

Application of Nega-cyclic Matrices to Generate Spreading Sequences

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Abstract: In the paper, we present a new class of orthogonal bipolar spreading sequences designed based on Goethals-Seidel construction with nega-cyclic matrices. The sequences can be designed for any length equal to $4 \pmod{8}$, and possess good correlation properties. In particular, their aperiodic autocorrelation characteristics are very good. That can be traded off for improvement in the cross-correlation performance using a diagonal modification method, as shown in the example.

1. Introduction

Orthogonal bipolar sequences are of a great practical interest for the current and future direct sequence (DS) code-division multiple-access (CDMA) systems where the orthogonality principle can be used for channels separation, e.g. [1]. The most commonly used sets of bipolar sequences are Walsh-Hadamard sequences [2], as they are easy to generate and simple to implement. However, they exist only for sequence lengths being an integer power of 2, which can be a limiting factor in some applications. In the paper, we describe a technique to generate sets of bipolar sequences of order $N \equiv 4 \pmod{8}$ based on Goethals-Seidel construction with nega-cyclic matrices used instead of circulant matrices in filling the array [3]. The resultant sets of sequences possess very good autocorrelation properties that make them amenable to synchronization requirements.

It is well known, e.g. [4, 5, 6], that if the sequences have good aperiodic cross-correlation properties, the transmission performance can be improved for those CDMA systems where different propagation delays exist. Wysocki and Wysocki in [7] proposed a technique to modify bipolar Walsh-Hadamard sequences to achieve changes in their correlation characteristics without compromising orthogonality. In this paper, we apply the same technique to improve cross-correlation properties

of sequences generated from nega-cyclic matrices. As it is always the case, the improvement is achieved at the expense of slightly worsening the autocorrelation properties. However, the overall autocorrelation properties of the modified sequence sets are still significantly better than those of Walsh-Hadamard sequences of comparable lengths.

The paper is organized as follows. In Section 2, we introduce principles of constructing Hadamard matrices using nega-cyclic matrices and the Goethals-Seidel construction. Section 3 introduces some correlation measures that can be used to compare different sets of spreading, and shows the values of those parameters for an example sequence set generated from nega-cyclic matrices of lengths 5. In Section 4, we briefly describe the method used to modify correlation characteristics of sequence sets, and show the results when applied to the examples given in Section 3. Section 5 concludes the paper.

2. Sequence construction technique

In [3] Ang, Seberry, and Wysocki have introduced a construction technique that allows designing of Hadamard matrices using nega-cyclic matrices. An Hadamard matrix \mathbf{H} of order n is a square matrix of '1s' and '-1s' having inner product of distinct rows equal to zero. Hence, we have $\mathbf{H}\mathbf{H}^T = n\mathbf{I}_n$, where \mathbf{H}^T means transposition of the matrix \mathbf{H} , and \mathbf{I}_n is a unit matrix of order n . We note that n can take values of 1, 2, or $n \equiv 0 \pmod{4}$.

Circulant matrices of order n are polynomials in the shift matrix \mathbf{S} , given by:

$$\mathbf{S} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (1)$$

If \mathbf{A} is a circulant matrix of odd order, then \mathbf{XAX} , where $\mathbf{X} = \text{diag}(1, -1, 1, -1, \dots, 1)$ is said to be a nega-cyclic matrix. Nega-cyclic matrices of order n are polynomials in the nega-shift matrix \mathbf{N} .

The back-diagonal matrix \mathbf{R} of order n is the matrix whose elements r_{ij} are given by:

$$r_{ij} = \begin{cases} 1 & \text{if } i + j = n + 1 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

where $i, j = 1, 2, \dots, n$.

It can be shown (see [3]) that if there exist four nega-cyclic matrices \mathbf{A} , \mathbf{B} , \mathbf{C} , and \mathbf{D} of order n satisfying the condition:

$$\mathbf{AA}^T + \mathbf{BB}^T + \mathbf{CC}^T + \mathbf{DD}^T = 4n\mathbf{I}_n \quad (3)$$

Then

$$\mathbf{GS} = \begin{bmatrix} \mathbf{A} & \mathbf{BR} & \mathbf{CR} & \mathbf{DR} \\ -\mathbf{BR} & \mathbf{A} & \mathbf{D}^T\mathbf{R} & -\mathbf{C}^T\mathbf{R} \\ -\mathbf{CR} & -\mathbf{D}^T\mathbf{R} & \mathbf{A} & \mathbf{B}^T\mathbf{R} \\ -\mathbf{DR} & \mathbf{C}^T\mathbf{R} & -\mathbf{B}^T\mathbf{R} & \mathbf{A} \end{bmatrix} \quad (4)$$

is a Hadamard matrix of order $4n$.

This construction is very similar to the original Goethals-Seidel arrays. However, instead of cyclic matrices we use here nega-cyclic ones. In [3], a method to choose the matrices \mathbf{A} , \mathbf{B} , \mathbf{C} , and \mathbf{D} has been shown, and the number of Hadamard matrices that can be generated this way is very large, particularly for $n \geq 5$.

3. Desired Characteristics of CDMA Spreading Sequences

For bipolar spreading sequences $\{s_n^{(i)}\}$ and $\{s_n^{(l)}\}$ of length N , the normalized discrete aperiodic correlation function is defined as [5]:

$$c_{i,l}(\tau) = \begin{cases} \frac{1}{N} \sum_{n=0}^{N-1-\tau} s_n^{(i)} s_{n+\tau}^{(l)}, & 0 \leq \tau \leq N-1 \\ \frac{1}{N} \sum_{n=0}^{N-1+\tau} s_{n-\tau}^{(i)} s_n^{(l)}, & 1-N \leq \tau < 0 \\ 0, & |\tau| \geq N \end{cases} \quad (5)$$

When $\{s_n^{(i)}\} = \{s_n^{(l)}\}$, the above equation defines the normalized discrete aperiodic auto-correlation function.

In order to evaluate the performance of a whole set of M spreading sequences, the average mean-square value of cross-correlation for all sequences in the set, denoted by R_{CC} , was introduced by Oppermann and Vucetic [5] as a measure of the set cross-correlation performance:

$$R_{CC} = \frac{1}{M(M-1)} \sum_{i=1}^M \sum_{\substack{k=1 \\ k \neq i}}^M \sum_{\tau=1-N}^{N-1} |c_{i,k}(\tau)|^2 \quad (6)$$

A similar measure, denoted by R_{AC} was introduced there for comparing the auto-correlation performance:

$$R_{AC} = \frac{1}{M} \sum_{i=1}^M \sum_{\substack{\tau=1-N \\ \tau \neq 0}}^{N-1} |c_{i,i}(\tau)|^2 \quad (7)$$

The R_{AC} allows for comparison of the auto-correlation properties of the set of spreading sequences on the same basis as their cross-correlation properties.

It is highly desirable to have both R_{CC} and R_{AC} as low as possible, as the higher value of R_{CC} results in stronger multi-access interference (MAI), and an increase in the value of R_{AC} impedes code acquisition process. Unfortunately, decreasing the value of R_{CC} causes increase in the value of R_{AC} , and vice versa.

Both R_{CC} and R_{AC} are very useful for large sequence sets and large number of active users, when the constellation of interferers (i.e. relative delays among the active users and the spreading codes used) changes randomly for every transmitted information symbol. However, for a more static situation, when the constellation of interferers stays constant for the duration of many information symbols, it is also important to consider the worst-case scenarios. This can be accounted for by analyzing the maximum value of peaks in the aperiodic cross-correlation functions over the whole set of sequences and in the aperiodic autocorrelation function for $\tau \neq 0$. Hence, one needs to consider two additional measures to compare the spreading sequence sets:

- Maximum value of the aperiodic cross-correlation functions C_{\max}

$$c_{\max}(\tau) = \max_{\substack{i=1,\dots,M \\ k=1,\dots,M \\ i \neq k}} |c_{i,k}(\tau)|; \quad \tau = (-N+1), \dots, (N-1)$$

$$C_{\max} = \max_{\tau} \{c_{\max}(\tau)\} \quad (8)$$

- Maximum value of the off-peak aperiodic autocorrelation functions A_{\max}

$$a_{\max}(\tau) = \max_{k=1,\dots,M} |c_{k,k}(\tau)|; \quad (9)$$

$$A_{\max} = \max_{\tau \neq 0} \{a_{\max}(\tau)\}$$

The known relationships between C_{\max} and A_{\max} are due to Welch [8] and Levenshtein [9].

The Welch bound and states that for any set of M bipolar sequences of length N

$$\max\{C_{\max}, A_{\max}\} \geq \sqrt{\frac{M-1}{2NM-M-1}} \quad (10)$$

A tighter Levenshtein bound is expressed by:

$$\max\{C_{\max}, A_{\max}\} \geq \sqrt{\frac{(2N^2+1)M-3N^2}{3N^2(MN-1)}} \quad (11)$$

It must be noted here that both Welch and Levenshtein bounds are derived for sets of bipolar sequences where the condition of orthogonality for perfect synchronization is not imposed. Hence, one can expect that by introducing the orthogonality condition, the lower bound for the aperiodic cross-correlation and aperiodic out-of-phase autocorrelation magnitudes must be significantly lifted.

As an example let us consider a set of spreading sequences designed using four nega-cyclic matrices of order 5, **A**, **B**, **C**, and **D** given below:

$$\mathbf{A} = \begin{bmatrix} - & - & - & - & - \\ + & - & - & - & - \\ + & + & - & - & - \\ + & + & + & - & - \\ + & + & + & + & - \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} + & - & + & - & - \\ + & + & - & + & - \\ + & + & + & - & + \\ - & + & + & + & - \\ + & - & + & + & + \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} + & - & - & - & + \\ - & + & - & - & - \\ + & - & + & - & - \\ + & + & - & + & - \\ + & + & + & - & + \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} + & - & + & + & - \\ + & + & - & + & + \\ - & + & + & + & + \\ - & - & + & + & - \\ + & - & - & + & + \end{bmatrix}.$$

In the matrices **A**, **B**, **C**, and **D**, we used the notation ‘+’ and ‘-’ instead of ‘1’ and ‘-1’, respectively. The resulting matrix **GS** equals to:

$$\mathbf{GS}_{20} = \begin{bmatrix} - & - & - & - & - & + & - & + & + & - & - & - & - & + & - & + & - & - & + \\ + & - & - & - & - & + & - & + & - & - & - & - & - & + & - & + & - & - & + \\ + & + & - & - & - & + & - & + & + & - & - & - & - & + & - & + & - & - & + \\ + & + & + & - & - & + & + & - & - & - & - & - & - & + & - & + & - & - & + \\ + & + & + & + & - & + & + & - & + & - & - & - & - & + & + & + & - & - & + \\ + & + & - & + & - & - & - & - & - & - & - & - & - & + & - & - & + & - & + \\ + & - & + & - & - & + & - & - & - & - & - & - & - & + & - & - & + & - & + \\ - & - & - & - & - & + & - & - & - & - & - & - & - & + & - & - & + & - & - \\ + & - & - & - & - & + & + & + & - & - & - & - & - & + & - & - & + & - & - \\ - & - & - & + & - & + & + & + & - & - & - & - & - & + & - & - & + & - & - \\ - & + & + & + & - & + & - & - & - & - & - & - & - & + & - & - & + & - & - \\ + & - & - & - & - & + & - & - & - & - & - & - & - & + & - & - & + & - & - \\ + & - & - & + & + & - & + & + & + & - & - & - & - & + & - & - & + & - & - \\ - & - & + & + & - & - & + & + & - & - & - & - & - & + & - & - & + & - & - \end{bmatrix}$$

The matrix **GS**₂₀ defines a set of 20 spreading sequences of length 20. The correlation parameters for this set of sequences are as follows:

$$R_{CC} = 0.9675$$

$$R_{AC} = 0.6180$$

$$C_{\max} = 0.8500$$

$$A_{\max} = 0.3500$$

The obtained values of correlation measures are very good. For the comparison, the corresponding values for the commonly applied Walsh-Hadamard sequences of length 32 are: 0.7873, 6.5938, 0.9688, and 0.9688, respectively. To illustrate the synchronization amenability of the set of sequences defined by the matrix **GS**₂₀, in Fig.1 we present the plot of $a_{\max}(\tau)$, and in Fig.2, we show the plot of $c_{\max}(\tau)$ to show that the cross-correlation properties of the designed sequence set are good, too.

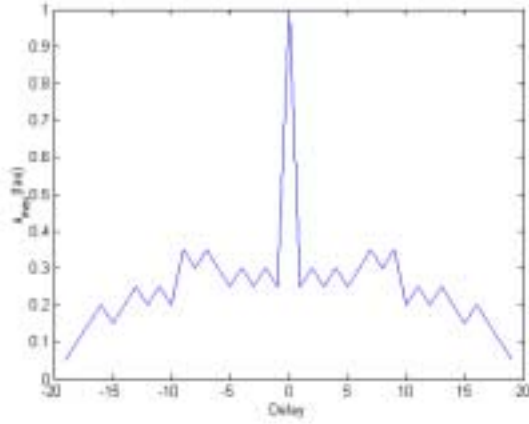


Fig. 1 Plot of the peak values in the aperiodic autocorrelation functions for the sequences given by the matrix \mathbf{GS}_{20} .

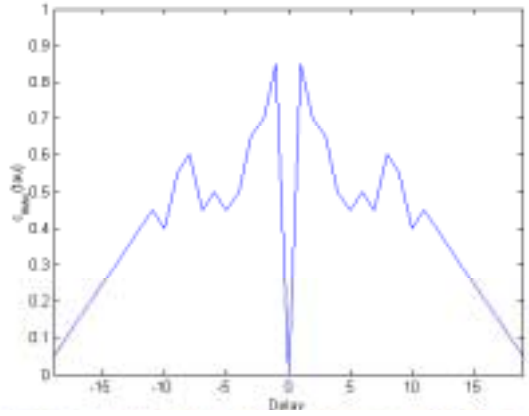


Fig. 2 Plot of peak values in the aperiodic cross-correlation functions for the sequences given by the matrix \mathbf{GS}_{20} .

4. Modification Method

Further improvement to the values of correlation parameters of the sequence sets based on Williamson-Hadamard matrices, can be obtained using the method introduced in [7] for Walsh-Hadamard sequences. That method is based on the fact that for a matrix \mathbf{H} to be orthogonal, it must fulfill the condition $\mathbf{H}\mathbf{H}^T = \mathbf{M}\mathbf{I}$, where \mathbf{H}^T is the transposed Hadamard matrix of order N , and \mathbf{I} is the $N \times N$ unity matrix. In the case of Williamson-Hadamard matrices, we have $N = 4n$. The modification is achieved by taking another orthogonal $N \times N$ matrix \mathbf{D}_N , and the new set of sequences is based on a matrix \mathbf{W}_N , given by:

$$\mathbf{W}_N = \mathbf{H}\mathbf{D}_N \quad (12)$$

Of course, the matrix \mathbf{W}_N is also orthogonal [7].

In [7], it has been shown that the correlation properties of the sequences defined by \mathbf{W}_N can be significantly different to those of the original sequences.

A simple class of orthogonal matrices of any order are diagonal matrices with their elements $d_{i,j}$ fulfilling the condition:

$$|d_{l,m}| = \begin{cases} 0 & \text{for } l \neq m \\ k & \text{for } l = m \end{cases}; \quad l, m = 1, \dots, N \quad (13)$$

To preserve the normalization of the sequences, the elements of \mathbf{D}_N , being in general complex numbers, must be of the form:

$$d_{l,m} = \begin{cases} 0 & \text{for } l \neq m \\ \exp(j\phi_l) & \text{for } l = m \end{cases}; \quad (14) \\ l, m = 1, \dots, N$$

From the implementation point of view, the best class of sequences is the one of binary sequences.

To find the best possible modifying diagonal matrix \mathbf{D}_N we can do an exhaustive search of all possible bipolar sequences of length N , and choose the one, which leads to the best performance of the modified set of sequences. However, this approach is very computationally intensive, and even for a modest values of N , e.g. $N = 20$, it is rather impractical. Hence, other search methods, like a random search, must be considered, like Monte Carlo algorithm.

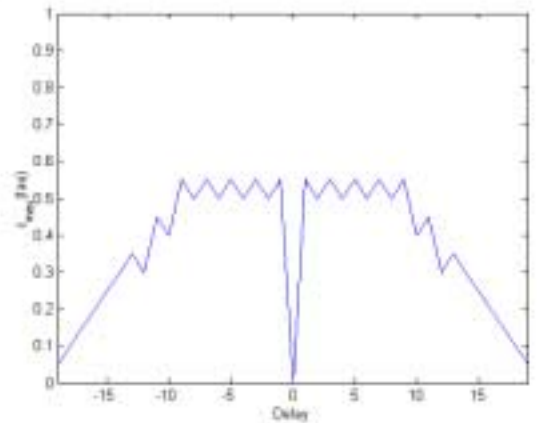


Fig. 3 Plot of peak values in the aperiodic cross-correlation functions for the modified sequences

In Fig. 3, we show the plots of $c_{\max}(\tau)$ for the modified sequence set, when the modifying sequence is:

$$D_{cc} = [+ - - - - + + + + - - - - + +].$$

The parameters of the modified sequence set are:

$$R_{CC} = 0.9548$$

$$R_{AC} = 0.8580$$

$$C_{\max} = 0.5500$$

$$A_{\max} = 0.5000$$

From the presented results it is visible that a significant improvement in the cross-correlation characteristics has been obtained at the expense of worsening the autocorrelation properties. This potential trade-off must always be taken to account when optimizing parameters of spreading sequences

5. Conclusions

In the paper, we presented a method to generate orthogonal spreading sequences of lengths being equal to 4 (mod 8). The technique, based on the newly discovered properties of nega-cyclic matrices combined with Goethals-Seidel construction allows for generation of spreading sequences having good correlation properties. Further improvement to their correlation performance can be obtained by applying the diagonal modification method. Because the technique allows to generate a large multitude of non-equivalent sequences, further research is required to search for the best sequence sets, in particular for larger lengths.

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