Some results on Kharaghani type orthogonal designs

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Abstract

In this paper we give a general theorem which can be used to multiply the length of amicable sequences keeping the amicability property and the type of the sequences. As a consequence we have that if there exist two, four or eight amicable sequences of length m and type (a_1, a_2), (a_1, a_2, a_3, a_4) or $(a_1, a_2, \ldots, a_8)$ then there exist amicable sequences of length $\ell \equiv\ 0\ (mod\ m)$ and of the same type. We also present a theorem that produces a set of $2v$ amicable sequences from a set of v (not necessary amicable) sequences and a construction method for amicable sequences of type $(a_1, a_1, a_2, a_2, \ldots, a_v, a_v)$ from v pairs of disjoint $(0, \pm 1)$ amicable sequences.

Using these results we can obtain many infinite classes of Kharaghani type orthogonal designs. Actually, if there exists an Kharaghani type orthogonal design of order n and of type $(a_1, a_2, \ldots, a_v)$, which is constructed from sequences, then there exists an infinite family of Kharaghani type orthogonal designs of the same type which is constructed from appropriate sequences.

Key words and phrases: Sequences, orthogonal designs, Kharaghani type orthogonal designs, amicable sets, Hall polynomial.

AMS Subject Classification: Primary 05B15, 05B20.

1 Introduction

An *orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$ denoted $OD(n; s_1, s_2, \ldots, s_u)$ in the variables $x_1, x_2, \ldots, x_u$, is a matrix A of order n with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$AA^T = \sum_{i=1}^{u} (s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n. Let A_1, A_2 be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2\}$ satisfying $A_1 A_1^T + A_2 A_2^T = (s_1 x_1^2 + s_2 x_2^2) I_n$. Then

$$D = \begin{pmatrix} A_1 & A_2 \\ -A_2^T & A_1^T \end{pmatrix} \quad \text{or} \quad D = \begin{pmatrix} A_1 & A_2 R \\ -A_2 R & A_1 \end{pmatrix}. \tag{1}$$

is an $OD(2n; s_1, s_2)$.

Let B_i, $i = 1, 2, 3, 4$ be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$\sum_{i=1}^{4} B_i B_i^T = \sum_{i=1}^{u} (s_i x_i^2) I_n.$$

Then the Goethals-Seidel array

$$G = \begin{pmatrix} B_1 & B_2 R & B_3 R & B_4 R \\ -B_2 R & B_1 & B_4^T R & -B_3^T R \\ -B_3 R & -B_4^T R & B_1 & B_2^T R \\ -B_4 R & B_3^T R & -B_2^T R & B_1 \end{pmatrix} \tag{2}$$

where R is the back-diagonal identity matrix, is an $OD(4n; s_1, s_2, \ldots, s_u)$. See page 107 of [1] for details.

A pair of matrices A, B is said to be amicable (anti-amicable) if $AB^T - BA^T = 0$ ($AB^T + BA^T = 0$). Following [4] a set $\{A_1, A_2, \ldots, A_{2n}\}$ of square real matrices is said to be *amicable* if

$$\sum_{i=1}^{n} \left(A_{\sigma(2i-1)} A_{\sigma(2i)}^T - A_{\sigma(2i)} A_{\sigma(2i-1)}^T \right) = 0 \tag{3}$$

for some permutation σ of the set $\{1, 2, \ldots, 2n\}$. For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. So

$$\sum_{i=1}^{n} \left(A_{2i-1} A_{2i}^T - A_{2i} A_{2i-1}^T \right) = 0. \tag{4}$$

Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. Throughout the paper R_k denotes the back diagonal identity matrix of order k.

A set of matrices $\{B_1, B_2, \ldots, B_n\}$ of order m with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ is said to satisfy an additive property of type $(s_1, s_2, \ldots, s_u)$ if

$$\sum_{i=1}^{n} B_i B_i^T = \sum_{i=1}^{u} (s_i x_i^2) I_m. \tag{5}$$

Let $\{A_i\}_{i=1}^{8}$ be an amicable set of circulant matrices (or type 1) of type $(s_1, s_2, \ldots, s_u)$ of order t. Then the Kharaghani array from [4]

$$H = \begin{pmatrix} A_1 & A_2 & A_4R_n & A_3R_n & A_6R_n & A_5R_n & A_8R_n & A_7R_n \\ -A_2 & A_1 & A_3R_n & -A_4R_n & A_5R_n & -A_6R_n & A_7R_n & -A_8R_n \\ -A_4R_n & -A_3R_n & A_1 & A_2 & -A_8^TR_n & A_7^TR_n & A_6^TR_n & -A_5^TR_n \\ -A_3R_n & A_4R_n & -A_2 & A_1 & A_7^TR_n & A_8^TR_n & -A_5^TR_n & -A_6^TR_n \\ -A_6R_n & -A_5R_n & A_8^TR_n & -A_7^TR_n & A_1 & A_2 & -A_4^TR_n & A_3^TR_n \\ -A_5R_n & A_6R_n & -A_7^TR_n & -A_8^TR_n & -A_2 & A_1 & A_3^TR_n & A_4^TR_n \\ -A_8R_n & -A_7R_n & -A_6^TR_n & A_5^TR_n & A_4^TR_n & -A_3^TR_n & A_1 & A_2 \\ -A_7R_n & A_8R_n & A_5^TR_n & A_6^TR_n & -A_3^TR_n & -A_4^TR_n & -A_2 & A_1 \end{pmatrix} \tag{6}$$

is a Kharaghani type orthogonal design $OD(8m; s_1, s_2, \ldots, s_u)$.

The Kharaghani array has been used in a number of papers [2, 3, 4, 5, 6] to obtain infinitely many families of Kharaghani type orthogonal designs.

A set $\{A_i\}_{i=1}^{4}$ is said to be a *short amicable set* of length m and type (u_1, u_2, u_3, u_4) if (4) and (5) are satisfied for $n = 4$ and $u \leq 4$. Short amicable sets can be used in either the Goethals-Seidel array or the *short Kharaghani array*

$$\begin{bmatrix} A & B & CR & DR \\ -B & A & DR & -CR \\ -CR & -DR & A & B \\ -DR & CR & -B & A \end{bmatrix} \tag{7}$$

to form an Goethal-Seidel type orthogonal design $OD(4m; u_1, u_2, u_3, u_4)$.

A set of sequences $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, \ldots, 2v$ where $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_p\}$, $j = 0, 1, \ldots, m-1$ and $k = 1, 2, \ldots, 2v$ is said to be a set of $2v$ *amicable sequences* of length m and type $(u_1, u_2, \ldots, u_p)$ if the corresponding circulant matrices which are constructed from these sequences satisfy the equations (4) and (5). On the other hand, it is clear that, if we have a set of circulant amicable matrices then their first rows can be considered as a set of amicable sequences. Therefore, throughout this paper we use either circulant amicable matrices or amicable sequences.

Given the sequence $A = \{a_1, a_2, \ldots, a_n\}$ of length n the *non-periodic autocorrelation function (NPAF)* $N_A(s)$ is defined as

$$N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \quad s = 0, 1, \ldots, n-1, \tag{8}$$

Given A as above of length n the *periodic autocorrelation function (PAF)* $P_A(s)$ is defined, reducing $i+s$ modulo n, as

$$P_A(s) = \sum_{i=1}^{n} a_i a_{i+s}, \quad s = 0, 1, \ldots, n-1. \tag{9}$$

We define the $NPAF$ (PAF) of a set of sequences the sum of the corresponding $NPAF$ (PAF) of the individual sequences.

Suppose $C = circ(c_0, c_1, \ldots, c_{n-1})$ is a circulant matrix of order n. Let

$$T_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

of order n, be the shift matrix. Then we can write $C = c_0 I + c_1 T_n + \ldots + c_{n-1} T_n^{n-1}$. Note that $T_n^n = I$ the identity matrix of order n. We say the Hall polynomial of C is $\sum_{i=0}^{n-1} c_i x^i$. The Hall polynomial of C^T is $\sum_{i=0}^{n-1} c_i x^{n-i}$.

2 Multiplication of the length of amicable sets of sequences

Theorem 1 *Let* $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, \ldots, 2v$ *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_p\}$, $j = 0, 1, \ldots, m-1$ *and* $k = 1, 2, \ldots, 2v$ *be a set of* $2v$ *amicable sequences of length* m *and type* $(u_1, u_2, \ldots, u_p)$. *Then there exist a set of* $2v$ *amicable sequences of length* $\ell \equiv 0 \ (\mathit{mod}\ m) = mi$ *for all* $i = 1, 2, \ldots$ *and type* $(u_1, u_2, \ldots, u_p)$.

Proof. Let i be a constant integer. We use the map T_m^k to define sequences A_k and the map $S_\ell^k = T_m^k$ to define sequences B_k

$$B_k = \sum_{j=0}^{m-1} a_{k,j} S_\ell^j, \ k = 1, 2, \ldots, 2v$$

Now

$$\sum_{k=1}^{2v} A_k A_k^T = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{2v} \left(a_{k,j} a_{k,x} T_m^{j-x} \right) = \left(\sum_{k=1}^{p} u_k x_k^2 \right) I_m.$$

Thus we have that

(i) If m is odd then the coefficients of T_m^{σ}, $\sigma = -(m-1), \ldots, -1, 1, \ldots, m-1$ is zero, and the coefficient of T_m^0 is $\sum_{k=1}^{p} u_k x_k^2$.That means

$$\sum^{m-1} j, x = 0j - x = \sigma \sum_{k=1}^{2v} a_{k,j} a_{k,x} = 0 \quad \text{and} \quad \sum_{j=0}^{m-1} \sum_{k=1}^{2v} a_{k,j}^2 = \sum_{k=1}^{p} u_k x_k^2 \tag{10}$$

(ii) If m is even, $m = 2n$ then we have that $T_m^n = T_m^{-n}$ and so the coefficients of T_m^{σ}, $\sigma = -(2n-1), \ldots, -(n+1), -(n-1), \ldots, -1, 1, \ldots, n-1, n+1, \ldots, 2n-1$ are zero, the coefficient of T_m^n plus the coefficient of T_m^{-n} is zero and the coefficient of T_m^0 is $\sum_{k=1}^{p} u_k x_k^2$.That means

$$\sum^{m-1} j, x = 0j - x = \sigma\sigma \neq \pm n \sum_{k=1}^{2v} a_{k,j} a_{k,x} = 0, \sum^{m-1} j, x = 0j - x = \pm n \sum_{k=1}^{2v} a_{k,j} a_{k,x} = 0 \text{ and } \sum_{j=0}^{m-1} \sum_{k=1}^{2v} a_{k,j}^2 = \sum_{k=1}^{p} u \tag{11}$$

Now

$$\sum_{k=1}^{2v} B_k B_k^T = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{2v} \left(a_{k,j} a_{k,x} S_{\ell}^{j-x} \right)$$

We have that the coefficients of S_{ℓ}^{σ} are equal to the coefficients of T_m^{σ} for all $\sigma = -(m-1), \ldots, m-1$, and so using equations (10) or (11) we obtain

$$\sum_{k=1}^{2v} B_k B_k^T = \left(\sum_{k=1}^{p} u_k x_k^2 \right) I_{2mi} = \left(\sum_{k=1}^{p} u_k x_k^2 \right) I_{\ell} \tag{12}$$

Moreover

$$\sum_{k=1}^{v} \left(A_{2k-1} A_{2k}^T - A_{2k} A_{2k-1}^T \right) = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{v} \left((a_{2k-1,j} a_{2k,x} - a_{2k,j} a_{2k-1,x}) T_m^{j-x} \right)$$
$$= 0$$

and from these we have that

(i) if m odd, then the coefficients of T_m^{σ}, $\sigma = -(m-1), \ldots, m-1$ are zero.That means

$$\sum^{m-1} j, x = 0j - x = \sigma \sum_{k=1}^{v} (a_{2k-1,j} a_{2k,x} - a_{2k,j} a_{2k-1,x}) = 0 \tag{13}$$

(ii) if m is even, $m = 2n$ then the coefficients of T_m^{σ}, $\sigma = -(2n-1), \ldots, -(n+1), -(n-1), \ldots, n-1, n+1, \ldots, 2n-1$ are zero and the coefficient of T_m^n plus the the coefficient of T_m^{-n} is zero. That means

$$\sum_{\substack{j, x = 0\\ j - x = \sigma\\ \sigma \neq \pm n}}^{m-1} \sum_{k=1}^{v} (a_{2k-1,j}a_{2k,x} - a_{2k,j}a_{2k-1,x}) = 0 \text{and} \sum^{m-1} j, x = 0 j - x = \pm n \sum_{k=1}^{v} (a_{2k-1,j}a_{2k,x} - a_{2k,j}a$$

(14)

Now

$$\sum_{k=1}^{v} \left(B_{2k-1}B_{2k}^T - B_{2k}B_{2k-1}^T \right) = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{v} \left((a_{2k-1,j}a_{2k,x} - a_{2k,j}a_{2k-1,x}) S_\ell^{j-x} \right)$$

We have that the coefficients of S_ℓ^{σ} are equal to the coefficients of T_m^{σ} for all $\sigma = -(m-1), \ldots, m-1$ and so using equations (13) or equations (2) we obtain

$$\sum_{k=1}^{v} \left(B_{2k-1}B_{2k}^T - B_{2k}B_{2k-1}^T \right) = 0 \tag{15}$$

Equations (12) and (15) show that $\{B_k\}_{k=1}^{2v}$ is an amicable set of matrices (sequences) of length $\ell \equiv 0 (mod\ m)$, $\ell = mi$, $i = 1, 2, \ldots$ and type $(u_1, u_2, \ldots, u_p)$. □

Corollary 1 *Let $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2$ where $a_{k,j} \in \{0, \pm x_1, \pm x_2\}$, $j = 0, 1, \ldots, m-1$ and $k = 1, 2$ be a set of two amicable sequences of length m and type (u_1, u_2). Then there exist a set of two amicable sequences of length $\ell \equiv\ 0\ (\ mod\ m) = mi$ and type (u_1, u_2).*

Proof. Use Theorem 1 with $2v = 2$ and $p = 2$.

Example 1 We have that $A_1 = 0T_4^0 + aT_4^1 + bT_4^2 - aT_4^3$ and $A_2 = 0T_4^0 + aT_4^1 + 0T_4^2 + aT_4^3$ is a set of two amicable matrices (sequences) of length $m = 4$ and type $(1, 4)$. Corollary 1 gives a set of two amicable sequences of length $m = 4i$ and type $(1, 4)$ for all $i = 1, 2, \ldots$.

Corollary 2 *Let $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, 3, 4$ where $a_{k,j} \in \{0, \pm x_1, \pm x_2, \pm x_3,\ \pm x_4\}$, $j = 0, 1, \ldots, m-1$ and $k = 1, 2, 3, 4$ be a set of four amicable sequences of length m and type (u_1, u_2, u_3, u_4). Then there exist a set of four amicable sequences of length $\ell \equiv\ 0\ (\ mod\ m) = mi$ and type (u_1, u_2, u_3, u_4).*

Proof. Use Theorem 1 with $2v = 4$ and $p = 4$.

Example 2 We have that $A_1 = aT_3^0 - bT_3^1 + aT_3^2$, $A_2 = bT_3^0 + aT_3^1 + bT_3^2$ and $A_3 = aT_3^0 + aT_3^1 - aT_3^2$, $A_4 = bT_3^0 + bT_3^1 + bT_3^2$ is a set of four amicable matrices (sequences) of length $m = 3$ and type $(6, 6)$. Corollary 2 gives a set of four amicable sequences of length $m = 3i$ and type $(6, 6)$ for all $i = 1, 2, \ldots$.

Corollary 3 *Let* $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, \ldots, 8$ *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_8\}$, $j = 0, 1, \ldots, m-1$ *and* $k = 1, 2, \ldots, 8$ *be a set of eight amicable sequences of length m and type* $(u_1, u_2, \ldots, u_8)$. *Then there exist a set of eight amicable sequences of length* $\ell \equiv 0 \ (mod\ m) = mi$ *and type* $(u_1, u_2, \ldots, u_8)$.

Proof. Use Theorem 1 with $2v = 8$ and $p = 8$.

Example 3 We have that $A_1 = -aT_7^0 + aT_7^1 + aT_7^2 + gT_7^3 + aT_7^4 + eT_7^5 + cT_7^6$, $A_2 = -fT_7^0 + fT_7^1 + fT_7^2 - hT_7^3 + fT_7^4 + bT_7^5 - dT_7^6$, $A_3 = -gT_7^0 + gT_7^1 + gT_7^2 - aT_7^3 + gT_7^4 + cT_7^5 - eT_7^6$, $A_4 = -hT_7^0 + hT_7^1 + hT_7^2 + fT_7^3 + hT_7^4 + dT_7^5 + bT_7^6$, $A_5 = -eT_7^0 + eT_7^1 + eT_7^2 - cT_7^3 + eT_7^4 - aT_7^5 + gT_7^6$, $A_6 = -dT_7^0 + dT_7^1 + dT_7^2 - bT_7^3 + dT_7^4 - hT_7^5 + fT_7^6$, $A_7 = -bT_7^0 + bT_7^1 + bT_7^2 + dT_7^3 + bT_7^4 - fT_7^5 - hT_7^6$ and $A_8 = -cT_7^0 + cT_7^1 + cT_7^2 + eT_7^3 + cT_7^4 - gT_7^5 - aT_7^6$ is a set of eight amicable matrices (sequences) of length $m = 7$ and type $(7, 7, 7, 7, 7, 7, 7, 7)$. Corollary 3 gives a set of eight amicable sequences of length $m = 7i$ and type $(7, 7, 7, 7, 7, 7, 7, 7)$ for all $i = 1, 2, \ldots$.

Remark 1 Using Corollaries 1, 2 and 3 as indicated by the examples and using array (1), (2) or (7) and (6) respectively we obtain many infinite classes of orthogonal designs.

3 Construction of amicable sets of sequences from non amicable sets of sequences

Lemma 1 *Let* $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, \ldots, v_1$, *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_p\}$, $j = 0, 1, \ldots, m-1$ *and* $k = 1, 2, \ldots, v_1$ *be a set of* v_1 *amicable sequences of length* m *and type* $(u_1, u_2, \ldots, u_p)$ *and* $B_r = \{b_{r,0}, b_{r,1}, \ldots, b_{r,m-1}\}$, $r = 1, 2, \ldots, v_2$, *where* $b_{r,s} \in \{0, \pm y_1, \pm y_2, \ldots, \pm y_q\}$, $s = 0, 1, \ldots, m-1$ *and* $r = 1, 2, \ldots, v_2$ *be a set of* v_2 *amicable sequences of length* m *and type* $(t_1, t_2, \ldots, t_q)$.
Then there exist a set of $v_1 + v_2$ *amicable sequences of length* m *and type* $(u_1, u_2, \ldots, u_p, t_1, t_2, \ldots, t_q)$.

Proof. These are the sequences A_k, $k = 1, 2, \ldots, v_1$ and B_k, $k = 1, 2, \ldots, v_2$ together. □

Corollary 4 *Let* $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m_1-1}\}$, $k = 1, 2, \ldots, v_1$, *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_p\}$, $j = 0, 1, \ldots, m_1 - 1$ *and* $k = 1, 2, \ldots, v_1$ *be a set of* v_1 *amicable sequences of length* m_1 *and type* $(u_1, u_2, \ldots, u_p)$ *and* $B_r = \{b_{r,0}, b_{r,1}, \ldots, b_{r,m_2-1}\}$, $r = 1, 2, \ldots, v_2$, *where* $b_{r,s} \in \{0, \pm y_1, \pm y_2, \ldots, \pm y_q\}$, $s = 0, 1, \ldots, m_2 - 1$ *and* $r = 1, 2, \ldots, v_2$ *be a set of* v_2 *amicable sequences of length* m_2 *and type* $(t_1, t_2, \ldots, t_q)$.
Then there exist a set of $v_1 + v_2$ *amicable sequences of length* $\ell \cdot i$ *where* $\ell = [m_1, m_2]$ *is the least common multiple (l.c.m.) of* m_1 *and* m_2 *and type* $(u_1, u_2, \ldots, u_p, t_1, t_2, \ldots, t_q)$.

Proof. Since ℓ is the least common multiple of m_1 and m_2 then $\ell = m_1 \cdot i_1 = m_2 \cdot i_2$. Using theorem 1 we can construct a set of v_1 amicable sequences of length ℓ and type $(u_1, u_2, \ldots, u_p)$ and a set of v_2 amicable sequences of length ℓ and type $(t_1, t_2, \ldots, t_q)$. Now using Lemma 1 we obtain a set of $v_1 + v_2$ amicable sequences of length ℓ and type $(u_1, u_2, \ldots, u_p, t_1, t_2, \ldots, t_q)$.

Using theorem 1 again in the derived sequences we have the result. □

Example 4 We have that $A_1 = \{e, f\}$, $A_2 = \{e, -f\}$, $A_3 = \{e, 0\}$, $A_4 = \{f, 0\}$ is a short amicable set of length 2 and type $(3, 3)$. We also have that $A_1 = \{a, a, b, -b\}$, $A_2 = \{c, c, d, -d\}$, $A_3 = \{d, d, -c, c\}$, $A_4 = \{b, b, -a, a\}$ is a short amicable set of length 4 and type $(4, 4, 4, 4)$. Now $\ell = [4, 2] = 4$ and thus from corollary 4 we obtain eight amicable sequences of length $\ell \cdot i$ and type $(3, 3, 4, 4, 4, 4)$ for all $i = 1, 2, \ldots$.

Theorem 2 (Doubling the number of sequences) *Let* $A_k = \{a_{k,0}, a_{k,1}, \ldots, a_{k,m-1}\}$, $k = 1, 2, \ldots v$ *where* $a_{k,j} \in \{0, \pm x_1, \pm x_2, \ldots, \pm x_p\}$, $j = 0, 1, \ldots, m-1$ *and* $k = 1, 2, \ldots, v$ *be* v *sequences with PAF=0 (or NPAF=0) of length* m *and type* $(u_1, u_2, \ldots, u_p)$. *Then there exist a set of* $2v$ *amicable sequences of length* m *and type* $(2u_1, 2u_2, \ldots, 2u_p)$ *with PAF=0 (or NPAF=0).*

Proof. Set $B_{2k-1} = B_{2k} = circ(A_k),\ k = 1, 2, \ldots, v$. Then

$$\sum_{k=1}^{2v} B_k B_k^T = 2 \cdot \sum_{k=1}^{v} A_k A_k^T = \left(\sum_{i=1}^{p} 2u_i x_i^2 \right) I_m$$

and

$$B_{2k-1} B_{2k}^T - B_{2k} B_{2k-1}^T = A_k A_k^T - A_k A_k^T = 0,\ k = 1, 2, \ldots, v.$$

Thus $\{B_k\}_{k=1}^{2v}$ is a set of $2v$ amicable matrices (sequences) of length m and type $(2u_1, 2u_2, \ldots, 2u_p)$. □

4 More Constructions

Theorem 3 *Let $(X_k, Y_k),\ k = 1, 2, \ldots, v$ be v pairs of sequences of lengths m_k with the properties*

$$Z_k Z_k^T + W_k W_k^T = p_k I_{m_k} \tag{16}$$

$$Z_k W_k^T - W_k Z_k^T = 0 \tag{17}$$

$$Z_k * W_k = 0 \tag{18}$$

for all $k = 1, 2, \ldots, v$, where $Z_k = circ(X_k)$ and $W_k = circ(Y_k)$. Then there exist a set of $2v$ amicable sequences of length $\ell \equiv 0 \mod [m_1, m_2, \ldots, m_v]$, where $[m_1, m_2, \ldots, m_v]$ is the least common multiple (l.c.m.) of $m_1, m_2, \ldots, m_v$ and of type $(p_1, p_1, p_2, p_2, \ldots, p_v, p_v)$ on the set $\{a_1, a_2, \ldots, a_{2v}\}$ of commuting variables.

Proof. Set

$$B_k = a_{2k} X_k + a_{2k-1} Y_k, \quad \text{and} \quad C_k = -a_{2k-1} X_k + a_{2k} Y_k, \quad k = 1, 2, \ldots, v$$

Condition (18) gives that $B_k,\ \ k = 1, 2, \ldots, v$ and $C_k,\ k = 1, 2, \ldots, v$ are sequences of lengths $m_k,\ k = 1, 2, \ldots, v$ and type $(p_1, p_1, p_2, p_2, \ldots, p_v, p_v)$.

For any k and by simple calculations using conditions (16) and (17) we have that

$$B_k B_k^T + C_k C_k^T = (p_k a_{2k-1}^2 + p_k a_{2k}^2) I_{m_k} \quad \text{and} \quad B_k C_k^T - C_k B_k^T = 0$$

Now from theorem 1, there are sequences D_k and E_k of length $\ell \equiv 0 \mod [m_1, m_2, \ldots, m_v],\ k = 1, 2, \ldots, v$, with the desirable properties. By lemma 1 we have the result. □

Example 5 Set $Z_1 = \{1\}$, $W_1 = \{0\}$, $Z_2 = \{1, 0\}$, $W_2 = \{0, 1\}$, $Z_3 = \{1, 1, 1, -1\}$, $W_3 = \{0, 0, 0, 0\}$, $Z_4 = \{0, 1, 0, -1, 0, 1\}$ and $W_4 = \{0, 0, 1, 0, 1, 0\}$. These are four pair of sequences of lengths $1, 2, 4$ and 6 satisfying conditions (16), (17) and (18) with $p_1 = 1$, $p_2 = 2$, $p_3 = 4$ and $p_4 = 5$. We have that $[1, 2, 4, 6] = 12$ and from theorem 3 we obtain eight sequences of length $\ell \equiv 0$ $(mod\ 12)$ and of type $(1, 1, 2, 2, 4, 4, 5, 5)$ on the set $\{a_1, a_2, \ldots, a_8\}$ of commuting variables which can be used in the Kharaghani array (6) to obtain an infinite class of Kharaghani type orthogonal designs $OD(8\ell; 1, 1, 2, 2, 4, 4, 5, 5)$.

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