Orthogonal Designs of Kharaghani Type: I

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Abstract

We use an array given in H. Kharaghani, Arrays for orthogonal designs, *J. Combin. Designs*, 8 (2000), 166-173, to obtain infinite families of 8-variable Kharaghani type orthogonal designs, $OD(8t; k_1, k_1, k_1, k_1, k_2, k_2, k_2, k_2)$, where k_1 and k_2 must be the sum of two squares. In particular we obtain infinite families of 8-variable Kharaghani type orthogonal designs, $OD(8t; k, k, k, k, k, k, k, k)$. For odd t orthogonal designs of order $\equiv 8(\bmod 16)$ can have at most eight variables.

Key words and phrases: Sequences, zero autocorrelation, orthogonal designs, amicable set of matrices, Kharaghani array, Kharaghani type orthogonal designs.

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1 Preliminaries

An *orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$ denoted $OD(n; s_1, s_2, \ldots, s_u)$ in the variables $x_1, x_2, \ldots, x_u$, is a matrix A of order n with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$AA^T = \sum_{i=1}^{u} (s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n. Let B_i, $i = 1, 2, 3, 4$ be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$\sum_{i=1}^{4} B_i B_i^T = \sum_{i=1}^{u} (s_i x_i^2) I_n.$$

Then the Goethals-Seidel array

$$G = \begin{pmatrix} B_1 & B_2R & B_3R & B_4R \\ -B_2R & B_1 & B_4^TR & -B_3^TR \\ -B_3R & -B_4^TR & B_1 & B_2^TR \\ -B_4R & B_3^TR & -B_2^TR & B_1 \end{pmatrix}$$

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where R is the back-diagonal identity matrix, is an $OD(4n; s_1, s_2, \ldots, s_u)$. See page 107 of [1] for details.

Plotkin [5] showed that there is an Hadamard matrix of order $2t$, then there is an $OD(8t; t, t, t, t, t, t, t, t)$. In the same paper he also constructed an $OD(24; 3, 3, 3, 3, 3, 3, 3, 3)$. This OD has appeared in [1], [6] and in [7]. It is conjectured that there is an $OD(8n; n, n, n, n, n, n, n, n)$ for each odd integer n. Until recently, none except the original for $n = 3$ found by Plotkin, had been constructed in the ensuing twenty eight years. Holzmann and Kharaghani [2] using a new method constructed many new Plotkin ODs of order 24 and two new Plotkin ODs of order 40 and 56. Actually their construction provides many new orthogonal designs in 6, 7 and 8 variables which include the Plotkin ODs of order 40 and 56.

A pair of matrices A, B is said to be amicable (anti-amicable) if $AB^T - BA^T = 0$ ($AB^T + BA^T = 0$). Following [3] a set $\{A_1, A_2, \ldots, A_{2n}\}$ of square real matrices is said to be *amicable* if

$$\sum_{i=1}^{n} \left(A_{\sigma(2i-1)} A^T_{\sigma(2i)} - A_{\sigma(2i)} A^T_{\sigma(2i-1)} \right) = 0 \tag{1}$$

for some permutation σ of the set $\{1, 2, \ldots, 2n\}$. For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. So

$$\sum_{i=1}^{n} \left(A_{2i-1} A^T_{2i} - A_{2i} A^T_{2i-1} \right) = 0. \tag{2}$$

Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. Throughout the paper R_k denotes the back diagonal identity matrix of order k.

An amicable set of matrices $\{B_1, B_2, \ldots, B_n\}$ of order m with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ is said to be *amicable plug-in*, $AP(m; s_1, s_2, \ldots, s_u)$, in the variables $x_1, x_2, \ldots, x_u$ if it satisfies an additive property

$$\sum_{i=1}^{n} B_i B_i^T = \sum_{i=1}^{u} (s_i x_i^2) I_m. \tag{3}$$

Let $\{A_i\}_{i=1}^{8}$ be an amicable plug-in set of circulant matrices of type $(s_1, s_2, \ldots, s_u)$ of order t. Then the Kharaghani array

$$H = \begin{pmatrix}
A_1 & A_2 & A_4R_n & A_3R_n & A_6R_n & A_5R_n & A_8R_n & A_7R_n \\
-A_2 & A_1 & A_3R_n & -A_4R_n & A_5R_n & -A_6R_n & A_7R_n & -A_8R_n \\
-A_4R_n & -A_3R_n & A_1 & A_2 & -A_8^TR_n & A_7^TR_n & A_6^TR_n & -A_5^TR_n \\
-A_3R_n & A_4R_n & -A_2 & A_1 & A_7^TR_n & A_8^TR_n & -A_5^TR_n & -A_6^TR_n \\
-A_6R_n & -A_5R_n & A_8^TR_n & -A_7^TR_n & A_1 & A_2 & -A_4^TR_n & A_3^TR_n \\
-A_5R_n & A_6R_n & -A_7^TR_n & -A_8^TR_n & -A_2 & A_1 & A_3^TR_n & A_4^TR_n \\
-A_8R_n & -A_7R_n & -A_6^TR_n & A_5^TR_n & A_4^TR_n & -A_3^TR_n & A_1 & A_2 \\
-A_7R_n & A_8R_n & A_5^TR_n & A_6^TR_n & -A_3^TR_n & -A_4^TR_n & -A_2 & A_1
\end{pmatrix}$$

is a Kharaghani type orthogonal design $OD(8m; s_1, s_2, \ldots, s_u)$.

We use the construction of Holzmann and Kharaghani [2] for an $OD(56; 7, 7, 7, 7, 7, 7, 7, 7)$ to find an infinite family of 8-variable Kharaghani type orthogonal designs $OD(8t; k_1, k_1, k_1, k_1, k_2, k_2, k_2, k_2)$, and $OD(8t; k, k, k, k, k, k, k, k)$, where k_1, k_2 and k must be the sum of two squares.

Given a set of ℓ sequences $A_j = \{a_{j1}, a_{j2}, \ldots, a_{jn}\}$, $j = 1, \ldots, \ell$, of length n the *non-periodic autocorrelation function*, denoted $NPAF$, $N_A(s)$ is defined as

$$N_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n-s} a_{ji} a_{j,i+s}, \quad s = 0, 1, \ldots, n-1, \tag{4}$$

If $A_j(z) = a_{j1} + a_{j2}z + \ldots + a_{jn}z^{n-1}$ is the associated polynomial of the sequence A_j, then

$$A(z)A(z^{-1}) = \sum_{j=1}^{\ell} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ji} a_{jk} z^{i-k} = N_A(0) + \sum_{j=1}^{\ell} \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}), z \neq 0. \tag{5}$$

Given A_ℓ, as above, of length n the *periodic autocorrelation function*, denoted PAF, $P_A(s)$ is defined, reducing $i + s$ modulo n, as

$$P_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n} a_{ji} a_{j,i+s}, \quad s = 0, 1, \ldots, n-1. \tag{6}$$

We note NPAF sequences imply PAF sequences exist, the NPAF sequences being padded at the end with sufficient zeros to make longer lengths. Hence NPAF sequences can give more general results. If two NPAF sequences have differing lengths then sufficient zeros are added to the end of each to make all the sequences the same length. In all cases NPAF and PAF sequences can be used to make circulant matrices satisfying the additive property (see [2, 3]); if NPAF sequences of lengths n_1 and n_2 are used, then by padding, circulant matrices for all orders $n \geq max(n_1, n_2)$ will exist; if PAF sequences of lengths n are used, then circulant matrices of order n exist.

2 Using sequences with zero NPAF to make ODs

We now consider the use of sequences with zero non-periodic autocorrelation function to make an amicable set of eight matrices. We refer the reader to [6, 7] for any undefined terms.

Theorem 1 *Let X_i, Y_i be two pairs of disjoint $(0, \pm 1)$ sequences with zero non-periodic autocorrelation function of length n_i and weight k_i, $i = 1, 2$. We pad the end of these sequences with zeros to obtain sequences of length $n \geq max(n_1, n_2)$. Let a, b, c, d, e, f, g, h be commuting variables and write aV_i, bW_i for the circulant (type 1 or group generated also suffice) matrices of order n formed by using the first rows with the elements of X_i multiplied by a and the elements of Y_i multiplied by b respectively.*

Let A_i be the circulant matrices of order n given by

$$\begin{array}{llll} A_1 = aV_1 + bW_1 & A_3 = dV_1 - cW_1 & A_5 = eV_2 + fW_2 & A_7 = hV_2 - gW_2 \\ A_2 = cV_1 + dW_1 & A_4 = bV_1 - aW_1 & A_6 = gV_2 + hW_2 & A_8 = fV_2 - eW_2 \end{array} \tag{7}$$

then $\{A_i\}_{i=1}^{8}$ *is an amicable plug-in set satisfying*

$$\sum_{i=1}^{4}\left(A_{2i-1}A_{2i}^T - A_{2i}A_{2i-1}^T\right) = 0, \tag{8}$$

and the additive property

$$\sum_{i=1}^{8}\left(A_iA_i^T\right) = (k_1(a^2+b^2+c^2+d^2)+k_2(e^2+f^2+g^2+h^2))I_n. \tag{9}$$

Proof: Now $A_1 = aV_1 + bW_1$, where $V_1,\ W_1$ are disjoint $(0,\pm1)$ circulant (type 1) matrices of order n which satisfy $V_1V_1^T + W_1W_1^T = k_1I_n$, and similarly for the other A_j, $j = 2, \cdots 8$.

Then

$$A_1A_1^T = (aV_1+bW_1)(aV_1^T+bW_1^T) = a^2V_1V_1^T + b^2W_1W_1^T + ab(V_1W_1^T + W_1V_1^T).$$

Hence

$$\begin{aligned} \textstyle\sum_{i=1}^{4}\left(A_iA_i^T\right) &= (a^2+b^2+c^2+d^2)(V_1V_1^T + W_1W_1^T) \\ &= k_1(a^2+b^2+c^2+d^2)I_n. \end{aligned}$$

Similarly

$$\begin{aligned} \textstyle\sum_{i=5}^{8}\left(A_iA_i^T\right) &= (e^2+f^2+g^2+h^2)(V_2V_2^T + W_2W_2^T) \\ &= k_2(e^2+f^2+g^2+h^2)I_n. \end{aligned}$$

Now

$$\begin{aligned} A_1A_2^T - A_2A_1^T &= (aV_1+bW_1)(cV_1^T+dW_1^T) - (bV_1-aW_1)(aV_1^T+bW_1^T) \\ &= (ad-bc)V_1W_1^T + (-ad+bc)W_1V_1^T. \end{aligned}$$

We also see that

$$\begin{aligned} A_3A_4^T - A_4A_3^T &= (dV_1-cW_1)(bV_1^T-aW_1^T) - (bV_1-aW_1)(dV_1^T-cW_1^T) \\ &= (-ad+bc)V_1W_1^T + (ad-bc)W_1V_1^T. \end{aligned}$$

Thus summing over the eight A_i we obtain equations (8) and (9). □

Corollary 1 *Let* $X_i,\ Y_i$ *be two disjoint* $(0,\pm1)$ *sequences with zero non-periodic autocorrelation function of length* n_i *and weight* k_i, $i = 1, 2$ *and* $n \geq max(n_1, n_2)$. *Then there exists a Kharaghani type orthogonal design* $OD(8s; k_1, k_1, k_1, k_1, k_2, k_2, k_2, k_2)$, $s \geq n$.

Proof: Use the sequences as in the theorem to form an amicable plug-in set with the additive property. Then use this set in the Kharaghani array to obtain the required Kharaghani type orthogonal design. □

Example 1 We use the sequences of length $n \geq 6 = max(6,4)$ and weights 5 and 4, $X_1 = \{1,0,1,0,0,0\}$ and $Y_1 = \{0,0,0,1,1,-1\}$, $X_2 = \{1,1,0,0\}$ and $Y_2 = \{0,0,1,-1\}$, and the sequence 0_s of s zeros, to form the circulant matrices with first rows

$$\begin{array}{llllllllllllllll} A_1 = & (a & 0 & a & b & b & -b & 0_s) & A_3 = & (d & 0 & d & -c & -c & c & 0_s) \\ A_2 = & (c & 0 & c & d & d & -d & 0_s) & A_4 = & (b & 0 & b & -a & -a & a & 0_s) \\ A_5 = & (e & e & f & -f & 0 & 0 & 0_s) & A_7 = & (h & h & -g & g & 0 & 0 & 0_s) \\ A_6 = & (g & g & h & -h & 0 & 0 & 0_s) & A_8 = & (f & f & -e & e & 0 & 0 & 0_s). \end{array}$$

By the theorem these form an amicable plug-in set of eight matrices of order $s+6$ and weights 5 and 4 which can be used in the Kharaghani array to give a Kharaghani type orthogonal design $OD(8s+48; 4,4,4,4,5,5,5,5)$ for every order $s \geq 0$. □

Let P, Q be two $(0,\pm1)$ sequences with zero non-periodic autocorrelation function of length n and weight k. Then the sequences $X = \{P, 0_n\}$ and $Y = \{0_n, Q\}$ are two $(0,\pm1)$ disjoint sequences with zero non-periodic autocorrelation function of length $2n$ and weight k.

Let $\alpha,\beta,\gamma,\delta,\epsilon,\phi,\psi,\ \mu,\nu$ non-negative integers. Koukouvinos and Seberry [4, p. 160] show that we have two $(0,\pm1)$ disjoint sequences with zero non-periodic autocorrelation function of lengths $\geq n$, $n \in N = \{2\times2^\alpha 6^\beta 10^\gamma 9^\delta 14^\epsilon 18^\phi 26^\psi 24^\mu 34^\nu\}$ and weights $2^\alpha 5^\beta 10^\gamma 13^\delta 17^\epsilon 25^\phi 26^\psi 34^\mu 50^\nu$.

Corollary 2 *Let $Z = \{z_1, z_2, \ldots, z_n\}$, $W = \{w_1, w_2, \ldots, w_n\}$ be two disjoint $(0,\pm1)$ sequences with zero periodic autocorrelation function of length n and weight k. Then there exists a Kharaghani type $OD(8s; k,k,k,k,k,k,k,k)$ for all $s = mn$, $m = 1,2,\ldots$.*

Proof: Set $X = \{z_1, 0_{m-1}, z_2, 0_{m-1}, \ldots, z_n, 0_{m-1}\}$ and $Y = \{0_{m-1}, w_1, 0_{m-1}, w_2, \ldots, 0_{m-1}, w_n\}$. These are two disjoint sequences of length $s = mn$ and weight k and can be used in Theorem 1 to form an amicable set of eight matrices with the additive property. Then we can use these eight matrices in the Kharaghani array to obtain the required orthogonal design of Kharaghani type. □

We give some examples of the Kharaghani type orthogonal designs we obtain in the Table:

OD	Lengths n	OD	Lengths n
$OD(8n;1,1,1,1,1,1,1,1)$	≥ 1	$OD(8n;1,1,1,1,2,2,2,2)$	≥ 2
$OD(8n;1,1,1,1,4,4,4,4)$	≥ 4	$OD(8n;1,1,1,1,5,5,5,5)$	≥ 6
$OD(8n;2,2,2,2,4,4,4,4)$	≥ 4	$OD(8n;2,2,2,2,5,5,5,5)$	≥ 6
$OD(8n;4,4,4,4,5,5,5,5)$	≥ 6	$OD(8n;4,4,4,4,8,8,8,8)$	≥ 8
$OD(8n;5,5,5,5,5,5,5,5)$	≥ 6	$OD(8n;5,5,5,5,8,8,8,8)$	≥ 8
$OD(8n;5,5,5,5,10,10,10,10)$	≥ 10	$OD(8n;5,5,5,5,13,13,13,13)$	≥ 18
$OD(8n;5,5,5,5,17,17,17,17)$	≥ 26	$OD(8n;13,13,13,13,17,17,17,17)$	≥ 26
$OD(8n;20,20,20,20,25,25,25,25)$	≥ 36	$OD(8n;25,25,25,25,25,25,25,25)$	≥ 36

3 Using sequences with zero PAF to make ODs

Provided the sequences used in the theorem all have the same length the corollary can be extended to include sequences with zero PAF.

Corollary 3 *Let X_i, Y_i be two pairs of disjoint $(0, \pm 1)$ sequences with zero periodic or non-periodic autocorrelation function of length n and weight $k_i, i = 1, 2$. Then there exists a Kharaghani type orthogonal design $OD(8s; k_1, k_1, k_1, k_1, k_2, k_2, k_2, k_2)$, $s \geq n$.*

Example 2 We use the sequences of length $n = 11$, and weights 4 and 9,

$$X_1 = \text{circ}\{1,1,0,0,0,0,0,0,0,0,0\} \text{ and } Y_1 = \text{circ}\{0,0,1,-,0,0,0,0,0,0,0\}$$
$$X_2 = \text{circ}\{0,1,0,1,1,0,0,1,0,-,0\} \text{ and } Y_2 = \text{circ}\{0,0,0,0,0,1,-,0,-,0,1\}$$

to form the circulant matrices with first rows which can be used in the theorem to give an amicable plug-in set of matrices of order 11 and weights 4 and 9 which can be used in the Kharaghani array to obtain Kharaghani type orthogonal designs $OD(88; 4,4,4,4,4,4,4,4)$, $OD(88; 4,4,4,4,9,9,9,9)$ and $OD(88; 9,9,9,9,9,9,9,9)$.

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