

An Algorithm to find Formulae and Values of Minors for Hadamard matrices: II

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Abstract

An algorithm computing the $(n-j) \times (n-j)$, $j = 1, 2, \dots$ minors of Hadamard matrices of order n is presented. Its implementation is analytically described step by step for several values of n and j . For $j = 7$ the values of minors are computed for the first time. A formulae estimating all the values of $(n-j) \times (n-j)$ minors is predicted.

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1 Minors of Hadamard matrices

Hadamard matrices of order n have determinant $n^{\frac{n}{2}}$. Sharpe [7] observed that all the $(n-1) \times (n-1)$ minors of an Hadamard matrix of order n are zero or $n^{\frac{n}{2}-1}$, and that all the $(n-2) \times (n-2)$ minors are zero or $2n^{\frac{n}{2}-2}$, and that all the $(n-3) \times (n-3)$ minors are zero or $4n^{\frac{n}{2}-3}$. We note that the maximum determinant corresponds to having the submatrix

$$\begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & - \\ 1 & - & 1 \end{bmatrix}$$

in the upper lefthand corner of the Hadamard matrix for $n-2$ and $n-3$ respectively.

Theorem 1 [4] *The $(n-3) \times (n-3)$ minors of a Hadamard matrix of order n are zero or $4n^{\frac{n}{2}-3}$.*

Theorem 2 [4] *The $(n-4) \times (n-4)$ minors of a Hadamard matrix of order n are zero, $8n^{\frac{n}{2}-4}$ or $16n^{\frac{n}{2}-4}$.*

Estimations for the values of $(n-j) \times (n-j)$, $j = 1, 2, 3, 4, 5, 6$ minors of Hadamard matrices of order n are also given in [5].

In [4] was outlined a method evaluating the $(n-5), (n-6), \dots, (n-j)$ minors. Throughout the paper the $(n-j) \times (n-j)$ minors are denoted by M_{n-j} whereas the symbols $+$ and $-$ denote $+1$ and -1 respectively. For a given matrix A the symbol m_{ij} denotes the inner product of its rows i and j .

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Lemma 1 *The Distribution Lemma [4]*

Let H be any Hadamard matrix, of order $n > 2$. Then for every triple of rows of H there are precisely $\frac{n}{4}$ columns which are

(a) $(1, 1, 1)^T$ or $(-, -, -)^T$

(b) $(1, 1, -)^T$ or $(-, -, 1)^T$

(c) $(1, -, 1)^T$ or $(-, 1, -)^T$

(d) $(1, -, -)^T$ or $(-, 1, 1)^T$

□

Corollary 1 Let H be any Hadamard matrix, of order $n > 2$. Let us denote with K a $k \times k$ submatrix of H , of order $k > 3$. Then for every triple of rows of K there are at most $\frac{n}{4}$ columns of the form described in the Distribution Lemma.

□

If we are considering the $(n - j) \times (n - j)$ minors of an Hadamard matrix of order n , then the first j rows, ignoring the upper lefthand $j \times j$ matrix, have 2^{j-1} potentially different first j elements in each column. Let $\underline{x}_{\beta+1}^T$ the vectors containing the binary representation of each integer $\beta + 2^{j-1}$ for $\beta = 0, \dots, 2^{j-1} - 1$. Replace all zero entries of $\underline{x}_{\beta+1}^T$ by -1 and define the $j \times 1$ vectors

$$\underline{u}_k = \underline{x}_{2^{j-1}-k+1}, \quad k = 1, \dots, 2^{j-1} \quad (1)$$

Let u_k indicate the number of columns beginning with the vectors \underline{u}_k , $k = 1, \dots, 2^{j-1}$. Let U_j be the submatrix containing the first j rows of the matrix and ignoring the upper lefthand $j \times j$ matrix. Then

$$U_j = \begin{matrix} & \overbrace{1\dots 1}^{u_1} & \overbrace{1\dots 1}^{u_2} & \dots & \overbrace{1\dots 1}^{u_{2^{j-1}-1}} & \overbrace{1\dots 1}^{u_{2^{j-1}}} \\ 1\dots 1 & 1\dots 1 & \dots & 1\dots 1 & 1\dots 1 \\ 1\dots 1 & 1\dots 1 & \dots & -\dots- & -\dots- \\ \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \dots & \cdot & \cdot \\ 1\dots 1 & 1\dots 1 & \dots & 1\dots 1 & -\dots- \\ 1\dots 1 & -\dots- & \dots & -\dots- & -\dots- \end{matrix} \quad (2)$$

From the above distribution lemma we have

$$\sum_{i=1}^{2^{j-1}} u_i = n - j. \quad (3)$$

Then it can be proved that [4]

$$M_{n-j} = n^{n-2^{j-1}-j} \det D \quad (4)$$

where D is the following $z \times z$, $z = 2^{j-1}$ matrix.

$$D = \begin{bmatrix} n - ju_1 & u_2(-\underline{u}_1 \cdot \underline{u}_2) & u_3(-\underline{u}_1 \cdot \underline{u}_3) & \cdots & u_z(-\underline{u}_1 \cdot \underline{u}_z) \\ u_1(-\underline{u}_2 \cdot \underline{u}_1) & n - ju_2 & u_3(-\underline{u}_2 \cdot \underline{u}_3) & \cdots & u_z(-\underline{u}_2 \cdot \underline{u}_z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_1(-\underline{u}_z \cdot \underline{u}_1) & u_2(-\underline{u}_z \cdot \underline{u}_2) & u_3(-\underline{u}_z \cdot \underline{u}_3) & \cdots & n - ju_z \end{bmatrix} \quad (5)$$

where $(\cdot \cdot)$ denotes the inner product.

We note if $n = 4t$, the orthogonality of the rows of the Hadamard matrix gives

$$\begin{aligned} t - j \leq u_1 + u_2 + \dots + u_{2^{j-3}} \leq t, \quad t - j \leq u_{2^{j-3}+1} + \dots + u_{2^{j-2}} \leq t \\ t - j \leq u_{2^{j-2}+1} + \dots + u_{2^{j-2}+2^{j-3}} \leq t, \quad t - j \leq u_{2^{j-2}+2^{j-3}+1} + \dots + u_{2^{j-1}} \leq t \end{aligned}$$

Each of these equations can be rewritten so the constraints become

$$\begin{aligned} 0 \leq t - u_1 - u_2 - \dots - u_{2^{j-3}} \leq j, \quad 0 \leq t - u_{2^{j-3}+1} - \dots - u_{2^{j-2}} \leq j \\ 0 \leq t - u_{2^{j-2}+1} - \dots - u_{2^{j-2}+2^{j-3}} \leq j, \quad 0 \leq t - u_{2^{j-2}+2^{j-3}+1} - \dots - u_{2^{j-1}} \leq j \end{aligned}$$

In order to implement formulae (4) we must first compute the numbers u_i , $i = 1, 2, \dots, 2^{j-1}$ required in (5). In the present paper, we develop explicitly the algorithm computing M_{n-j} , $j = 1, 2, \dots$. For the appropriate implementation of the algorithm the following notions are required.

(i) *Hadamard submatrix*

Let H be an $n \times n$ Hadamard matrix. Let us suppose that M is a matrix with entries ± 1 , of order j . We form the general matrix

$$N = [M \ U_j] \quad (6)$$

where U_j is given from (2).

For every triple of rows (i, k, l) , $1 \leq i < k < l \leq j$ of N , consider the submatrix N_1 formed from them. Let v_1, v_2, v_3, v_4 be the number of columns starting correspondingly with:

- (a) $(1, 1, 1)^T$ or $(-, -, -)^T$
- (b) $(1, 1, -)^T$ or $(-, -, 1)^T$
- (c) $(1, -, 1)^T$ or $(-, 1, -)^T$
- (d) $(1, -, -)^T$ or $(-, 1, 1)^T$

By the orthogonality of these three lines and relation (3) we form the equations:

$$\begin{aligned} v_1 + v_2 - v_3 - v_4 &= -m_{ik} \\ v_1 - v_2 + v_3 - v_4 &= -m_{il} \\ v_1 - v_2 - v_3 + v_4 &= -m_{kl} \\ v_1 + v_2 + v_3 + v_4 &= n - j \end{aligned}$$

This system can be solved uniquely

$$\begin{aligned}
v_1 &= \frac{1}{4}(n - j - m_{ik} - m_{il} - m_{kl}) \\
v_2 &= \frac{1}{4}(n - j - m_{ik} + m_{il} + m_{kl}) \\
v_3 &= \frac{1}{4}(n - j + m_{ik} - m_{il} + m_{kl}) \\
v_4 &= \frac{1}{4}(n - j + m_{ik} + m_{il} - m_{kl})
\end{aligned} \tag{7}$$

If the solution of the above system has integer coordinates and satisfies Corollary 1, then the matrix N_1 is called a *Hadamard submatrix*.

(ii) *Inner product equivalent matrices*

Definition 1 (*The inner product (IP) profile vector.*)

Let us suppose that M is a matrix with entries ± 1 , of order j . We call inner product (IP) profile vector for a given matrix M , the vector

$$IP = (m_{12}, m_{13}, \dots, m_{1j}, m_{23}, m_{24}, \dots, m_{2j}, \dots, m_{j-1,j})$$

□

Definition 2 Two matrices will be called inner product (IP) equivalent if they have the same inner product (IP) profile vector.

□

Example 1 Let us consider the matrices

$$N_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - \\ 1 & 1 & 1 & - & 1 \\ 1 & 1 & 1 & - & - \\ 1 & 1 & - & 1 & - \end{bmatrix}, N_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & 1 \\ 1 & 1 & 1 & 1 & - \\ 1 & 1 & - & - & 1 \\ 1 & 1 & 1 & - & - \end{bmatrix}, N_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & 1 & 1 \\ 1 & 1 & - & 1 & 1 \\ 1 & - & - & 1 & 1 \\ 1 & - & 1 & 1 & - \end{bmatrix}.$$

Then the corresponding IP profile vectors are:

$$IP_1 = (3, 3, 1, 1, 1, 3, 3, 3, -1, 1), IP_2 = (3, 3, 1, 1, 1, 3, 3, -1, 3, 1) \text{ and } IP_3 = (3, 3, 1, 1, 1, 3, 3, 3, -1, 1).$$

We note that although N_1, N_2, N_3 are equivalent matrices only N_1 and N_3 are inner product equivalent matrices, because $IP_1 = IP_3$.

□

Definition 3 Two matrices with entries ± 1 are called column equivalent if one can be obtained from the other by permuting columns and/or multiplying columns by -1 .

2 The Algorithm computing the $(n - j) \times (n - j)$ minors of an $n \times n$ Hadamard matrix.

Let H be a Hadamard matrix of order n . The following algorithm computes the values of M_{n-j} , $j = 1, 2, \dots$

Step1: Generate all (IP) inequivalent ± 1 matrices M , of order j with first row and column all $+1$.

Step2: Form the general matrix, $N = [M \ U_j]$, of size $j \times n$ for the first j rows of an $n \times n$ Hadamard matrix H , where U_j is given from (2)

Step3: For each M test if it can be a *Hadamard submatrix*

Step 4: For each M specified in Step 3 do the following:

Set $s = 0$

For $k = 3, 4, \dots, j$

Step 5 Take the first k lines of N

Set $s = s + 1$

Set $u_l^{(s)}$ the number of columns starting with

the vectors $\underline{u}_l = \underline{x}_{2^{k-1-l+1}}$, $l = 1, \dots, 2^{k-1}$

Using the orthogonality and relation (3), form

$\binom{k}{2} + 1$ equations which have 2^{k-1} variables $u_l^{(s)}$

satisfying Corollary 1. Search for all the feasible

solutions to the system of different equations produced,

taking into account relation (8).

Step 6 For each feasible solution found in Step 5, use the matrix D , given from (5), to find all possible values of the $(n - j) \times (n - j)$ minors.

Implementation of the algorithm

Step 1: *Generate efficiently(IP) inequivalent matrices.*

We create first all possible ± 1 matrices M of order j with first row and column all $+1$. Instead of creating all $2^{j-1} \times 2^{j-1}$ matrices M , taking into account Proposition 1, we create only

the column inequivalent matrices M which are the combinations with repetition $\begin{bmatrix} 2^{j-1} \\ j-1 \end{bmatrix} =$

$\binom{2^{j-1} + j - 2}{2^{j-1} - 1}$. Since two column inequivalent matrices might be (IP) equivalent, we need

to check for different (IP) profile vectors after generating all column inequivalent matrices M .

For example let $j = 6$. Then the number of all possible matrices M is $2^{25} = 33554432$, but

the number of column inequivalent M is only 376992. Finally the number of (IP) inequivalent

matrices is 368672.

Step2:

$$M(1:3,:) \begin{matrix} u_1^{(1)} & u_2^{(1)} & u_3^{(1)} & u_4^{(1)} \\ 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \end{matrix}$$

that contains the three first rows of N . $M(1:3,:)$ is the matrix containing the three first rows of M . Since $\sum_{i=1}^4 u_i^{(1)} = n - 6$ and by the orthogonality we form the equations:

$$\begin{aligned} u_1^{(1)} + u_2^{(1)} - u_3^{(1)} - u_4^{(1)} &= -m_{12} \\ u_1^{(1)} - u_2^{(1)} + u_3^{(1)} - u_4^{(1)} &= -m_{13} \\ u_1^{(1)} - u_2^{(1)} - u_3^{(1)} + u_4^{(1)} &= -m_{23} \\ u_1^{(1)} + u_2^{(1)} + u_3^{(1)} + u_4^{(1)} &= n - 6 \end{aligned}$$

This system can be solved uniquely

$$\begin{aligned} u_1^{(1)} &= \frac{1}{4}(n - 6 - m_{12} - m_{13} - m_{23}) \\ u_2^{(1)} &= \frac{1}{4}(n - 6 - m_{12} + m_{13} + m_{23}) \\ u_3^{(1)} &= \frac{1}{4}(n - 6 + m_{12} - m_{13} + m_{23}) \\ u_4^{(1)} &= \frac{1}{4}(n - 6 + m_{12} + m_{13} - m_{23}) \end{aligned} \tag{11}$$

Corollary 1 gives that $0 \leq u_i^{(1)} \leq \frac{n}{4}$, $i = 1, 2, 3, 4$ and $u_i^{(1)}$ must be integer. Since in the next step for $k = 4$ we will have 7 equations with 8 unknowns we need one coordinate of the solution $u_i^{(1)}$, $i = 1, 2, 3, 4$ according to which we will express the range of values of one of the 8 unknowns variables of the next step, following relation (8). Let us suppose that for each (IP) we keep the coordinate $u_4^{(1)}$.

(II) $k=4$

Similarly to (I) let us create the matrix

$$M(1:4,:) \begin{matrix} u_1^{(2)} & u_2^{(2)} & u_3^{(2)} & u_4^{(2)} & u_5^{(2)} & u_6^{(2)} & u_7^{(2)} & u_8^{(2)} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & - & 1 & - & 1 & - & 1 & - \end{matrix}$$

that contains the four first rows of N . $M(1:4,:)$ contains the first four rows of M . Using the orthogonality and since $\sum_{i=1}^8 u_i^{(2)} = n - 6$ we create a system of 7 equations with 8 unknowns. We solve the system symbolically and we suppose that the solution $u_i^{(2)}$, $i = 1, 2, \dots, 8$ is expressed as a function of the variable $u_8^{(2)}$.

$$u_1^{(2)} = 1/4 \cdot (n - 6 - m_{23} - m_{24} - m_{34}) - u_8^{(2)}$$

$$\begin{aligned}
u_2^{(2)} &= 1/4 \cdot (m_{34} - m_{13} + m_{24} - m_{12}) + u_8^{(2)} \\
u_3^{(2)} &= 1/4 \cdot (m_{34} - m_{14} + m_{23} - m_{12}) + u_8^{(2)} \\
u_4^{(2)} &= 1/4 \cdot (n - 6 - m_{34} + m_{13} + m_{14}) - u_8^{(2)} \\
u_5^{(2)} &= 1/4 \cdot (m_{24} - m_{14} + m_{23} - m_{13}) + u_8^{(2)} \\
u_6^{(2)} &= 1/4 \cdot (n - 6 - m_{24} + m_{12} + m_{14}) - u_8^{(2)} \\
u_7^{(2)} &= 1/4 \cdot (n - 6 - m_{23} + m_{12} + m_{13}) - u_8^{(2)}
\end{aligned} \tag{12}$$

Each $u_i^{(2)}$ is a positive integer satisfying $0 \leq u_i^{(2)} \leq \frac{n}{4}$, $i = 1, 2, \dots, 8$. For all possible (IP) and the solution $u_4^{(1)}$ as found from (11) we must specify the possible values of the parameter $u_8^{(2)}$. Since $u_7^{(2)} + u_8^{(2)} = u_4^{(1)}$ and thus $u_8^{(2)} = u_4^{(1)} - u_7^{(2)} \leq u_4^{(1)}$ ($u_7^{(2)}$ is always a non negative number) the range of values for $u_8^{(2)}$ is from 0 to $u_4^{(1)}$. We now compute, for all the possible values of $u_8^{(2)}$ the integer values of the remaining $u_i^{(2)}$ satisfying Corollary 1. Since in the next step for $k = 5$ we will have 11 equations with 16 unknowns we need four coordinates of the solution $u_i^{(2)}$, $i = 1, 2, \dots, 8$ according to which we will express the range of values of four of the 16 unknowns variables of the next step, following relation (8). Let us suppose that for each (IP) we keep the coordinates $u_4^{(2)}, u_6^{(2)}, u_7^{(2)}, u_8^{(2)}$.

(III) $k=5$

Similarly to (I),(II) we create the matrix containing the first five rows of N . By orthogonality, the relation $\sum_{i=1}^{16} u_i^{(3)} = n - 6$ and by expressing symbolically, as a function of the variables $u_8^{(3)}, u_{12}^{(3)}, u_{14}^{(3)}, u_{15}^{(3)}, u_{16}^{(3)}$, the solution of the 11 equations produced with 16 unknowns we get:

$$\begin{aligned}
u_1^{(3)} &= 1/2 \cdot (n - 6) - 1/4 \cdot (m_{45} + m_{25} + m_{35} - m_{15} - m_{12} + m_{23} - m_{13} \\
&\quad + m_{24} + m_{34} - m_{14}) - u_8^{(3)} - u_{12}^{(3)} - u_{14}^{(3)} - u_{15}^{(3)} - 3 \cdot u_{16}^{(3)} \\
u_2^{(3)} &= -1/4 \cdot (n - 6 - m_{45} - m_{25} - m_{35} + m_{15} + m_{12} + m_{13} + m_{14}) \\
&\quad + u_8^{(3)} + u_{12}^{(3)} + u_{14}^{(3)} + 2 \cdot u_{16}^{(3)} \\
u_3^{(3)} &= -1/4 \cdot (n - 6 - m_{45} + m_{15} + m_{12} + m_{13} - m_{24} - m_{34} + m_{14}) \\
&\quad + u_8^{(3)} + u_{12}^{(3)} + u_{15}^{(3)} + 2 \cdot u_{16}^{(3)} \\
u_4^{(3)} &= 1/4 \cdot (n - 6 - m_{45} + m_{15} + m_{14}) - u_8^{(3)} - u_{12}^{(3)} - u_{16}^{(3)} \\
u_5^{(3)} &= -1/4 \cdot (n - 6 - m_{35} + m_{15} + m_{12} - m_{23} + m_{13} - m_{34} + m_{14}) \\
&\quad + u_8^{(3)} + u_{14}^{(3)} + u_{15}^{(3)} + 2 \cdot u_{16}^{(3)} \\
u_6^{(3)} &= 1/4 \cdot (n - 6 - m_{35} + m_{15} + m_{13}) - u_8^{(3)} - u_{14}^{(3)} - u_{16}^{(3)} \\
u_7^{(3)} &= 1/4 \cdot (n - 6 - m_{34} + m_{13} + m_{14}) - u_8^{(3)} - u_{15}^{(3)} - u_{16}^{(3)} \\
u_9^{(3)} &= -1/4 \cdot (n - 6 - m_{25} + m_{15} + m_{12} + m_{13} - m_{23} - m_{24} + m_{14}) \\
&\quad + u_{12}^{(3)} + u_{14}^{(3)} + u_{15}^{(3)} + 2 \cdot u_{16}^{(3)} \\
u_{10}^{(3)} &= 1/4 \cdot (n - 6 + m_{15} - m_{25} + m_{12}) - u_{12}^{(3)} - u_{14}^{(3)} - u_{15}^{(3)}
\end{aligned} \tag{13}$$

$$\begin{aligned}
u_{11}^{(3)} &= 1/4 \cdot (n - 6 + m_{14} - m_{24} + m_{12}) - u_{12}^{(3)} - u_{15}^{(3)} - u_{16}^{(3)} \\
u_{13}^{(3)} &= 1/4 \cdot (n - 6 + m_{13} - m_{23} + m_{12}) - u_{14}^{(3)} - u_{15}^{(3)} - u_{16}^{(3)}
\end{aligned}$$

It holds that $u_{2i-1}^{(3)} + u_{2i}^{(3)} = u_i^{(2)}$, $i = 1, 2, \dots, 8$, where each $u_i^{(3)}$ is a positive integer satisfying $0 \leq u_i^{(3)} \leq \frac{n}{4}$, $i = 1, 2, \dots, 16$. For each (IP) and the coordinates $u_4^{(2)}, u_6^{(2)}, u_7^{(2)}, u_8^{(2)}$ found from (12), we specify the possible values of the parameters $u_8^{(3)}, u_{12}^{(3)}, u_{14}^{(3)}, u_{16}^{(3)}$. These are in the following range $0 \leq u_8^{(3)} \leq u_4^{(2)}$, $0 \leq u_{12}^{(3)} \leq u_6^{(2)}$, $0 \leq u_{14}^{(3)} \leq u_7^{(2)}$ and $0 \leq u_{16}^{(3)} \leq u_8^{(2)}$. Since $u_{15}^{(3)} + u_{16}^{(3)} = u_8^{(2)}$ and thus $u_{15}^{(3)} = u_8^{(2)} - u_{16}^{(3)}$, the range of values of $u_{15}^{(3)}$ is specified from $u_{16}^{(3)}$. In the sequel, for all the possible values of $u_8^{(3)}, u_{12}^{(3)}, u_{14}^{(3)}, u_{15}^{(3)}$ and $u_{16}^{(3)}$ we compute the integer values of the rest $u_i^{(3)}$ satisfying Corollary 1. Since in the next step for $k = 6$ we will have 16 equations with 32 unknowns, we need eleven coordinates of the solution $u_i^{(3)}$, $i = 1, 2, \dots, 16$ according to which we will express the range of values of ten out of the 32 unknowns variables of the next step, following relation (8). Let us suppose that for each (IP) we keep the coordinates $u_4^{(3)}, u_6^{(3)}, u_7^{(3)}, u_8^{(3)}, u_{10}^{(3)}, u_{11}^{(3)}, u_{12}^{(3)}, u_{13}^{(3)}, u_{14}^{(3)}, u_{15}^{(3)}, u_{16}^{(3)}$.

(IV) $k=6$

Similarly to (I),(II),(III) let us create the matrix containing the first six rows of N . By orthogonality, the relation $\sum_{i=1}^{32} u_i^{(4)} = n - 6$, and by solving symbolically the system of the produced equations as a function of the variables $u_8^{(4)}, u_{12}^{(4)}, u_{14}^{(4)}, u_{15}^{(4)}, u_{16}^{(4)}, u_{20}^{(4)}, u_{22}^{(4)}, u_{23}^{(4)}, u_{24}^{(4)}$, and $u_i^{(4)}$, $i = 26, 27, \dots, 32$ we get:

$$\begin{aligned}
u_1^{(4)} &= (n - 6) + 1/2 \cdot (m_{13} + m_{14} + m_{15} + m_{16} + m_{12}) + \frac{1}{4} \cdot (-m_{23} - m_{24} \\
&\quad - m_{25} - m_{26} - m_{34} - m_{35} - m_{36} - m_{45} - m_{46} - m_{56}) \\
&\quad - u_{26}^{(4)} - u_8^{(4)} - u_{12}^{(4)} - u_{14}^{(4)} - u_{15}^{(4)} - u_{29}^{(4)} - u_{23}^{(4)} - u_{27}^{(4)} - u_{20}^{(4)} - u_{22}^{(4)} \\
&\quad - 3(u_{16}^{(4)} - u_{31}^{(4)} - u_{24}^{(4)} - u_{30}^{(4)} - u_{28}^{(4)}) - 6u_{32}^{(4)} \\
u_2^{(4)} &= -1/2 \cdot (n - 6 + m_{16}) + \frac{1}{4} \cdot (-m_{13} - m_{14} - m_{15} + m_{26} + m_{36} + m_{46} + m_{56} - m_{12}) \\
&\quad + u_8^{(4)} + u_{12}^{(4)} + u_{14}^{(4)} + u_{20}^{(4)} + u_{22}^{(4)} + u_{26}^{(4)} + 2(u_{16}^{(4)} + u_{24}^{(4)} + u_{28}^{(4)} + u_{30}^{(4)}) + 3u_{32}^{(4)} \\
u_3^{(4)} &= -1/2 \cdot (n - 6 - m_{15}) + \frac{1}{4} \cdot (-m_{13} - m_{14} - m_{16} + m_{25} + m_{35} + m_{45} + m_{56} - m_{12}) \\
&\quad + u_8^{(4)} + u_{12}^{(4)} + u_{15}^{(4)} + u_{20}^{(4)} + u_{23}^{(4)} + u_{27}^{(4)} + 2(u_{16}^{(4)} + u_{24}^{(4)} + u_{28}^{(4)} + u_{31}^{(4)}) + 3u_{32}^{(4)} \\
u_4^{(4)} &= \frac{1}{4} \cdot (n - 6 + m_{15} + m_{16} - m_{56}) - u_8^{(4)} - u_{12}^{(4)} - u_{16}^{(4)} - u_{20}^{(4)} - u_{24}^{(4)} - u_{28}^{(4)} - u_{32}^{(4)} \\
u_5^{(4)} &= -\frac{1}{2} \cdot (n - 6 - m_{14}) + \frac{1}{4} \cdot (-m_{13} - m_{15} - m_{16} + m_{24} + m_{34} + m_{45} + m_{46} - m_{12}) \\
&\quad + u_8^{(4)} + u_{14}^{(4)} + u_{15}^{(4)} + u_{22}^{(4)} + u_{23}^{(4)} + u_{29}^{(4)} + 2(u_{16}^{(4)} + u_{24}^{(4)} + u_{30}^{(4)} + u_{31}^{(4)}) + 3u_{32}^{(4)} \\
u_6^{(4)} &= \frac{1}{4} \cdot (n - 6 + m_{14} + m_{16} - m_{46}) - u_8^{(4)} - u_{14}^{(4)} - u_{16}^{(4)} - u_{22}^{(4)} - u_{24}^{(4)} - u_{30}^{(4)} - u_{32}^{(4)} \\
u_7^{(4)} &= \frac{1}{4} \cdot (n - 6 + m_{14} + m_{15} - m_{45}) - u_8^{(4)} - u_{15}^{(4)} - u_{16}^{(4)} - u_{23}^{(4)} - u_{24}^{(4)} - u_{31}^{(4)} - u_{32}^{(4)}
\end{aligned} \tag{14}$$

$$\begin{aligned}
u_9^{(4)} &= -\frac{1}{2} \cdot (n - 6 + m_{13}) + \frac{1}{4} \cdot (-m_{14} - m_{15} - m_{16} + m_{23} + m_{34} + m_{35} + m_{36} - m_{12}) \\
&\quad + u_{12}^{(4)} + u_{14}^{(4)} + u_{15}^{(4)} + u_{26}^{(4)} + u_{27}^{(4)} + u_{29}^{(4)} + 2(u_{16}^{(4)} + u_{28}^{(4)} + u_{30}^{(4)} + u_{31}^{(4)}) + 3u_{32}^{(4)} \\
u_{10}^{(4)} &= \frac{1}{4} \cdot (n - 6 + m_{13} + m_{16} - m_{36}) - u_{12}^{(4)} - u_{14}^{(4)} - u_{16}^{(4)} - u_{26}^{(4)} - u_{28}^{(4)} - u_{30}^{(4)} - u_{32}^{(4)} \\
u_{11}^{(4)} &= \frac{1}{4} \cdot (n - 6 + m_{13} + m_{15} - m_{35}) - u_{12}^{(4)} - u_{15}^{(4)} - u_{16}^{(4)} - u_{27}^{(4)} - u_{28}^{(4)} - u_{31}^{(4)} - u_{32}^{(4)} \\
u_{13}^{(4)} &= \frac{1}{4} \cdot (n - 6 + m_{13} + m_{14} - m_{34}) - u_{14}^{(4)} - u_{15}^{(4)} - u_{16}^{(4)} - u_{29}^{(4)} - u_{30}^{(4)} - u_{31}^{(4)} - u_{32}^{(4)} \\
u_{17}^{(4)} &= -\frac{1}{2} \cdot (n - 6 + m_{12}) + \frac{1}{4} \cdot (-m_{13} - m_{14} - m_{15} - m_{16} + m_{23} + m_{24} + m_{25} + m_{26}) \\
&\quad + u_{20}^{(4)} + u_{22}^{(4)} + u_{23}^{(4)} + u_{26}^{(4)} + u_{27}^{(4)} + u_{29}^{(4)} + 2 \cdot (u_{24}^{(4)} + u_{28}^{(4)} + u_{30}^{(4)} + u_{31}^{(4)}) + 3u_{32}^{(4)} \\
u_{18}^{(4)} &= \frac{1}{4} \cdot (n - 6 + m_{16} - m_{26} + m_{12} - u_{26}^{(4)} - u_{20}^{(4)} - u_{22}^{(4)} - u_{24}^{(4)} - u_{28}^{(4)} - u_{30}^{(4)} - u_{32}^{(4)}) \\
u_{19}^{(4)} &= \frac{1}{4} \cdot (n - 6 + m_{15} - m_{25} + m_{12} - u_{27}^{(4)} - u_{20}^{(4)} - u_{23}^{(4)} - u_{24}^{(4)} - u_{28}^{(4)} - u_{31}^{(4)} - u_{32}^{(4)}) \\
u_{21}^{(4)} &= \frac{1}{4} \cdot (n - 6 + m_{14} - m_{24} + m_{12} - u_{29}^{(4)} - u_{22}^{(4)} - u_{23}^{(4)} - u_{24}^{(4)} - u_{30}^{(4)} - u_{31}^{(4)} - u_{32}^{(4)}) \\
u_{25}^{(4)} &= \frac{1}{4} \cdot (n - 6 + m_{13} - m_{23} + m_{12} - u_{29}^{(4)} - u_{26}^{(4)} - u_{27}^{(4)} - u_{28}^{(4)} - u_{30}^{(4)} - u_{31}^{(4)} - u_{32}^{(4)})
\end{aligned}$$

It holds that $u_{2i-1}^{(4)} + u_{2i}^{(4)} = u_i^{(3)}$, $i = 1, 2, \dots, 16$, where each $u_k^{(4)}$ is a positive integer satisfying $0 \leq u_k \leq \frac{n}{4}$, $k = 1, 2, \dots, 32$. For each (IP) and $u_4^{(3)}, u_7^{(3)}, u_8^{(3)}, u_{10}^{(3)}, u_{11}^{(3)}, u_{12}^{(3)}, u_{13}^{(3)}, u_{14}^{(3)}, u_{15}^{(3)}, u_{16}^{(3)}$ found from (13) we specify the possible values of the parameters of (14). The following inequalities hold:

$$\begin{aligned}
u_{31}^{(4)} &= u_{16}^{(3)} - u_{32}^{(4)} \quad \text{and} \quad 0 \leq u_{32}^{(4)} \leq u_{16}^{(3)} \\
u_{29}^{(4)} &= u_{15}^{(3)} - u_{30}^{(4)} \quad \text{and} \quad 0 \leq u_{30}^{(4)} \leq u_{15}^{(3)} \\
u_{27}^{(4)} &= u_{14}^{(3)} - u_{28}^{(4)} \quad \text{and} \quad 0 \leq u_{28}^{(4)} \leq u_{14}^{(3)} \\
&\quad 0 \leq u_{26}^{(4)} \leq u_{13}^{(3)} \\
u_{23}^{(4)} &= u_{12}^{(3)} - u_{24}^{(4)} \quad \text{and} \quad 0 \leq u_{24}^{(4)} \leq u_{12}^{(3)} \\
&\quad 0 \leq u_{22}^{(4)} \leq u_{11}^{(3)} \\
&\quad 0 \leq u_{20}^{(4)} \leq u_{10}^{(3)} \\
u_{15}^{(4)} &= u_8^{(3)} - u_{16}^{(4)} \quad \text{and} \quad 0 \leq u_{16}^{(4)} \leq u_8^{(3)} \\
&\quad 0 \leq u_{14}^{(4)} \leq u_7^{(3)} \\
&\quad 0 \leq u_{12}^{(4)} \leq u_6^{(3)} \\
&\quad 0 \leq u_8^{(4)} \leq u_4^{(3)}
\end{aligned}$$

3 Conclusions and open problems

The results obtained from the proposed algorithm, after testing it up to the $(n-7)$ case, produced the values of minors presented in the following table:

order	Values of Minors
n-1	$0, n^{\frac{n}{2}-1}$
n-2	$0, 2n^{\frac{n}{2}-2}$
n-3	$0, 4n^{\frac{n}{2}-3}$
n-4	$0, 8n^{\frac{n}{2}-4}, 16n^{\frac{n}{2}-4}$
n-5	$0, 16n^{\frac{n}{2}-5}, 32n^{\frac{n}{2}-5}, 48n^{\frac{n}{2}-5}$
n-6	$0, 32n^{\frac{n}{2}-6}, 64n^{\frac{n}{2}-6}, 96n^{\frac{n}{2}-6}, 128n^{\frac{n}{2}-6}, 160n^{\frac{n}{2}-6}$
n-7	$0, 64n^{\frac{n}{2}-7}, 128n^{\frac{n}{2}-7}, 192n^{\frac{n}{2}-7}, 256n^{\frac{n}{2}-7}, 320n^{\frac{n}{2}-7}, 384n^{\frac{n}{2}-7}, 448n^{\frac{n}{2}-7}, 512n^{\frac{n}{2}-7}, 576n^{\frac{n}{2}-7}$

Table 1

Remark 1 We see that as the value of $n-j$ decreases the range of values of the corresponding minors M_{n-j} increases. We notice that the following formulae holds:

$$M_{n-j} = 0, \quad \text{or} \quad p \cdot n^{\frac{n}{2}-j}, \quad j = 0, 1, 2, \dots$$

where for the evaluation of the coefficient p the following procedure is adopted:

```

Set  $p = 2^{j-1}$ 
Set  $s = \max|\det(A)|$ 
   where  $A$  an  $j \times j$  matrix with all elements  $\pm 1$ 's.
Set  $k = 1$ 
repeat
    $p = k \cdot p$ 
    $k = k + 1$ 
until
    $p = s$ .

```

Actually, as we can see from the above Table 1, the coefficients of the last value of each minor are 1, 2, 4, 16, 48, 160, 576 which are the maximum values of the determinants of the ± 1 matrices of orders 1, 2, 3, 4, 5, 6, and 7 respectively, see for example [3] and the references therein. Thus the number of the values appearing for each M_{n-j} will be $\frac{s}{2^{j-1}}$. We conjecture that the $n-8$ minors can take the values $k \cdot 2^7 \cdot n^{\frac{n}{2}-8}$, $k = 1, 2, \dots, \frac{4096}{2^7} = 32$ since the maximum determinant of an 8×8 matrix with all elements ± 1 's is 4096, which is the determinant of a Hadamard matrix of order 8.

The algorithm theoretically can proceed to the computation of any $n-j$ minor. As j increases a difficult step of the algorithm concerns the construction of all possible $j \times j$ matrices M with entries ± 1 . An efficient procedure to construct all the inequivalent ± 1 matrices M is under

investigation. A parallel implementation of the above algorithm is also under research. If we can manage to implement the algorithm up to the $n - 15$ case, then we can get an estimation of the maximum determinant of the 15×15 matrix with all elements ± 1 's, which is an unsolved problem. However, it has been conjectured in [1] that this maximum determinant should be $2^{14} \cdot 3^6 \cdot 35$, and an example of such a matrix attaining this value was also given in [1].

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