The Maximal Determinant and Subdeterminants of ±1 Matrices

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Abstract

In this paper we study the maximal absolute values of determinants and subdeterminants of ±1 matrices, especially Hadamard matrices. It is conjectured that the determinants of ±1 matrices of order n can have only the values $k \cdot p$, where p is specified from an appropriate procedure. This conjecture is verified for small values of n. The question of what principal minors can occur in a completely pivoted ±1 matrix is also studied. An algorithm to compute the $(n-j) \times (n-j)$, $j = 1, 2, \ldots$ minors of Hadamard matrices of order n is presented, and these minors are determined for j =1,. . . ,4.

Key words and phrases: Minors, Hadamard matrices, subdeterminants, completely pivoted.

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1 Minors of ±1 matrices

An *Hadamard matrix* of order n is a square $n \times n$ matrix, has entries ±1 and its distinct row and column vectors are orthogonal, and it said to be *normalized* if it has its first row and column all 1's. The following famous conjecture specifies the existence of Hadamard matrices.

Hadamard Conjecture There exists an Hadamard matrix of order $4t$ for every positive integer t.

Hadamard matrices satisfy the Hadamard's famous inequality, that if a matrix $X = (x_{ij})$ has entries on the unit disk then

$$|\det(X)| \leq \left(\prod_{j=1}^{n} \sum_{i=1}^{n} (x_{ij})^2 \right)^{\frac{1}{2}} \leq n^{\frac{n}{2}}. \tag{1}$$

Hadamard matrices of order n have absolute value of determinant $n^{\frac{n}{2}}$, and the inequalities in (1) are sharp if and only if X is a Hadamard matrix.

Throughout this paper we use $-$ for -1, and when we are saying the determinant of a matrix we mean the absolute value of the determinant.

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It is a famous unsolved problem to determine the maximum determinant for all matrices of order n with entries ± 1. Koukouvinos, Mitrouli and Seberry [13] give a lower bound for the upper bound of every ± 1 matrix provided the Hadamard conjecture is true.

We note that the $(n-j) \times (n-j)$ minors of an Hadamard matrix of order n are:

1. for $j = 1$, zero or $n^{\frac{n}{2}-1}$, Sharpe [19];

2. for $j = 2$ zero or $2n^{\frac{n}{2}-2}$, Sharpe [19];

3. for $j = 3$ zero or $4n^{\frac{n}{2}-3}$, Sharpe [19];

4. for $j = 4$ zero or $8n^{\frac{n}{2}-4}$ or $16n^{\frac{n}{2}-4}$, Koukouvinos, Mitrouli and Seberry [14].

A restricted list of possible values of $(n-j) \times (n-j)$ minors, j $= 1,\ldots,6$, for Hadamard matrices of order n, was given by Day and Peterson in [5].

Definition 1 *A D-optimal design of order n is an $n \times n$ matrix with entries ± 1 having maximum determinant.*

Let $\mathcal{X}_n$ be the set of all ± 1 matrices of order n. For $n \equiv 1 (mod\ 4)$ it was proved by Ehlich [9] that for all $X \in \mathcal{X}_n$,

$$\det(X) \leq (2n-1)^{\frac{1}{2}} (n-1)^{\frac{(n-1)}{2}} \tag{2}$$

and in order for maximum equality it is necessary that $2n - 1$ be a square and that there exists an $X \in \mathcal{X}_n$ with $XX^T = (n-1)I_n + J_n$, where J_n is the $n \times n$ matrix all of whose entries are equal to one, and I_n is the $n \times n$ identity matrix.

For $n \equiv 2 (mod\ 4)$ Ehlich [9], and independently Wojtas [20], proved that for all $X \in \mathcal{X}_n$,

$$\det(X) \leq (2n-2)(n-2)^{\frac{n}{2}-1}. \tag{3}$$

Moreover, the equality in (3) holds if and only if there exists $X \in \mathcal{X}_n$ such that

$$XX^T = X^T X = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix},$$

where $L = (n-2)I + 2J$ is an $\frac{n}{2} \times \frac{n}{2}$ matrix. A further necessary condition for equality to hold is that $n - 1$ is the sum of two squares. For $n = 6$ a matrix such as

$$\begin{bmatrix} 1 & 1 & 1 & - & 1 & 1 \\ 1 & 1 & 1 & 1 & - & 1 \\ 1 & 1 & 1 & 1 & 1 & - \\ 1 & - & - & 1 & 1 & 1 \\ - & 1 & - & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 & 1 \end{bmatrix}$$

which is called a D-optimal design matrix, has maximum determinant.

Ehlich [10] investigated the case $n \equiv 3 (mod\ 4)$ which appears the most difficult case. Assume $n \equiv 3 (mod\ 4)$ and $n \geq 63$. Ehlich proved that for all $X \in \mathcal{X}_n$,

$$\det(X) \le (\frac{4 \cdot 11^6}{7^7}(n-3)^{n-7}n^7)^{\frac{1}{2}}.$$
(4)

Moreover, for the equality to hold it is necessary that $n = 7m$ and that there exists $X \in \mathcal{X}_n$ with

$$XX^T = I_7 \otimes [(n-3)I_m + 4J_m] - J_n.$$
(5)

The corresponding bounds for $det(X)$ when $n \equiv 3 (mod\ 4)$, $n < 63$, are also given by Ehlich, as are structures of XX^T for normalized maximal examples X. The formula for values $n < 63$ is the same as in (5). A ± 1 matrix X has maximal determinant if XX^T has block structure with the blocks along the diagonal of the form $(n-3)I + 3J$ and the off-diagonal blocks equal to $-J$.

It is obvious that for $n = 22,\ 34,\ 58,\ 70,\ 78,\ 94\ (n \le 100)$ the upper bound given in (3) cannot be attained as $n - 1$ is not the sum of two squares.

Koukouvinos, Mitrouli and Seberry [13] showed.

Theorem 1 *[13] Suppose 4t is the order of an Hadamard matrix. Write $v = 4t - 1$. Then there are ± 1 matrices whose*

- *$v \times v$ determinants have magnitude $(4t)^{2t-1}$;*

- *$(v-1) \times (v-1)$ determinants have magnitude $2(4t)^{2t-2}$;*

- *$(v-2) \times (v-2)$ determinants have magnitude $4(4t)^{2t-3}$.*

We now give a brief explanation of the usefulness of ± 1 matrices in some statistical applications. Consider an experimental situation in which a response y depends on k factors $x_1, \ldots, x_k$ with the first order relationship of the form $E(\underline{y}) = X\underline{\beta}$, where $\underline{y}$ is an $n \times 1$ vector of observations, the design matrix X is $n \times (k+1)$ whose jth row is of the form $(1, x_{j1}, x_{j2}, \ldots, x_{jk})$, $j = 1, 2, \ldots, n$ and $\underline{\beta}$ is the (k+1)$\times$1 vector of coefficients to be estimated. In a two-level factorial design, each x_i can be coded as ± 1. The design is then determined by the $n \times (k+1)$ matrix of elements ± 1. The ith column gives the sequence of factor levels for factor x_i, each row constitutes a run. When $k = n - 1$, the design is called a saturated design and the design matrix is an $n \times n$ square matrix. Note that $n = k + 1$ is the minimal number of points (rows) required to estimate all coefficients of interest (the β_i's).

Several criteria have been advanced for the purpose of comparing designs and for constructing optimal designs. One of the most popular is the D-optimality criterion, which seeks to maximize $\det(X^T X)$.

Definition 2 *Let X be a ± 1 design of order n and X^* be the D-optimal design of the same order. The ratio*
$$d = detX/detX^*$$
is called efficiency of the design and forms a measure of comparing these designs.

For more details see [12] and the references therein.
We compare these results to evaluate the efficiency of some of these designs for comparison.

1. For $n = 15$ the upper bound given by Ehlich [10] is $2^{14} \cdot 26284$, but this bound cannot be attained. Cohn [2] found an almost D-optimal design with determinant $2^{14} \cdot 25515 = 2^{14} \cdot 3^6 \cdot 5 \cdot 7$, and efficiency > 0.97.

2. For $n = 17$ the upper bound given by (2) cannot be attained. Moyssiadis and Kounias [16] obtain a matrix with maximum determinant $16^7 \cdot 80$.

3. For $n = 19$ the upper bound given by Ehlich [10] is $2^{18} \cdot 3499393$, but this bound cannot be attained. Cohn [2] found an almost D-optimal design with determinant $2^{18} \cdot 3411968 = 2^{30} \cdot 7^2 \cdot 17$, and efficiency > 0.975.

4. For $n = 21$ the upper bound given by (2) cannot be attained. Chadjipantelis, Kounias, and Moyssiadis [1] obtain a matrix with maximum determinant $20^9 \cdot 116$.

5. For $n = 22$ the upper bound given by (3) is is $2^{21} \cdot 205078125$, but this bound cannot be attained. Cohn [2] found an almost D-optimal design with determinant $2^{21} \cdot 184769649 = 2^{21} \cdot 3^2 \cdot 23^2 \cdot 197^2$, and efficiency > 0.90.

6. For $n = 29$ the upper bound given by (2) is $\sqrt{57} \cdot 2^{28} 7^{14}$. Koukouvinos et al [13] obtained a matrix with determinant 2^{67}. However, Koukouvinos [12] obtained a matrix with determinant $2^{28} 7^{13} 43$, and efficiency > 0.7947.

7. For $n = 34$ the upper bound given by (3) is $32^{16} \cdot 66$. By theory this cannot be obtained. Koukouvinos et al [13] obtain a matrix with determinant $36^{16} \cdot 2$. The efficiency of this matrix is > 0.20 (very small).

8. For $n = 58$ the upper bound given by (3) is $56^{28} \cdot 114$. By theory this cannot be obtained. Koukouvinos et al [13] obtain a matrix with determinant $60^{28} \cdot 2$. The efficiency of this design is > 0.12 (very small).

9. For $n = 70$ the upper bound given by (3) is $68^{34} \cdot 138$. By theory this cannot be obtained. Koukouvinos et al [13] obtain a matrix with determinant $72^{34} \cdot 2$. The efficiency of this design is > 0.10 (very small).

In this paper we study the existence of ± 1 matrices with maximum absolute value determinant. The following facts are known. See Day and Peterson [5] for references.

Proposition 1 *Let B be an $n \times n$ matrix with elements ± 1. Then*

1. det B is an integer and 2^{n-1} divides det B;

2. when $n \leq 6$ the only possible values for det B are the following, and they all do occur,

n	1	2	3	4	5	6
$\det B$	1	0, 2	0, 4	0, 8, 16	0, 16, 32, 48	0, 32, 64, 96, 128, 160

Following Day and Peterson [5], we study all ± 1 matrices of order $n = 2, 3, 4, 5, 6$, and the distribution of their determinant's absolute values, in order to refine our algorithm (given later).

Conjecture 1 *Let A be an $n \times n$ matrix with entries ± 1. Then the absolute value determinant of A is 0, or p where for the evaluation of the coefficient p the following procedure is adopted:*

Set $p = 2^{n-1}$
Set $s = max|det(A)|$
    where A an $n \times n$ matrix with all elements ± 1's.
Set $k = 1$
repeat
    $p = k \cdot p$
    $k = k + 1$
until
    $p = s$.

We first list $n \times n$ matrices with elements ± 1 of order $n = 2, 3, 4, 5$ which have maximal determinant.

$$
\begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix},
\begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix},
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix},
\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 \\ 1 & 1 & - & - & - \\ 1 & - & - & 1 & 1 \\ 1 & - & 1 & 1 & - \end{bmatrix}
$$

Definition 3 *Two matrices of order n with entries ± 1 are called equivalent if one can be obtained from the other by permutation of rows and/or columns and multiplication of rows and/or columns by -1.*

It is true that the maximum determinant for $n = 2$, 3, 4, and 5 occurs only for matrices equivalent to the above matrices.

In order to study the existence of maximum determinants, we can search to see if the four matrices above exist as submatrices of a matrix with entries ± 1. In this paper we study if the above matrices exist as submatrices of Hadamard matrices.

What is a CP matrix and how does it differ?
Let $A = [a_{ij}] \in \mathcal{R}^{n \times n}$. We reduce A to upper triangular form by using Gaussian Elimination (GE) operations. Let $A^{(k)} = [a_{ij}^{(k)}]$ denote the matrix obtained after the first k pivoting steps, so $A^{(n-1)}$ is the final upper triangular matrix. A diagonal entry of that final matrix will be called a pivot. Matrices with the property that no exchanges are actually needed during GE with complete pivoting are called completely pivoted (CP) or feasible. Thus when Gaussian elimination is performed on a CP matrix, after pivoting on the (i, i) entry, no entry in the remaining submatrix has absolute value greater than $|a_{ii}^{(i-1)}|$.

We now give a new proof of a result of Day and Peterson [5]. This proof can be extended to say more about CP Hadamard matrices.

Theorem 2 *Every Hadamard matrix of order ≥ 4 contains a submatrix equivalent to*

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix} \qquad (6)$$

Every CP Hadamard matrix has this as its upper left corner 4×4 submatrix.

Proof. The columns and rows of a Hadamard matrix of order $4t$ can be scaled by ± 1, and the columns rearranged, so that the first three rows are equivalent to

$$\begin{array}{cccc} 1...1 & 1...1 & 1...1 & 1...1 \\ 1...1 & 1...1 & -...- & -...- \\ 1...1 & -...- & 1...1 & -...- \end{array} .$$

With the same type of operations, the first three columns can be changed to be the transpose of these three rows.

Now move the first, $2t + 1$st, $t + 1$st and $3t + 1$st rows to the top and those same columns to the left. Arrange the columns as shown below and write x, y, z and w for the number of columns of each type as determined by row four. Thus we have

$$\begin{array}{ccccccccc} \overbrace{}^{4} & \overbrace{}^{x} & \overbrace{}^{t-1-x} & \overbrace{}^{y} & \overbrace{}^{t-1-y} & \overbrace{}^{z} & \overbrace{}^{t-1-z} & \overbrace{}^{w} & \overbrace{}^{t-1-w} \\ 1\ 1\ 1\ 1 & 1...1 & 1...1 & 1...1 & 1...1 & 1...1 & 1...1 & 1...1 & 1...1 \\ 1\ -\ 1\ - & 1...1 & 1...1 & 1...1 & 1...1 & -...- & -...- & -...- & -...- \\ 1\ 1\ -\ - & 1...1 & 1...1 & -...- & -...- & 1...1 & 1...1 & -...- & -...- \\ 1\ -\ -\ a & 1...1 & -...- & 1...1 & -...- & 1...1 & -...- & 1...1 & -...- \end{array}$$

By considering the inner product of row four with the first three rows we obtain

$$2x + 2y + 2z + 2w = 4t - 3 - a$$
$$2x + 2y - 2z - 2w = a - 1$$
$$2x - 2y + 2z - 2w = a - 1.$$

So

$$x = t - 1 - z, \quad y = z \ \text{ and } \ w = t - z - \frac{1}{2}(a + 1).$$

We wish to prove that the required submatrix exists in the Hadamard matrix. So we assume the contrary that $a = -1$. This means that $w = 0$ as otherwise we would have chosen $a = 1$ from the w columns that would give the submatrix. This means $z = y = t - \frac{1}{2}(a + 1)$ and $x = -1 + \frac{1}{2}(a + 1)$. Since, by assumption, $a = -1$, this means $x = -1$ which is a contradiction.

Hence the required submatrix (or its Hadamard equivalent) always exists in any Hadamard matrix of order ≥ 4.

Since the CP property is not destroyed if we change signs of rows and columns to make the first row and column positive, the same proof can be used to show that a CP Hadamard matrix contains precisely this submatrix as its upper left 4×4 submatrix. $\qquad \square$

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In order to further study embedded submatrices we specify all possible 5×5 matrices with elements ± 1 that contain this 4×4 part and also have the maximum possible value of the determinant. Thus we extend this matrix to all the possible 5×5 matrices M with elements ± 1 i.e.

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & * \\ 1 & 1 & - & - & * \\ 1 & - & - & 1 & * \\ 1 & * & * & * & * \end{bmatrix}$$

where * can take the values 1 or -1.

In the remainder of this section we classify all these 2^7 matrices according to their determinant and the CP property. Actually, we are interested only in the matrices with determinants 48 and 32 as, in a CP Hadamard matrix, the 5×5 left upper part matrix with determinant less than 32 cannot appear.

We observe that

1. of the 2^7 matrices only 4 had determinant 48. All of them were CP and all were equivalent to the following D-optimal design:

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & - \\ 1 & 1 & - & - & - \\ 1 & - & - & 1 & - \\ 1 & - & - & - & 1 \end{bmatrix}. \tag{7}$$

2. of the 2^7 matrices, 48 matrices had determinant 32. All of them were CP and all were equivalent to the following matrix:

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & - \\ 1 & 1 & - & - & - \\ 1 & - & - & 1 & - \\ 1 & 1 & 1 & 1 & - \end{bmatrix}. \tag{8}$$

3. 28 of the remaining matrices had determinant 16; and 48 had determinant 0.

Embedding 5×5 $(1, -1)$ matrices of determinant 48 (D-optimal designs)
Since we know from Day and Peterson [5] (our theorem 2) that the 4×4 submatrix given in (6) always exists in an Hadamard matrix, we wish to specify which Hadamard matrices have the D-optimal design of order 5 embedded. Edelman and Mascarenhas [6] proved the D-optimal design of order 5 exists embedded in the 12×12 Hadamard matrix. We give here another proof of this, using a method that can be adapted to the search for embedded matrices in Hadamard matrices of any order. The following lemma appears in the proof of Proposition 5.8 in [5], and was rediscovered in [14], and it will be used in the rest of the paper.

Lemma 1 (The distribution lemma) *Let H be any Hadamard matrix, of order $n > 2$. Then for every triple of rows of H there are precisely $\frac{n}{4}$ columns which are*

(a) $(1, 1, 1)^T$ *or* $(-, -, -)^T$

(b) $(1, 1, -)^T$ *or* $(-, -, 1)^T$

(c) $(1, -, 1)^T$ *or* $(-, 1, -)^T$

(d) $(1, -, -)^T$ *or* $(-, 1, 1)^T$

$\square$

Proposition 2 *If H is a 12×12 Hadamard matrix then the D-optimal design of order 5 is embedded in it.*

Proof. We note that, up to equivalence, in any five rows of a Hadamard matrix only the following columns can appear. Here u_i denotes the number of each type:

$$
\begin{array}{cccccccccccccccc}
u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 & u_9 & u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - \\
1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - \\
1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - \\
\end{array}
\tag{9}
$$

Next we try to specify the 12 columns that can appear. The columns will be specified according to the following restrictions. The order of the matrix gives

$$
\sum_{i=1}^{16} u_i = 12.
\tag{10}
$$

From the distribution lemma we have that

$$
0 \le u_i \le \frac{12}{4}.
\tag{11}
$$

We also use the restriction that the matrix given in (6) will exist among the selected columns. On the other hand, from the orthogonality of the rows we obtain 10 equations. Thus we have in total 11 equations in 16 unknowns. By solving the above system, we see that only the following sets of 12 columns can appear in any five rows of the matrix H:

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Set

1	u_1	u_1	u_4	u_6	u_7	u_8	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}
2	u_1	u_2	u_3	u_5	u_8	u_8	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}
3	u_1	u_2	u_3	u_6	u_7	u_8	u_9	u_{12}	u_{12}	u_{13}	u_{14}	u_{15}
4	u_1	u_2	u_3	u_6	u_7	u_8	u_{10}	u_{11}	u_{12}	u_{13}	u_{13}	u_{16}
5	u_1	u_2	u_4	u_5	u_7	u_8	u_9	u_{11}	u_{12}	u_{14}	u_{14}	u_{15}
6	u_1	u_2	u_4	u_5	u_7	u_8	u_{10}	u_{11}	u_{11}	u_{13}	u_{14}	u_{16}
7	u_1	u_2	u_4	u_6	u_7	u_7	u_9	u_{11}	u_{12}	u_{13}	u_{14}	u_{16}
8	u_1	u_3	u_4	u_5	u_6	u_8	u_9	u_{10}	u_{12}	u_{14}	u_{15}	u_{15}
9	u_1	u_3	u_4	u_5	u_6	u_8	u_{10}	u_{10}	u_{11}	u_{13}	u_{15}	u_{16}
10	u_1	u_3	u_4	u_6	u_6	u_7	u_9	u_{10}	u_{12}	u_{13}	u_{15}	u_{16}
11	u_1	u_4	u_4	u_5	u_6	u_7	u_9	u_{10}	u_{11}	u_{14}	u_{15}	u_{16}

We see that the sets numbered $1, 2, 3, 5$ and 8 of the above sets of columns directly contain the columns of the D-optimal design of order 5 given in (7), i.e. u_1, u_8, u_{12}, u_{14} and u_{16}. The remaining sets of columns numbered $4, 6, 7, 9, 10$ and 11 contain at least one 5×5 matrix with determinant 48 which is equivalent to the D-optimal design of order 5 given in (7). $\qquad\square$

Lemma 2 *If H is a 12×12 CP Hadamard matrix, then its leading principal minor of order five takes the maximum value 48.*

Proof. Since the matrix is CP, the D-optimal design of order 5 will appear as its leading principal minor of order 5 and thus its value will be equal to 48. $\qquad\square$

Remark 1 The above result helped in resolving the unique pivot structure of the 12×12 Hadamard matrix [6]. Our method can be used to locate embedded matrices with specific determinants in other Hadamard matrices. A direct result of this, combined with the algorithm of the next section, will be the specification of the pivot structure of Hadamard matrices, which still remains an open problem in Numerical Analysis [3], [5], [7].

$\qquad\square$

2 Algorithm for $(n - j) \times (n - j)$ minors of an $n \times n$ Hadamard matrix.

Any $(n - j) \times (n - j)$ minor will be denoted by M_{n-j}.

If we are considering the $(n - j) \times (n - j)$ minors, then the first j rows, ignoring the upper lefthand $j \times j$ matrix, have 2^{j-1} potentially different first j elements in each column. Let $\underline{x}_{\beta+1}^T$ be the vectors containing the binary representation of each integer $\beta + 2^{j-1}$ for $\beta = 0, \ldots, 2^{j-1} - 1$. Replace all the zero entries of $\underline{x}_{\beta+1}^T$ by -1 and define the $j \times 1$ vectors

$$\underline{u}_k = \underline{x}_{2^{j-1}-k+1}, \quad k = 1, \ldots, 2^{j-1} \tag{12}$$

Let u_k indicate the number of columns beginning with the vectors $\underline{u}_k$, $k = 1, \ldots, 2^{j-1}$.

We note

$$\sum_{i=1}^{2^{j-1}} u_i = n - j. \tag{13}$$

Then it holds that [14]

$$M_{n-j} = n^{n-2^{j-1}-j} \det D \tag{14}$$

where D is the following $2^{j-1} \times 2^{j-1}$ matrix.

$$D = \begin{bmatrix} n - ju_1 & u_2 m_{12} & u_3 m_{13} & \cdots & u_z m_{1z} \\ u_1 m_{21} & n - ju_2 & u_3 m_{23} & \cdots & u_z m_{2z} \\ \vdots & \vdots & \vdots & & \vdots \\ u_1 m_{z1} & u_2 m_{z2} & u_3 m_{z2} & \cdots & n - ju_z \end{bmatrix}$$

where $(m_{ik}) = (-\underline{u}_i \cdot \underline{u}_k)$, with $\cdot$ the inner product.

Based on formula (14), the following algorithm computes the $(n - j) \times (n - j)$ minors of an $n \times n$ Hadamard matrix H.

Step1: Generate all ± 1 matrices M, of order j with first row and column all $+1$.
Step2: Form the general matrix, $N = [M \ \ U_j]$, of size $j \times n$ for the first j rows of an
$n \times n$ Hadamard matrix H, where

$$U_j = \begin{array}{cccccc} \overbrace{1...1}^{u_1} & \overbrace{1...1}^{u_2} & \ldots & \overbrace{1...1}^{u_{2^{j-1}-1}} & \overbrace{1...1}^{u_{2^{j-1}}} \\ 1...1 & 1...1 & \ldots & 1...1 & 1...1 \\ 1...1 & 1...1 & \ldots & -...- & -...- \\ . & . & \ldots & . & . \\ . & . & \ldots & . & . \\ 1...1 & 1...1 & \ldots & 1...1 & -...- \\ 1...1 & -...- & \ldots & -...- & -...- \end{array}$$

Step3: For each M

 Consider all $\binom{j}{3}$ subsets of three rows of N and use the distribution
 lemma with $\sum_{i=1}^{2^{j-1}} u_i = n - j$ to form 4 equations in the variables
 $u_1, \ldots, u_{2^{j-1}}$ for each subset. If a feasible solution is found
 keep this matrix M.
Step 4: For each M found in Step 3 keep only the matrices having different inner products of rows.
Step 5: For each M specified in Step 4 do the following:
 For $k = 3, 4, \ldots, j$
  Step 6 Take the first k lines of N and using
   the orthogonality and the order form
   $\binom{k}{2} + 1$ equations which in practice have 2^{k-1}
   variables with values $\leq \frac{n}{4}$ satisfying the Distribution
   Lemma (proved in [14]).
   Search for all the feasible solution to the produced system
   of different equations.

Step 7 For each feasible solution found in Step 6
use the matrix D to find all possible
values of the $(n - j) \times (n - j)$ minors.

3 Algorithm to Find the Minors of Hadamard Matrices: More Depth

In Koukouvinos et al [15] methods are outlined to evaluate the $(n-5), (n-6), \ldots, (n-j)$ minors of Hadamard matrices of order n in the most efficient way.

3.1 Finding the Upper Left Hand Corner Matrices M

Here we consider the first step of the algorithm, that is:

Generate all ± 1 matrices M, of order j with first row and column all $+1$.
As the algorithm of Koukouvinos, Mitrouli and Seberry [14] requires the matrices M to have first row and column all $+1$ we considered the $2^{(n-1)^2}$ such matrices for various n. These gave exactly the same values of the determinants.

So we then shifted to only considering the determinants of submatrices which could lead to CP matrices. Explicitly then we conjecture

Conjecture 2 *Consider all ± 1 matrices of order n where*

1. *permutation of the rows and/or columns is not permitted;*

2. *multiplication of the rows and/or columns by -1 is not permitted;*

3. *when Gaussian Elimination is applied, the diagonal element is never zero and the maximal value appears always in the diagonal i.e. the matrix is CP.*

Then the determinant of such matrices only assumes a small number of values compared with n.

This is also motivated by the following conjecture told to one of us (Seberry) by D. H. Lehmer and Emma Lehmer in 1975. A few papers written on this topic are [4, 8, 17].

Conjecture 3 *Consider all $(0, 1)$ matrices of order n. Then the permanent of such matrices only assumes a small number of values compared with n.*

3.2 The Upper Left Hand Corner of CP Matrices

While we noted in section 1 the upper left hand corner of CP Hadamard matrices for $n - 2$, $n - 3$ and $n - 4$ all contain one another, but this differs for $n - 5$.

Here we have two possible upper left hand corners for CP Hadamard matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 \\ 1 & 1 & - & - & 1 \\ 1 & - & - & 1 & 1 \\ 1 & 1 & 1 & - & - \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 \\ 1 & 1 & - & - & 1 \\ 1 & - & - & 1 & - \\ 1 & 1 & 1 & - & - \end{bmatrix}.$$

Both contain the 4×4 subdeterminant with maximum determinant in their upper left hand corner. However their determinants are 32 and 48 respectively. An open problem concerns determining if these matrices exist embedded in a given Hadamard matrix. According to the pivot patterns of certain 16×16 Hadamard matrices given in Day and Peterson [5], each of these matrices can occur as their upper left hand corner 5×5 submatrix.

4 Determinants of an Inequivalence Class of Circulant ± 1 Matrices

To obtain more evidence that the determinants of ± 1 matrices of order n are multiples of 2^{n-1}, we considered the subset of circulant matrices that had first element $+1$, at most $\lfloor \frac{n}{2} \rfloor$ elements -1 and which could not be translated into one another by cyclic shifts. We obtained the following results which supported our hypothesis.

Order	First Row	Determinant
$n = 2$	11	0
	1−	0
$n = 3$	111	0
	11−	4
$n = 4$	1111	0
	111−	16
	1 − 1−	0
$n = 5$	11111	0
	1111−	48
	111 − −	16
	11 − 1−	16
$n = 6$	111111	0
	11111−	128
	1111 − −	0
	111 − 1−	128
	111 − −−	0
$n = 7$	1111111	0
	111111−	320
	11111 − −	192
	1111 − 1−	192
	1111 − −−	64
	111 − 1 − −	512
	11 − 11 − −	64
	11 − 1 − 1−	64

5 Conclusions and Open Problems

- many open problems exist in finding the ± 1 matrix with maximum determinant for all positive integer values n;

- we have seen that in small cases the determinant of all ± 1 matrices of order n assumes only a small number of values;

- for a small ± 1 matrix with maximum determinant the upper left hand corner of the matrix can assume only a small number of values;

- for a small ± 1 CP matrix the upper left hand corner of the matrix can assume only a small number of values;

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