective parameters are ≥ 1.

Implementation of the Algorithm Exist 1

Next we demonstrate the application of the above described Algorithm for various values of k.

1. Existence of 3×3 Matrices (k=3)

We want to establish whether the matrix

$$B_1 = \begin{bmatrix} + & + & + \\ + & - & + \\ + & + & - \end{bmatrix}$$

always exists in a $W(n, n-1)$. First we note that the upper left 2×2 submatrix of B_1 $\begin{bmatrix} + & + \\ + & - \end{bmatrix}$ always occurs in any $W(n, n-1)$, due to the orthogonality of the first two rows.

1. We have $A = B_1,\ B = \begin{bmatrix} + & + \\ + & - \end{bmatrix}$

2. We create

$$Z = \left[\begin{array}{cc|cccc|ccc} + & + & \overbrace{+}^{x_1} & \overbrace{+}^{x_2} & \overbrace{+}^{x_3} & \overbrace{+}^{x_4} & 0 & + & + \\ + & - & + & + & - & - & u & 0 & w \\ \cline{1-2} + & z & + & - & + & - & x & y & 0 \end{array}\right]$$

where u, w, x, y and z are ± 1.

3. **Case 1,** $z = 1$

For z=1, the system is

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= n - 5 \\ x_1 + x_2 - x_3 - x_4 &= -w \\ x_1 - x_2 + x_3 - x_4 &= -2 - y \\ x_1 - x_2 - x_3 + x_4 &= -ux \end{aligned} \tag{2}$$

The system has exactly one solution, as we have 4 equations with 4 unknowns. The solution is:

$$\begin{aligned} x_1 &= \tfrac{1}{4}(-y + n - ux - 7 - w) \\ x_2 &= \tfrac{1}{4}(y + n + ux - 3 - w) \\ x_3 &= \tfrac{1}{4}(-7 + w - y + n + ux) \\ x_4 &= \tfrac{1}{4}(y + n - ux - 3 + w) \end{aligned} \tag{3}$$

We need to specify whether $x_2 \geq 1$, since the other two columns of A, $[+,+,+]^T$ *and* $[+,-,+]^T$, are the first two columns of Z. The minimum value of x_2 is $\frac{n}{4} - \frac{6}{4}$. We have

$$x_2 \geq \frac{n}{4} - \frac{6}{4} \geq 1 \Leftrightarrow x_2 \geq 1 \ for \ n \geq 10$$

Hence, we have that B_1 exists in any $W(n, n-1)$ with $n \geq 10$.

For $z = -1$, the system differs in the third and fourth equation. After calculating the solution, in this case we need to specify whether $x_2, \ x_3 \geq 1$, since the columns of A $[+,+,+]^T$ is the first column of Z. Similarly, we get $x_2, \ x_3 \geq 1$ for $n \geq 10$. Consequently B_1 exists in any $W(n, n-1)$ with $n \geq 10$.

With a similar argument we can prove that $B_2 = \begin{bmatrix} + & + & + \\ + & - & 0 \\ + & + & - \end{bmatrix}$ exists in any $W(n, n-1)$ with $n \geq 10$.

Lemma 4. *The matrices B_1 or B_2 always exist in a $W(n, n-1)$ with $n \geq 10$.*

Remark 1. The maximum value of the 3×3 minor of a $W(n, n-1)$ is equal, according to the previous results, to the absolute value of determinant of B_1 and B_2, which is 4.

With a similar argument we can prove the following Lemma:

Lemma 5. *The matrices* $A_1 = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$ *or* $A_2 = \begin{bmatrix} + & + & 0 & - \\ + & - & - & - \\ + & - & + & + \\ + & + & - & + \end{bmatrix}$ *always exist in a* $W(n, n-1)$ *with* $n \geq 10$.

Remark 2. The maximum values of the 4×4 minors of a $W(n, n-1)$ are equal, according to the previous results, to the absolute values of determinants of A_1 and A_2, which are 16 and 12 respectively.

2. Existence of 5×5 Matrices (k=5)

We want to establish whether the matrix $C_8 = \begin{bmatrix} + & + & 0 & - & + \\ + & - & - & - & - \\ + & - & + & + & + \\ + & + & - & + & - \\ + & + & + & - & - \end{bmatrix}$ always exists in a $W(n, n-1)$. First we note that the upper left 4×4 submatrix of C_8 is A_2, which was proved previously that always occurs in any $W(n, n-1)$.

1. We have $A = C_8,\ B = A_2$
2. We create

$$Z = \left[\begin{array}{cccc|} + & + & 0 & - \\ + & - & - & - \\ + & - & + & + \\ + & + & - & + \\ \hline + & z & + & w \end{array}\right] \quad U_5 \quad \left|\begin{array}{cccc} + & + & + & + \\ 0 & a & b & c \\ d & 0 & e & f \\ g & h & 0 & k \\ l & m & p & 0 \end{array}\right]$$

where
$U_5 =$

$$\begin{bmatrix} \overbrace{+}^{x_1} & \overbrace{+}^{x_2} & \overbrace{+}^{x_3} & \overbrace{+}^{x_4} & \overbrace{+}^{x_5} & \overbrace{+}^{x_6} & \overbrace{+}^{x_7} & \overbrace{+}^{x_8} & \overbrace{+}^{x_9} & \overbrace{+}^{x_{10}} & \overbrace{+}^{x_{11}} & \overbrace{+}^{x_{12}} & \overbrace{+}^{x_{13}} & \overbrace{+}^{x_{14}} & \overbrace{+}^{x_{15}} & \overbrace{+}^{x_{16}} \\ + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\ + & + & + & + & - & - & - & - & + & + & + & - & - & - & + & - \\ + & + & - & - & + & + & - & - & + & + & - & + & - & + & - & - \\ + & - & + & - & + & - & + & - & + & - & + & + & + & - & - & - \end{bmatrix}$$

and a, b, c, d, e, f, g, h, k, l, m, n, p, z and w, are ± 1. As described in Comment 5, the column $[0, -, +, -, +]^T$ is excluded from $Z(:, n-4:n)$ and the values of the variables in this column remain fixed.

3. **Case 1,** $z = 1$**,** $w = 1$
The system is

$$\begin{array}{l}
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} \\
+x_{15} + x_{16} = n - 8 \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 - x_9 - x_{10} - x_{11} - x_{12} - x_{13} - x_{14} \\
-x_{15} - x_{16} = -1 - a - b - c \\
x_1 + x_2 + x_3 + x_4 - x_5 - x_6 - x_7 - x_8 + x_9 + x_{10} + x_{11} - x_{12} - x_{13} - x_{14} \\
+x_{15} - x_{16} = 1 - d - e - f \\
x_1 + x_2 - x_3 - x_4 + x_5 + x_6 - x_7 - x_8 + x_9 + x_{10} - x_{11} + x_{12} - x_{13} + x_{14} \\
-x_{15} - x_{16} = -1 - g - h - k \\
x_1 - x_2 + x_3 - x_4 + x_5 - x_6 + x_7 - x_8 + x_9 - x_{10} + x_{11} + x_{12} + x_{13} - x_{14} \\
-x_{15} - x_{16} = -1 - l - m - p \\
x_1 + x_2 + x_3 + x_4 - x_5 - x_6 - x_7 - x_8 - x_9 - x_{10} - x_{11} + x_{12} + x_{13} + x_{14} \\
-x_{15} + x_{16} = -be - cf \\
x_1 + x_2 - x_3 - x_4 + x_5 + x_6 - x_7 - x_8 - x_9 - x_{10} + x_{11} - x_{12} + x_{13} - x_{14} \\
+x_{15} + x_{16} = -ah - ck \\
x_1 - x_2 + x_3 - x_4 + x_5 - x_6 + x_7 - x_8 - x_9 + x_{10} - x_{11} - x_{12} - x_{13} + x_{14} \\
+x_{15} + x_{16} = 2 - am - bp \\
x_1 + x_2 - x_3 - x_4 - x_5 - x_6 + x_7 + x_8 + x_9 + x_{10} - x_{11} - x_{12} + x_{13} - x_{14} \\
-x_{15} + x_{16} = -dg - fk \\
x_1 - x_2 + x_3 - x_4 - x_5 + x_6 - x_7 + x_8 + x_9 - x_{10} + x_{11} - x_{12} - x_{13} + x_{14} \\
-x_{15} + x_{16} = -2 - dl - ep \\
x_1 - x_2 - x_3 + x_4 + x_5 - x_6 - x_7 + x_8 + x_9 - x_{10} - x_{11} + x_{12} - x_{13} - x_{14} \\
+x_{15} + x_{16} = -2 - gl - hm
\end{array} \tag{4}$$

The system apparently has an infinite number of solutions, which depend on five parameters, as we have 11 equations with 16 unknowns. We have chosen between the five parameters x_1, x_8 and x_{12} because, in this case, the respective columns appear in Z and we want to make use of this fact by assuming $x_1,\ x_8,\ x_{12} \geq 1$. The other two parameters can be chosen arbitrary. The solution is:

$$\begin{array}{rl}
x_2 = & \frac{1}{4}(-f - a - b - fk - dg - ck - ah - c - d - e) - x_1 + x_{12} + x_{14} \\
x_3 = & \frac{1}{4}(-8 - be - cf - ep - dl - bp - am + n) - x_1 - x_{14} - x_{16} \\
x_4 = & \frac{1}{4}(ep + dl + fk + dg + bp + am + ck + ah) + x_1 - x_{12} + x_{16} \\
x_5 = & \frac{1}{4}(-g + ep + dl + fk + dg - l - h - k - m - p) + x_8 - x_{12} + x_{16} \\
x_6 = & \frac{1}{4}(-10 + f - ep - dl + l + n + d + e + m + p) - x_8 - x_{14} - x_{16} \\
x_7 = & \frac{1}{4}(g - a - b + be + cf - fk - dg - c + h + k) - x_8 + x_{12} + x_{14} \\
x_9 = & \frac{1}{4}(-12 - ep - dl - fk - dg - hm - gl + n) - x_1 - x_8 - x_{16} \\
x_{10} = & \frac{1}{4}(4 - g + a + b + ep + dl + fk + dg + ck + ah + hm + gl + c - h - k) \\
& +x_1 + x_8 - x_{12} - x_{14} + x_{16} \\
x_{11} = & \frac{1}{4}(10 - f + be + cf + ep + dl + fk + dg + bp + am + hm + gl - l - n \\
& -d - e - m - p) + x_1 + x_8 + x_{14} + 2x_{16} \\
x_{13} = & \frac{1}{4}(-8 + f + a + b - be - cf + n + d + c + e) - x_{12} - x_{14} - x_{16} \\
x_{15} = & \frac{1}{4}(-8 + g - ep - dl - fk - dg - bp - am - ck - ah - hm - gl + l + n \\
& +m + h + k + p) - x_1 - x_8 + x_{12} - 2x_{16}
\end{array} \tag{5}$$

We need to specify whether $x_7 \geq 1$, since the other columns of A appear in this case in Z. The minimum value of x_7 is $-\frac{10}{4} - x_8 + x_{12} + x_{14}$. We have

$$x_7 \geq -\frac{6}{4} - x_8 + x_{14} \geq 1 \Leftrightarrow x_{14} \geq 1 + \frac{6}{4} + x_8 \geq \frac{14}{4}$$

which means actually $x_{14} \geq 4$.

Hence, we have that C_8 exists in any $W(n, n-1)$ only if there exist at least 4 columns of the form $[+,-,-,+,-]^T$ or H-equivalent to it.
With similar arguments we deal with the other 3 cases and in every case it is proved that C_8 exists in any $W(n, n-1)$ if and only if $x_{14} \geq 4$.

Remark 3. It is obvious that for larger orders k the previous algorithm will encounter difficulties at extracting the wished results. Apart from this, results of the type "C_8 exists in any $W(n, n-1)$ if and only if $x_{14} \geq 4$" are not very general and consequently of less importance. So, we needed a more sophisticated technique which is more efficient in practice, provides more general results and can be used more easily for larger dimensions n.

2.2 Another Algorithm Specifying the Existence of $k \times k$ $(0, +, -)$ Submatrices in a $W(n, n-1)$

Notation. We denote by $U_{k,3}$ the first three rows of the previously defined matrix U_k.

$$U_{k,3} = \begin{array}{ccccc} x_1 & x_2 & \dots & x_{2^{k-1}-1} & x_{2^{k-1}} \\ 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & - & - \\ 1 & 1 & \dots & - & - \end{array}$$

Example 2. $U_{3,3} = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \end{array} = U_3$, $U_{4,3} = \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \end{array}$

The following algorithm specifies the existence of a $k \times k$ submatrix A in a $W(n, n-1)$, given that the upper left $(k-1) \times (k-1)$ submatrix B of A always exists in a $W(n, n-1)$.

Algorithm Exist 2
Step 1
Read the $k \times k$ matrix A and the $(k-1) \times (k-1)$ matrix B
Step 2
Denote with C the upper left $3 \times (k-1)$ submatrix of B
Step 3
Create the matrix Y_k

$$Y_k = \left[\begin{array}{c|c|cccccc} C & U_{k,3} & 0 & + & + & + & \cdots & + \\ & & y_{21} & 0 & y_{23} & y_{24} & \cdots & y_{2k} \\ & & y_{31} & y_{32} & 0 & y_{34} & \cdots & y_{3k} \end{array}\right], \text{ where } y_{ij} = \pm 1$$

Step 4
Compute the linear system resulting from the dimension and the orthogonality of the rows of Y_k and deduce the distribution of the number of columns of Y_k
Step 5
Find the maximum values for the x_i which correspond to the columns of A
Step 6
Formulate (if necessary) conditions for the order n:
the columns of A appear (the corresponding x_i are all ≥ 1)

Complexity. All the calculations of the algorithm are elementary and take place in Step 4. Precisely, we have totally $4 \cdot (2^{k-1} + 1)$ additions or subtractions, so the complexity is kept at low levels.

Comments.

1. Clearly, the first three rows of any arbitrary $W(n, n-1)$ can be written always in the form of the matrix Y_k and z_i, y_{ij} can be ± 1.
2. The matrix Y_k, which is created in Step 3, is in fact a submatrix of the matrix Z, as defined in the previous algorithm. In order to take advantage of the fact that B always exists, we include separately the first three rows of B in the matrix C.
3. In Step 4 is formulated a Distribution type Lemma, which gives the number of several columns appearing in a weighing matrix and will allow us to obtain bounds on the column structure of a weighing matrix. This Lemma results from the solution of the system, which is set up from counting of all columns and the inner products of every two distinct rows that must be zero.

Implementation of the Algorithm Exist 2

1. Existence of 5×5 Matrices (k=5)

We want to establish whether the matrix $C_1 = \begin{bmatrix} + & + & + & + & + \\ + & - & + & - & - \\ + & - & - & + & + \\ + & + & - & - & + \\ + & + & - & + & - \end{bmatrix} = [x_1\ x_{12}\ x_8\ x_{11}\ x_{10}]$

always exists in a $W(n, n-1)$. First we note that the upper left 4×4 submatrix of C_1 is A_1, which was proved previously that always occurs in any $W(n, n-1)$.

1. We have $A = C_1,\ B = A_1$
2. $C = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & - & - & + \end{bmatrix}$

3. We create
$$Y_5 = \left[\begin{array}{cccc|c|ccccc} + & + & + & + & & 0 & + & + & + & + \\ + & - & + & - & U_{5,3} & q & 0 & a & b & c \\ + & - & - & + & & r & d & 0 & e & f \end{array}\right]$$
where a, b, c, d, e, f, q and r are ± 1 and
$U_{5,3} =$
$$\left[\begin{array}{cccccccccccccccc} \overbrace{+}^{x_1} & \overbrace{+}^{x_2} & \overbrace{+}^{x_3} & \overbrace{+}^{x_4} & \overbrace{+}^{x_5} & \overbrace{+}^{x_6} & \overbrace{+}^{x_7} & \overbrace{+}^{x_8} & \overbrace{+}^{x_9} & \overbrace{+}^{x_{10}} & \overbrace{+}^{x_{11}} & \overbrace{+}^{x_{12}} & \overbrace{+}^{x_{13}} & \overbrace{+}^{x_{14}} & \overbrace{+}^{x_{15}} & \overbrace{+}^{x_{16}} \\ + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\ + & + & + & + & - & - & - & - & + & + & + & - & - & - & + & - \end{array}\right]$$

4. The Distribution Lemma for this case results from the following manipulation of the equations (they result from the dimension and the orthogonality):
$$x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}+x_{11}+x_{12}+x_{13}+x_{14}+x_{15}+x_{16} = n-9 \quad (6)$$
$$x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8-x_9-x_{10}-x_{11}-x_{12}-x_{13}-x_{14}-x_{15}-x_{16} = -a-b-c \quad (7)$$
$$x_1+x_2+x_3+x_4-x_5-x_6-x_7-x_8+x_9+x_{10}+x_{11}-x_{12}-x_{13}-x_{14}+x_{15}-x_{16} = -d-e-f \quad (8)$$
$$x_1+x_2+x_3+x_4-x_5-x_6-x_7-x_8-x_9-x_{10}-x_{11}+x_{12}+x_{13}+x_{14}-x_{15}+x_{16} = -qr-be-cf \quad (9)$$
$(6)+(7)+(8)+(9) : x_1+x_2+x_3+x_4 = \frac{1}{4}(n-9-a-b-c-g-h-k-qr-be-cf)$
$(6)+(7)-(8)-(9) : x_5+x_6+x_7+x_8 = \frac{1}{4}(n-9-a-b-c+d+e+f+qr+be+cf)$
$(6)-(7)+(8)-(9) : x_9+x_{10}+x_{11}+x_{12} = \frac{1}{4}(n-9+a+b+c-d-e-f+qr+be+cf)$
$(6)-(7)-(8)+(9) : x_{13}+x_{14}+x_{15}+x_{16} = \frac{1}{4}(n-9+a+b+c+d+e+f-qr-be-cf)$

Lemma 6. ***(Distribution Lemma)****. Let W be any $W(n, n-1)$ of order $n > 2$ with its first three rows written in the form of Y_5. Then the number of columns which are*

(a) $(+,+,+)^T$ *or* $(-,-,-)^T$ *is* $\frac{1}{4}(n-9-a-b-c-g-h-k-qr-be-cf)$
(b) $(+,+,-)^T$ *or* $(-,-,+)^T$ *is* $\frac{1}{4}(n-9-a-b-c+d+e+f+qr+be+cf)$
(c) $(+,-,+)^T$ *or* $(-,+,-)^T$ *is* $\frac{1}{4}(n-9+a+b+c-d-e-f+qr+be+cf)$
(d) $(+,-,-)^T$ *or* $(-,+,+)^T$ *is* $\frac{1}{4}(n-9+a+b+c+d+e+f-qr-be-cf)$

5. We have obviously
$$x_1 \le \frac{1}{4}(n-9-a-b-c-g-h-k-qr-be-cf)$$
$$x_{10}, x_{11}, x_{12} \le \frac{1}{4}(n-9+a+b+c-d-e-f+qr+be+cf)$$
$$x_8 \le \frac{1}{4}(n-9-a-b-c+d+e+f+qr+be+cf)$$
By assigning all possible values ± 1 to the variables, we get: x_1, x_8, x_{10}, x_{11}, $x_{12} \le \frac{1}{4}(n-4)$.
6. Hence, $1 \le x_1 \le \frac{1}{4}(n-4) \Leftrightarrow n \ge 8$. Similarly, $x_8, x_{10}, x_{11}, x_{12} \ge 1 \Leftrightarrow n \ge 8$. So we have that for $n \ge 8$ C_1 always exists. In a similar way can be proved the same result for the matrices in the following lemma.

Lemma 7. *One of the following matrices, named C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9 and C_{10} respectively,*
$$\begin{bmatrix} + & + & + & + & + \\ + & - & + & - & - \\ + & - & - & + & + \\ + & + & - & - & + \\ + & + & - & + & - \end{bmatrix} \begin{bmatrix} + & + & + & + & + \\ + & - & + & - & - \\ + & - & - & + & 0 \\ + & + & - & - & + \\ + & + & - & + & - \end{bmatrix} \begin{bmatrix} + & + & + & + & + \\ + & - & + & - & - \\ + & - & - & + & + \\ + & + & - & - & + \\ + & + & - & 0 & - \end{bmatrix} \begin{bmatrix} + & + & + & + & + \\ + & - & + & - & - \\ + & - & - & + & - \\ + & + & - & - & - \\ + & + & + & + & - \end{bmatrix} \begin{bmatrix} + & + & + & + & + \\ + & - & + & - & - \\ + & - & - & + & - \\ + & + & - & - & 0 \\ + & + & - & + & - \end{bmatrix}$$

$$\begin{bmatrix} + & + & + & + & + \\ + & - & + & - & - \\ + & - & - & + & - \\ + & + & - & - & + \\ + & + & 0 & - & - \end{bmatrix} \begin{bmatrix} + & + & + & + & + \\ + & - & + & - & - \\ + & - & - & + & + \\ + & + & - & - & + \\ + & + & - & + & 0 \end{bmatrix} \begin{bmatrix} + & + & 0 & - & + \\ + & - & - & - & - \\ + & - & + & + & + \\ + & + & - & + & - \\ + & + & + & - & - \end{bmatrix} \begin{bmatrix} + & + & 0 & - & + \\ + & - & - & - & - \\ + & - & + & + & 0 \\ + & + & - & + & - \\ + & + & + & - & - \end{bmatrix} \begin{bmatrix} + & + & 0 & - & + \\ + & - & - & - & - \\ + & - & + & + & + \\ + & + & - & + & 0 \\ + & + & + & 0 & - \end{bmatrix}$$

always exists in a $W(n, n-1)$ with $n \geq 8$.

Remark 4. The maximum values of the 5×5 minors of a $W(n, n-1)$ are equal, according to the previous results, to the absolute values of determinants of $C_1, \ldots, C_{10}$, which are 48, 40, 36 and 32.

Theorem 1. *Let W be a CP skew and symmetric conference matrix, of order $n \geq 8$ then if GE is performed on W the fifth pivot is 2 or 3 or $\frac{9}{4}$ or $\frac{10}{3}$ or $\frac{10}{4}$.*

Proof. It follows obviously from lemma 7, remark 4 and relationship (1). □

2.3 Conclusions

The object of our work was to find an algorithm able to decide if specific $(0, +, -)$ submatrices of order k exist embedded inside a CP weighing matrix $W(n, n-1)$. If such a submatrix exists, then, after a sequence of H-equivalent operations, it can appear on the upper left $k \times k$ block of the $W(n, n-1)$. So, the $k \times k$ principal minor of the $W(n, n-1)$ is equal to the determinant of this submatrix. Hence, according to relationship (1), the pivots of the $W(n, n-1)$ can be calculated.

By applying algorithms Exist 1 and Exist 2, we obtained the values for the fifth pivot of a CP skew and symmetric conference matrix of order $n \geq 8$. The use of symbolic algebra packages is required for the solution of the systems appearing in the implementation of the algorithms.

An issue open for research is to apply Algorithm Exist 2, or a more improved form of it, for the next orders of submatrices $k = 6, 7, \ldots$, in order to prove more appearing values in the pivot structure of a $W(n, n-1)$ for large enough n. Also the alteration of the parameters of the Algorithms, so that they can be used more generally for $W(n, n-p)$, $p = 2, 3, \ldots$, is interesting and is dealt with currently.

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