

# Some results on self-orthogonal and self-dual codes

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## Abstract

We use generator matrices  $G$  satisfying  $GG^T = aI + bJ$  over  $\mathbb{Z}_k$  to obtain linear self-orthogonal and self-dual codes. We give a new family of linear self-orthogonal codes over  $GF(3)$  and  $\mathbb{Z}_4$  and a new family of linear self-dual codes over  $GF(3)$ .

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## 1 Introduction

A linear code  $C$  of length  $n$  over  $\mathbb{Z}_k$  (or a  $\mathbb{Z}_k$ -code of length  $n$ ) is a  $\mathbb{Z}_k$ -submodule of  $\mathbb{Z}_k^n$ . If  $k = p$  is prime then  $\mathbb{Z}_p = GF(p)$  and a linear code of length  $n$  is a subspace of  $GF(p)$ . An element of  $C$  is called a codeword. We define the inner product on  $\mathbb{Z}_k^n$  by  $x \cdot y = x_1y_1 + \cdots + x_ny_n$ , where  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$ . The dual code  $C^\perp$  of  $C$  is defined as  $C^\perp = \{v \in \mathbb{Z}_k^n \mid v \cdot w = 0 \text{ for all } w \in C\}$ .

$w \in C$ }. A code  $C$  is *self-dual* if  $C = C^\perp$ . The Hamming weight ( $wt(c)$ ) of a codeword  $c$  is the number of non-zero components in the codeword. The *minimum weight* of a code  $C$  is the smallest weight among all codewords of  $C$ . The minimum distance of a linear code  $C$  is its minimum weight. We say that self-dual codes with the largest minimum weight among self-dual codes of that length are *optimal*. A linear code over  $GF(p)$  of length  $n$  with  $k$  independent rows in its generator matrix will be denoted as  $[n, k; p]$ . Furthermore, if its minimum distance is  $d$  it will be denoted as  $[n, k, d; p]$ .

Two codes over  $\mathbb{Z}_k$  are said to be *equivalent* if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates.

There has been a large amount of research recently devoted to self-orthogonal and self-dual codes over the ring  $\mathbb{Z}_4$ , [1, 3, 5, 7]. Patrick Solé's remark that the orthogonality of Hadamard matrices can naturally be interpreted as  $\mathbb{Z}_4$ -orthogonality was investigated in [4]. These self-orthogonal and self-dual codes over  $\mathbb{Z}_4$  were obtained from equivalence classes of Hadamard matrices.

## 2 The constructions

We give a general theorem which will be used later in the paper.

**Theorem 1** *Suppose  $A$  and  $B$  are two matrices of order  $n$  over  $\mathbb{Z}_k$  satisfying*

$$AA^T + BB^T = sI + rJ$$

*where  $s \equiv r \equiv 0 \pmod{k}$ . Then*

$$G = [A \ B]$$

*generates a linear self-orthogonal code over  $\mathbb{Z}_k$ , of length  $2n$  and with  $m$ ,  $m \leq \frac{n}{2}$  independent rows in its generator matrix.  $\square$*

The next corollary is a generalization of a construction given by Georgiou and Koukouvinos [6].

**Corollary 1** *Suppose  $A$  and  $B$  are two matrices of order  $n$  over  $\mathbb{Z}_k$  satisfying*

$$AA^T = a_1I + a_2J \text{ and } BB^T = b_1I + b_2J$$

where  $a_1 + b_1 \equiv a_2 + b_2 \equiv 0 \pmod{k}$ . Then

$$G = [A \ B]$$

generates a linear self-orthogonal code of length  $2n$  and with  $m$  independent rows in its generator matrix, over  $\mathbb{Z}_k$ ,  $m \leq \frac{n}{2}$ .  $\square$

**Theorem 2** Suppose  $A$  and  $B$  are two matrices of order  $n$  over  $\mathbb{Z}_k$  satisfying

$$AA^T = a_1I + a_2J \text{ and } BB^T = b_1I + b_2J$$

where  $a_2 + b_2 \equiv 0 \pmod{k}$  and  $a_1 + b_1 + a \equiv 0 \pmod{k}$  for some  $a \in \mathbb{Z}_k$ . Then

$$G_2 = \begin{bmatrix} & A & B \\ aI_{2n} & & \\ & B^T & -A^T \end{bmatrix}$$

generates a linear self-dual code of length  $4n$  and with  $2n$  independent rows in its generator matrix, over  $\mathbb{Z}_k$ .  $\square$

**Example 1** (i) Set  $A = B = \text{circ}(1, 1, 1, 1, 0)$ . We have that  $AA^T = BB^T = I + 3J$ . Then

$$G_2 = \begin{bmatrix} & A & B \\ I_{2n} & & \\ & B^T & -A^T \end{bmatrix}$$

generates an  $[20, 10, 6; 3]$  extremal self-dual code with weight enumerator

$$W(z) = 1 + 120z^6 + 4260z^9 + 26280z^{12} + 25728z^{15} + 2560z^{18}.$$

(ii) Set  $A = \text{circ}(-2, -2, 0, -1, 0)$  and  $B = \text{circ}(-1, -1, -1, -1, 1)$ . We have that  $AA^T = 5I + 4J$  and  $BB^T = 4I + J$ . Then

$$G_2 = \begin{bmatrix} & A & B \\ I & & \\ & B^T & -A^T \end{bmatrix}$$

generates an  $[20, 10, 8; 5]$  extremal self-dual code with weight enumerator

$$W(z) = 1 + 1280z^8 + 3200z^9 + 24848z^{10} + 58560z^{11} + 248480z^{12} + 464960z^{13} + 1175840z^{14} + 1568000z^{15} + 2267240z^{16} + 1896720z^{17} + 1398960z^{18} + 541760z^{19} + 115776z^{20}.$$

- (ii) Set  $A = circ(-2, -2, 0, -1, 0)$  and  $B = circ(-1, -1, -1, -1, 1)$ . We have that  $AA^T = 5I + 4J$  and  $BB^T = 4I + J$ . Then

$$G = [A \ B]$$

generates an  $[10, 5, 4; 5]$  self-dual code with weight enumerator

$$W(z) = 1 + 40z^4 + 44z^5 + 220z^6 + 760z^7 + 940z^8 + 740z^9 + 380z^{10}.$$

For the SBIBDs we use in the remainder of this paper, we refer the reader to the book of Beth, Jungnickel and Lenz [2]. By  $A = SBIBD(v, k, \lambda)$  we denote the  $v \times v$   $(0, 1)$  incidence matrix of the  $SBIBD(v, k, \lambda)$ .

- Example 2** 1. There exist  $A = SBIBD(31, 10, 3)$  and  $B = SBIBD(31, 15, 7)$ , so  $[A \ B]$  generates a linear self-orthogonal code of length 62 and with  $k_1$  independent rows in its generator matrix, over  $GF(5)$  with minimum distance  $d_1$  as

$$AA^T = 7I + 3J \text{ and } BB^T = 8I + 7J.$$

2. There exist  $A = SBIBD(71, 15, 3)$  and  $B = SBIBD(71, 21, 6)$ , so  $[A \ B]$  generates a linear self-orthogonal code of length 142 and with  $k_2$  independent rows in its generator matrix, over  $GF(3)$  with minimum distance  $d_2$  as

$$AA^T = 12I + 3J \text{ and } BB^T = 15I + 6J.$$

3. There exist  $A = SBIBD(133, 33, 8)$  and  $B = SBIBD(133, 12, 1)$ , so  $[A \ B]$  generates a linear self-orthogonal code of length 266 and with  $k_3$  independent rows in its generator matrix, over  $GF(3)$  with minimum distance  $d_3$  as

$$AA^T = 25I + 8J \text{ and } BB^T = 11I + J.$$

□

In the next theorems we use specific families to find linear self-orthogonal codes. We combine skew-Hadamard matrices or conference matrices with incidence matrices of projective planes to construct some linear self-orthogonal codes over  $\mathbb{Z}_k$ .

Details on skew-Hadamard matrices and conference matrices required for the next theorem can be found in Seberry and Yamada [9]. Appropriate details of the incidence matrices of projective planes can be found in Ryser [8].

**Theorem 3** *Let  $p + 1$  be the order of a skew-Hadamard matrix or a conference matrix. Suppose  $p = q^2 + q + 1$  for some prime power  $q$ . Then there exists a self-orthogonal code over  $\mathbb{Z}_k$  of length  $2p$ , with  $m$  independent rows in its generator matrix and minimum distance  $d$  whenever  $p + q = (q + 1)^2 \equiv 0 \pmod{k}$ .*

**Proof.** Write the skew-Hadamard matrix  $S + I$ , minus its diagonal entries, or conference matrix as

$$\begin{bmatrix} 0 & e \\ \pm e^T & P \end{bmatrix}$$

where  $e$  is the  $1 \times p$  matrix of ones. Then  $P$  is a  $p \times p$  matrix satisfying

$$PP^T = pI - J.$$

Write  $Q$  for an incidence matrix of the projective plane over  $GF(q)$ . Then  $Q$ , of order  $p = q^2 + q + 1$ , is circulant and satisfies

$$QQ^T = qI + J.$$

Now  $G_1 = [P \ Q]$  generates the required self-orthogonal code over  $\mathbb{Z}_k$  of length  $2p$  and with  $m$ ,  $m \leq p$  independent rows in its generator matrix as  $G_1 G_1^T = (p + q)I = (q + 1)^2 I \equiv 0$ .  $\square$

**Corollary 2** *Let  $p + 1$  be the order of a skew-Hadamard matrix or a conference matrix. Suppose  $p = q^2 + q + 1$  for some prime power  $q$ , and  $q \equiv 2 \pmod{3}$ . Then there exists a self-orthogonal  $[2p, m, d]$  ternary code with  $m \leq p - 1$ . Note that  $m = p$  iff  $q \equiv 1 \pmod{3}$  and thus  $G_1 = [P \ Q]$  is the generator matrix of a self-dual code.*

**Proof.** Use theorem 3. □

**Example 3** Let  $q = 2$ ,  $p = 7$ ,  $P = \text{circ}(0, 1, 1, -1, 1, -1, -1)$  and  $Q = \text{circ}(1, 1, 0, 1, 0, 0, 0)$ . We consider the matrix  $[P \ Q]$  and we remove its first row. Then the derived matrix is the generator matrix of a  $[14, 6, 6; 3]$  code with weight enumerator

$$W(z) = 1 + 84z^6 + 476z^9 + 168z^{12}.$$

**Theorem 4** The codes over  $GF(3)$  and  $\mathbb{Z}_4$  we obtain using  $G_1$  are

(i)  $[2p, p, d]$  for  $q \equiv 1(\text{mod } 3)$

(ii)  $[2p, p - 1, d]$  for  $q \equiv 0, 2(\text{mod } 3)$  and  $q \equiv 0, 1, 2, 3(\text{mod } 4)$ .

**Proof.** Consider the matrix  $P$  of order  $p = q^2 + q + 1$ . Now  $PP^T = (q^2 + q + 1)I - J$  and  $\det PP^T \equiv 0(\text{mod } 3)$  and  $0(\text{mod } 4)$ . Now consider  $P'$  with one row of  $P$  removed. Then the matrix  $P'$  has size  $(q^2 + q) \times (q^2 + q + 1)$  and so  $P'P'^T$  is of order  $q^2 + q$  and has the following form:

$$P'P'^T = \begin{bmatrix} q^2 + q & -1 & -1 & \cdots & -1 \\ -1 & q^2 + q & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & q^2 + q \end{bmatrix}$$

and  $\det P'P'^T = (1)(q^2 + q + 1)^{q^2+q-1} \not\equiv 0$  for  $q \equiv 0, 2(\text{mod } 3)$  and  $q \equiv 0, 1, 2, 3(\text{mod } 4)$ . Hence the rank of the matrix  $P'$  is  $p - 1$  for these cases.

Now the matrix  $Q$  satisfies  $QQ^T = qI + J$  and  $\det QQ^T = (q + 1)^2(q)^{q^2+q} \not\equiv 0(\text{mod } 3)$  for  $q \equiv 1(\text{mod } 3)$ . Hence the rank of the matrix  $Q$  is  $p$  for this case. □

**Remark 1** We recall that a self-orthogonal code,  $C$ , of length  $2p$ , with  $p$  independent rows in its generator matrix and distance  $d_1$  with  $C^\perp$  a self-orthogonal code of length  $2p$  and  $p$  independent rows in its generator matrix with distance  $d_2$  we have that  $C = C^T$  and so  $C$  is in fact self-dual.

**Theorem 5** *Let  $p + 1$  be the order of a skew-Hadamard matrix or a conference matrix. Suppose  $p = q^2 + q + 1$  for some prime power  $q$ . Then there exists a self-orthogonal  $\mathbb{Z}_k$ -code of length  $2p$ , with  $m$  independent rows in its generator matrix and minimum distance  $d$ , whenever  $p + q \equiv 0 \pmod{k}$ .*

**Proof.** Construct the matrices  $P$  and  $Q$  as in the proof of theorem 3. Set

$$G_3 = \begin{bmatrix} P & Q \\ Q^T & -P^T \end{bmatrix}.$$

We have that

$$G_3 G_3^T = \begin{bmatrix} P & Q \\ Q^T & -P^T \end{bmatrix} \begin{bmatrix} P^T & Q \\ Q^T & -P \end{bmatrix} = \begin{bmatrix} PP^T + QQ^T & PQ - QP \\ Q^T P^T - P^T Q^T & Q^T Q + P^T P \end{bmatrix}$$

If  $PQ = QP$  (for example, this is true if  $P$  is circulant, in which case  $p$  is prime) then this matrix generates the required self-orthogonal code of length  $2p$  with  $m$  independent rows in its generator matrix, as  $G_3 G_3^T = (q + 1)^2 I_m \equiv 0 \pmod{k}$ . □

**Theorem 6** *Let  $p + 1$  be the order of a skew-Hadamard matrix or a conference matrix. Suppose  $p = q^2 + q + 1$  for some prime power  $q$ . Then there exists a self-dual  $\mathbb{Z}_k$ -code of length  $4p$ , with  $2p$  independent rows in its generator matrix and minimum distance  $d$ , whenever  $p + q + a \equiv 0 \pmod{k}$  for some  $a \in \mathbb{Z}_k$ .*

**Proof.** Construct the matrices  $P, Q$  and  $G_3$  as in the proof of theorem 5. Set  $G_4 = [I_{2p} \ G_3]$ . If  $PQ = QP$  (for example, this is true if  $P$  is circulant, in which case  $p$  is prime) then the matrix  $G_4$  generates the required self-dual code of length  $4p$  with  $2p$  independent rows in its generator matrix, as  $G_4 G_4^T = (q + p + a) I_{2p}$ . □

We are able to use the considerable literature on the minimum distance of codes generated by skew-Hadamard matrices,  $I + S$ , minus its diagonal entries, to obtain lower bounds for the minimum distance of codes with generator matrix  $[P \ Q]$ , where  $P$  and  $Q$  are given in the proof of Theorem 3 via the following lemma:

**Lemma 1** Suppose  $A$  and  $B$  are two matrices of order  $n$  with elements from  $\mathbb{Z}_k$  and  $\det(A) \neq 0$ . We denote the minimum weights among all linear combinations of their rows (over  $\mathbb{Z}_k$ ) by  $d_A$  and  $d_B$  respectively. Then the code,  $C$ , with generator matrix  $[A \ B]$  has minimum Hamming distance  $d_C \geq d_A + d_B$ .

**Remark 2** There are many pairs  $(p, q)$  which satisfy the conditions of Theorem 3. The first few pairs are  $(7, 2)$ ,  $(13, 3)$ ,  $(31, 5)$ ,  $(73, 8)$ ,  $(91, 9)$ ,  $(183, 13)$ ,  $(307, 17)$ ,  $(757, 27)$ ,  $(1723, 41)$ .

**Example 4** 1. Let  $q = 3$ ,  $p = 13$ ,  $P = \text{circ}(0, 1, -1, 1, 1, -1, -1, -1, -1, 1, 1, -1, 1)$  and  $Q = \text{circ}(1, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0)$ . We consider the matrix  $[P \ Q]$  and we remove its first row. Then the derived matrix is the generator matrix of a self-orthogonal  $\mathbb{Z}_4$ -code of length 26, with 12 independent rows in its generator matrix and minimum distance 8 with weight enumerator

$$\begin{aligned} W(z) = & 1 + 390z^8 + 1716z^{10} + 40092z^{12} + 17056z^{13} + 226720z^{14} + \\ & + 422656z^{15} + 541593z^{16} + 2348320z^{17} + 1012440z^{18} + \\ & + 4010240z^{19} + 2425436z^{20} + 2384096z^{21} + 2247648z^{22} + \\ & + 559104z^{23} + 472680z^{24} + 56160z^{25} + 10868z^{26}. \end{aligned}$$

2. Let  $q = 5$ ,  $p = 31$ ,  $P = \text{circ}(0, -1, -1, 1, -1, -1, 1, -1, -1, -1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1)$  and  $Q = \text{circ}(1, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ . We consider the matrix  $[P \ Q]$  and we remove its first row. Then the derived matrix is the generator matrix of a self-orthogonal code over  $GF(3)$  of length 62, with 30 independent rows in its generator matrix and minimum distance 12. Thus we can obtain a  $[62, m, d; 3]$  code for all  $m \leq 30$  and with  $d(m) \geq 12$  by removing rows.  $\square$

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