

Regular Hadamard Matrices, Maximum Excess and SBIBD *

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Abstract

When $k = q_1, q_2, q_1 q_2, q_1 q_4, q_2 q_3 N, q_3 q_4 N, q_1, q_2$ and q_3 are prime power, where $q_1 \equiv 1 \pmod{4}$, $q_2 \equiv 3 \pmod{8}$, $q_3 \equiv 5 \pmod{8}$, $q_4 = 7$ or 23 , $N = 2^a 3^b t^2$, $a, b = 0$ or 1 , $t \neq 0$ is an arbitrary integer, we prove that there exist regular Hadamard matrices of order $4k^2$, and also there exist $SBIBD(4k^2, 2k^2 + k, k^2 + k)$. We find new $SBIBD(4k^2, 2k^2 + k, k^2 + k)$ for 233 values of k .

1 Preliminaries

An $n \times n$ matrix H is called Hadamard matrix (or H-matrix) if every entry of the matrix is 1 or -1 , and

$$HH^T = nI_n,$$

where I_n is an $n \times n$ identity matrix. In this paper we use H^T to denote the transpose of a matrix H .

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We denote the excess of a H -matrix $H = [a_{ij}]$ by $\sigma(H)$, where

$$\sigma(H) = \sum_{1 \leq i, j \leq n} a_{ij}.$$

Let $\sigma(n) = \max\{\sigma(H)\}$. The weight of a H -matrix H , denoted by $W(H)$, is the number of ones in the H . We define $W(n) = \max\{W(H)\}$. Note that the maximums are taken over all $n \times n$ H -matrices H . It is obvious that $\sigma(H) = 2W(H) - n^2$ and $\sigma(n) = 2W(n) - n^2$ (see [4], [5], [6], [7] for details).

M. R. Best [1] proved that

$$\sigma(n) \leq n\sqrt{n} \tag{1}$$

Definition 1 (*Regular Hadamard Matrix*) A regular Hadamard matrix has the sum of each column of the matrix and the sum of each row of the matrix constant.

Definition 2 (*SBIBD*) A symmetric balanced incomplete block design, called a $SBIBD(v, k, \lambda)$, is defined by a $v \times v$ matrix M , which has every entry 0 or 1. The sum of each column and the sum of each row of the matrix is k . For any two columns c_i, c_j (and two rows r_i, r_j), $1 \leq i \neq j \leq v$, the inner product of c_i and c_j (r_i and r_j) is λ (see [10]).

With the result of this paper and those of [4], [9] the status of the existence of $4k^2$ -Hadamard matrix and $SBIBD(4k^2, 2k^2 + k, k^2 + k)$ becomes that they exist for $k \in \{1, 3, 5, \dots, 45, 49, \dots, 69, 73, 75, 81, \dots, 101, 105, 107, 109, \dots, 125, 129, 131, 135, 137, 139, 143, \dots, 149, 153, \dots, 165, 169, \dots, 175, \dots, 189, 193, \dots, 197, 201, \dots, 207, 211, 215, 219, 221, 225, 227, 229, 233, 235, 241, \dots, 251, 257, 259, 261, 267, 269, 273, 275, 277, 281, \dots, 299, 303, 307, 313, \dots, 327, 331, \dots, 339, 343, \dots, 353, 361, 363, 371, 373, 375, 379, 387, 389, 391, 393, 397, 401, 405, \dots, 411, 415, 417, 419, 421, 427, 429, 433, 441, 443, 447, 449, 451, 457, 461, 467, 471, 475, 477, 489, 491, 495, 499, 507, 509, 511, 513, 519, 521, 523, 525, 529, 531, \dots, 543, 547, 549, 551, 557, 559, 563, 567, 569, 571, 575, 577, 579, 583, 587, 591, 593, 601, 603, 605, 609, 613, 617, \dots, 625, 633, 637, 641, 643, 645, 653, 655, 659, 661, 667, 671, 673, 675, 677, 679, 683, 687, 691, 695, 699, 701, 703, 707, 709, 723, 725, 729, 731, 733, 735, 739, 741, 747, 753, 757, 761, 763, 767, 769, 771, 773, 777, 779, 783, 787, 791, 797, 803, 807, 809, 811, 815, 819, 821, 827, 829, 831, 841, \dots, 859, 865, 867, 871, 875, 877, 879, 881, 883, 885, 891, 895, 897, 907, 909, 921, 925, 929, 931, 937, 939, 941, \dots, 947, 951, 953, 957, 959, 961, 963, 971, 975, 977, 979, 981, 993, 997, 999, q_1, q_2, q_1q_2, q_1q_4, q_2q_3N, q_3q_4N\}$, where q_1, q_2 and q_3 are prime power, $q_1 \equiv 1 \pmod{4}$, $q_2 \equiv 3 \pmod{8}$, $q_3 \equiv 5 \pmod{8}$, $q_4 = 7$ or 23 , $N = 2^a 3^b t^2$, $a, b = 0$ or 1 , $t \neq 0$ is an arbitrary integer, $r \geq 0$. This means we find 233 new values < 1000 .

Let G be an *Abelian* group with the addition \oplus and the subtraction \ominus . We denote by θ be the zero element in G . Consider the polynomials in the elements

of G over the field of rational number, $\sum_{g \in G} a(g)g$, where the integer $a(g)$ is the number of occurrences of g , and define the addition by

$$\sum_{g \in G} a(g)g + \sum_{g \in G} b(g)g = \sum_{g \in G} (a(g) + b(g))g.$$

We denote $\sum_{g \in A} g$ as A , $G = \sum_{g \in G} g$ and $G^* = G - \theta$. For any two subsets $A, B \subset G$, we define

$$A \ominus B = \sum_{a \in A, b \in B} (a \ominus b), \quad \Delta A = A \ominus A,$$

$$\Delta(A, B) = (A \ominus B) + (B \ominus A).$$

It is obviously that $\Delta(A, A) = 2\Delta A$. We define $\Delta\phi = 0$, $\Delta(\phi, A) = 0$ for any $A \subset G$.

Definition 3 (DS) Let $D = \{a_1, \dots, a_k\}$ be a subset of a group G of order v . If for every non-zero element $g \in G$ there are λ pairs (a_i, a_j) , $a_i, a_j \in D$, such that

$$a_i \oplus a_j = g,$$

we call D a (v, k, λ) -difference set (DS).

Definition 4 (Incidence matrix) The incidence matrix $A = (a_{ij})$ of a (v, k, λ) -difference set D is defined by ordering the elements of the group $G = \{g_i\}$, $i = 1, \dots, v$. and defining

$$a_{ij} = \begin{cases} 1, & g_j \ominus g_i \in D, \\ 0, & \text{otherwise} \end{cases}$$

Definition 5 (SDS) Let $D_i \subset G$, $|D_i| = k_i$, $i = 1, \dots, r$. If

$$\sum_{i=1}^r \Delta D_i = \left(\sum_{i=1}^r k_i - \lambda \right) \theta + \lambda G,$$

$\lambda \geq 0$, then D_1, \dots, D_r are $r - \{v; k_1, \dots, k_r; \lambda\}$ supplementary difference sets (SDS), where $v = |G|$.

If $k_1 = \dots = k_r = k$, we simplify D_1, \dots, D_r to $r - \{v; k; \lambda\}$ SDS. When $r = 1$, the SDS become the difference set (DS).

We only consider $r = 4$. Then we define $\lambda = \sum_{i=1}^4 k_i - v$ in this paper. In this case, we call D_1, D_2, D_3, D_4 type H -SDS.

Definition 6 (Type H_1) Let $D_1, D_2, D_3, D_4 \subset G$ be SDS of order v . $|D_i| = k_i$, $i = 1, 2, 3, 4$. $D_1, D_2, D_3, D_4 \in H_1$ iff

$$\sum_{i=1}^4 \Delta D_i = v\theta + \lambda G,$$

and

$$\Delta(D_1, D_2) + \Delta(D_3, D_4) = \lambda G,$$

where $\lambda = k_1 + k_2 + k_3 + k_4 - v$.

Definition 7 (*T-matrix*) Let T_1, T_2, T_3, T_4 be $n \times n$ matrices with entries $(0, \pm 1)$. Let I_n be an $n \times n$ identity matrix. Then we call T_1, T_2, T_3, T_4 *T-matrices* if

- (i) $T_i T_j = T_j T_i$, $1 \leq i, j \leq 4$, $i \neq j$,
- (ii) there exists an $n \times n$ monomial matrix R with $R^T = R$, $R^2 = I_n$, such that $(T_i R)^T = T_i R$, $i = 1, 2, 3, 4$,
- (iii) if $T_i = (t_{jk}^{(i)})$, $1 \leq j, k \leq n$, $i = 1, 2, 3, 4$, then $\sum_{i=1}^4 |t_{jk}^{(i)}| = 1$, $i \leq j, k \leq n$,
- (iv) $\sum_{i=1}^4 T_i T_i^T = n I_n$,

We use condition (i) and (ii) to replace the condition of circulant *T-matrices*, and matrix R may easily be found in *Abelian groups*.

Definition 8 (*C-partitions*) A_1, A_2, \dots, A_8 are called *C-partitions* of an abelian group G of order v , if the following three conditions are satisfied:

- (i) $A_i \cap A_j = \phi$, $i \neq j$;
- (ii) $\bigcup_{i=1}^8 A_i = G$;
- (iii) $\sum_{i=1}^8 \Delta A_i = v\theta + \sum_{i=1}^4 \Delta(A_i, A_{i+4})$.

Lemma 1 (*J. Seberry [7]*) The following conditions are equivalent:

- (i) There exists a Hadamard matrix of order $4k^2$ with maximum excess $8k^3$.
- (ii) There exists a regular Hadamard matrix of order $4k^2$.
- (iii) There exists SBIBD($4k^2, 2k^2 + k, k^2 + k$).

Some very useful methods to construct *Hadamard* matrices with maximum excess from *Willamson* matrices and *T-matrices* are given in [7].

Lemma 2 (*M. Y. Xia and G. Liu [11]*) Let q be a prime power, if $q \equiv 1 \pmod{4}$, there exist $4 - \{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ supplementary difference sets.

Lemma 3 (*M. Y. Xia and G. Liu [14]*) Let q be a power of a prime, $q \equiv 3 \pmod{8}$, then there exist $4 - \{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ supplementary difference sets.

Lemma 4 (*Y. Q. Chen [2], M. Y. Xia [12]*) Let $q = 2^a 3^b N^2$, $a, b = 0$ or 1 , N is an arbitrary integer, then there exist $(4q^2, 2q^2 + q, q^2 + q)$ difference sets and Williamson type matrices (type 1) A_1, A_2, A_3 and A_4 of order q^2 that satisfy

$$\begin{aligned} \sigma(A_1) &= \sigma(A_2) = \sigma(A_3) = q^3, \quad \sigma(A_4) = -q^3, \\ A_1^2 + A_2^2 + A_3^2 + A_4^2 &= 4q^2 I_{q^2}, \\ A_i A_j + A_k A_l &= 0, \quad \{i, j, k, l\} = \{1, 2, 3, 4\}. \end{aligned} \quad (2)$$

Lemma 5 (*M. Y. Xia and T. Xia [13]*) Let q_1 be a prime power, $q_1 \equiv 5 \pmod{8}$, $q_2 = 2^a 3^b N^2$, $a, b = 0$ or 1 , N is an arbitrary integer, then there exist $(1, -1)$ Williamson type matrices (type 1) A_1, A_2, A_3 and A_4 of order $(q_1 q_2)^2$ that satisfy:

$$\begin{aligned} \sigma(A_1) &= \sigma(A_2) = \sigma(A_3) = (q_1 q_2)^3, \quad \sigma(A_4) = -(q_1 q_2)^3, \\ A_1 A_1^T + A_2 A_2^T + A_3 A_3^T + A_4 A_4^T &= 4(q_1 q_2)^2 I_{(q_1 q_2)^2}, \\ A_1 A_2^T + A_2 A_1^T + A_3 A_4^T + A_4 A_3^T &= 0. \end{aligned} \quad (3)$$

Proposition 1 Let $p \equiv 5 \pmod{8}$ be a prime, $q \equiv 2^a 3^b N^2$, $a, b = 0$ or 1 , N be an arbitrary integer, for any integer $r \geq 1$, there exist $(1, -1)$ matrices A_1, A_2, A_3 and A_4 of order $(p^r q)^2$ that satisfy

$$\begin{aligned} \sigma(A_1) &= \sigma(A_2) = \sigma(A_3) = (p^r q)^3, \quad \sigma(A_4) = -(p^r q)^3, \\ \sum_{i=1}^4 A_i A_i^T &= 4(p^r q)^2 I_{(p^r q)^2}, \\ A_1 A_2^T + A_2 A_1^T + A_3 A_4^T + A_4 A_3^T &= 0 \end{aligned} \quad (4)$$

Proof. When $q_1 = p^{2r+1}$, then $q_1 \equiv 5 \pmod{8}$. Then from lemma 5, the result is true.

When $q_1 = p^{2r} = (p^r)^2$, from lemma 4 we have the result. This completes the proof. \square

Remark. By using definition 6 we can say when $p \equiv 5 \pmod{8}$, $q = 2^a 3^b N^2$, $a, b = 0$ or 1 , N is an arbitrary integer, for any integer $r \geq 1$, there exist SDS D_1, D_2, D_3 and D_4 of order $p^{2r} q^2$ and type H_1 . We say $H_1(p^{2r} q^2) \neq \phi$, wherever such SDS exist for order $p^{2r} q^2$.

In section 2 we use SDS to construct SBIBD In section 3 we use SDS and T -matrices to construct SBIBD. We find new results which give many new SBIBDs.

2 Construct SBIBD from SDS

Theorem 1 If there exist $4 - \{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ SDS on an Abelian group G of order q^2 , then there exist SBIBD $(4q^2, 2q^2 + q, q^2 + q)$.

Proof. Let D_1, D_2, D_3, D_4 be $4 - \{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ SDS on G , since we have

$$|D_1| = |D_2| = |D_3| = |D_4| = \frac{1}{2}q(q-1), \quad \sum_{i=1}^4 \Delta D_i = q^2 \theta + q(q-2)G.$$

Let g_1, \dots, g_{q^2} be the arbitrary order of G , set

$$A_i = (a_{jk}^{(i)})_{1 \leq j, k \leq q^2}, \quad a_{jk}^{(i)} = \begin{cases} -1 & \text{if } g_k \ominus g_j \in D_i, \\ 1 & \text{otherwise,} \end{cases} \quad i = 1, 2, 3, 4, \quad (5)$$

$$R = (r_{jk})_{1 \leq j, k \leq q^2}, \quad r_{jk} = \begin{cases} 1 & \text{if } g_j \oplus g_k = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

It is obvious that A_1, A_2, A_3, A_4 are matrices of type 1. In this case

- (i) $A_i A_j = A_j A_i, i \neq j, i, j = 1, 2, 3, 4,$
- (ii) $(A_i R)^T = A_i R, i = 1, 2, 3, 4,$
- (iii) $\sum_{i=1}^4 A_i A_i^T = 4q^2 I_{q^2}.$

Since $|D_i| = \frac{1}{2}q(q-1)$, there exist $\frac{1}{2}q(q+1)$ ones and $\frac{1}{2}q(q-1)$ negative ones in each row of $A_i, i = 1, 2, 3, 4,$ so $\sigma(A_i) = q^3, i = 1, 2, 3, 4.$ Set

$$H = \begin{pmatrix} -A_1 & A_2 R & A_3 R & A_4 R \\ A_2 R & A_1 & A_4^T R & -A_3^T R \\ A_3 R & -A_4^T R & A_1 & A_2^T R \\ A_4 R & A_3^T R & -A_2^T R & A_1 \end{pmatrix}. \quad (7)$$

It is easy to verify that $HH^T = 4q^2 I_{4q^2}, \sigma(A_i) = \sigma(A_i R) = \sigma(A_i^T R), i = 1, 2, 3, 4.$ So we have

$$\sigma(H) = 2\{\sigma(A_1) + \sigma(A_2) + \sigma(A_3) + \sigma(A_4)\} = 8q^3.$$

From lemma 1, $\frac{1}{2}(H + J)$ is a $SBIBD(4q^2, 2q^2 + q, q^2 + q).$ This completes the proof. \square

Proposition 2 *Let q be a prime power, $q \equiv 1 \pmod{4}$ or $q \equiv 3 \pmod{8}.$ There exist $SBIBD(4q^2, 2q^2 + q, q^2 + q).$*

Proof. From lemma 2, lemma 3 and theorem 1 the conclusion is true. \square

Remark. When $q \equiv 1 \pmod{4}$ is a prime power, there exist *Williamson* type matrices A_1, A_2, A_3 and A_4 of order $q^2,$ that make the matrix H of (7) have maximum excess and the form

$$H = \begin{pmatrix} -A_1 & A_2 & A_3 & A_4 \\ A_2 & A_1 & A_4 & -A_3 \\ A_3 & -A_4 & A_1 & A_2 \\ A_4 & A_3 & -A_2 & A_1 \end{pmatrix}. \quad (8)$$

Lemma 6 *Let $q = 2^a 3^b N^2, a, b = 0$ or $1, N$ be arbitrary integer. There exists $SBIBD(4q^2, 2q^2 + q, q^2 + q).$*

Proof. From lemma 4, there exist DS of type $(4q^2, 2q^2 + q, q^2 + q)$, the $(0, 1)$ incidence matrix B of the DS is an $SBIBD(4q^2, 2q^2 + q, q^2 + q)$. The proof is completed. \square

Remark. From lemma 4 we know that there exist *Williamson* type matrices A_1, A_2, A_3, A_4 of order q^2 that satisfy (2). In this case, the matrix H of order $4q^2$ with maximum excess has the following form

$$H = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_1 & A_4 & A_3 \\ A_3 & A_4 & A_1 & A_2 \\ A_4 & A_3 & A_2 & A_1 \end{pmatrix}, \quad (9)$$

or

$$H = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_3 & A_4 & A_1 \\ A_3 & A_4 & A_1 & A_2 \\ A_4 & A_1 & A_2 & A_3 \end{pmatrix}. \quad (10)$$

Lemma 7 Let $p \equiv 5(\text{mod } 8)$ be a prime, $q = 2^a 3^b p^c N^2$, $a, b, c = 0$ or 1 , N be an arbitrary integer. Then there exists $SBIBD(4q^2, 2q^2 + q, q^2 + q)$.

Proof. When $c = 0$, from lemma 6, the lemma 7 is true. When $c = 1$, from lemma 5 that there exist $(1, -1)$ matrices (type 1) A_1, A_2, A_3 and A_4 of order q^2 that satisfy (3). Let

$$H = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2 & A_1 & A_4 & A_3 \\ -A_3^T & -A_4^T & A_1^T & A_2^T \\ -A_4^T & -A_3^T & A_2^T & A_1^T \end{pmatrix}, \quad (11)$$

then $HH^T = 4q^2 I_{4q^2}$, $\sigma(H) = 4(\sigma(A_1) + \sigma(A_2)) = 8q^3$. So the matrix H of (11) is an *Hadamard* matrix with maximum excess. In this case, from lemma 1 there exists $SBIBD(4q^2, 2q^2 + q, q^2 + q)$. The proof is completed. \square

Proposition 3 Let $q = 2^{r_1} 3^{r_2} p^{r_3} N^2$, $p \equiv 5(\text{mod } 8)$ be a prime, r_1, r_2, r_3 be integers and $r_1, r_2, r_3 \geq 0$, N be an arbitrary integer. Then lemma 7 still holds.

Proof. Let $r_i = 2m_i + a_i$, $0 \leq a_i \leq 1$, $i = 1, 2, 3$, then $q = 2^{a_1} 3^{a_2} p^{a_3} (2^{m_1} 3^{m_2} p^{m_3} N)^2$. From lemma 7 the result is true. \square

3 Construct SBIBD from SDS and T-matrices

More details of *T-matrices* are discussed in [3]. In this paper we refer to the paper [15].

Theorem 2 *If there exist $4 - \{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ SDS D_1, D_2, D_3, D_4 of order q^2 in an Abelian group G , and every entry of G appears an even number of times in D_1, D_2, D_3, D_4 , then there exist T-matrices T_1, T_2, T_3, T_4 that satisfy*

$$\sigma(T_1) = q^3, \quad \sigma(T_2) = \sigma(T_3) = \sigma(T_4) = 0.$$

Proof. Let

$$\begin{aligned} E_1 &= G \setminus (D_1 \cup D_2 \cup D_3 \cup D_4), & E_2 &= (D_1 \cap D_2) \setminus E_5, \\ E_3 &= (D_1 \cap D_3) \setminus E_5, & E_4 &= (D_1 \cap D_4) \setminus E_5, \\ E_5 &= D_1 \cap D_2 \cap D_3 \cap D_4, & E_6 &= (D_3 \cap D_4) \setminus E_5, \\ E_7 &= (D_2 \cap D_4) \setminus E_5, & E_8 &= (D_2 \cap D_3) \setminus E_5. \end{aligned}$$

From [15] we know

$$\begin{aligned} E_i \cap E_j &= \phi, \quad i \neq j, \quad 1 \leq i, j \leq 8, \\ G &= \cup_{i=1}^8 E_i, \\ \sum_{i=1}^8 \Delta E_i &= q^2 \theta + \sum_{i=1}^4 \Delta(E_i, E_{i+4}), \end{aligned}$$

and

$$\begin{aligned} D_1 &= E_5 \cup E_2 \cup E_3 \cup E_4, & D_2 &= E_5 \cup E_2 \cup E_7 \cup E_8 \\ D_3 &= E_5 \cup E_3 \cup E_6 \cup E_8, & D_4 &= E_5 \cup E_4 \cup E_6 \cup E_7. \end{aligned}$$

Set $|E_i| = e_i, i = 1, \dots, 8$, we have

$$\begin{aligned} |D_1| &= e_2 + e_3 + e_4 + e_5, & |D_2| &= e_2 + e_5 + e_7 + e_8, \\ |D_3| &= e_3 + e_5 + e_6 + e_8, & |D_4| &= e_4 + e_5 + e_6 + e_7. \end{aligned}$$

Since $|D_1| = |D_2| = |D_3| = |D_4| = \frac{1}{2}q(q-1)$, then

$$e_2 - e_6 = e_3 - e_7 = e_4 - e_8 = 0.$$

Since

$$\begin{aligned} q^2 &= |G| = |\cup_{i=1}^8 E_i| = \sum_{i=1}^8 e_i = e_1 + e_5 + 2(e_2 + e_3 + e_4) \\ &= e_1 - e_5 + 2(e_2 + e_3 + e_4 + e_5) = e_1 - e_5 + q(q-1), \end{aligned}$$

then $e_1 - e_5 = q$. Let g_1, \dots, g_{q^2} be an arbitrary order of entries of G , and

$$T_i = \left(t_{jk}^{(i)} \right)_{1 \leq j, k \leq q^2}, t_{jk}^{(i)} = \begin{cases} 1 & \text{if } g_k \ominus g_j \in E_i, \\ -1 & \text{if } g_k \ominus g_j \in E_{i+4}, \quad i = 1, 2, 3, 4. \\ 0 & \text{otherwise,} \end{cases}$$

T_1, T_2, T_3, T_4 are T-matrices of order q^2 and

$$\sigma(T_1) = q^3, \quad \sigma(T_2) = \sigma(T_3) = \sigma(T_4) = 0.$$

This completes the proof. \square

Proposition 4 *Let q be a prime power and $q \equiv 3 \pmod{8}$, then there exist T-matrices T_1, T_2, T_3 and T_4 of order q^2 that satisfy theorem 2.*

Theorem 3 *If there exist T-matrices T_1, T_2, T_3 and T_4 of order t^2 , and $\sigma(T_1) = t^3$, $\sigma(T_2) = \sigma(T_3) = \sigma(T_4) = 0$, then there exists SBIBD($4k^2, 2k^2 + k, k^2 + k$), $k = tq$, where $q \equiv 1 \pmod{4}$ is any prime power.*

Proof. When $q \equiv 1 \pmod{4}$ is a prime power, from lemma 2 we know there exist $4 - \{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ SDS. In this case from theorem 1 we have Williamson type (type 1) matrices A_1, A_2, A_3 and A_4 of order q^2 which are satisfy

- (i) $A_i = A_i^T, A_i A_j = A_j A_i, 1 \leq i, j \leq 4, i \neq j,$
- (ii) $\sum_{i=1}^4 A_i^2 = 4q^2 I_{q^2},$
- (iii) $\sigma(A_1) = \sigma(A_2) = \sigma(A_3) = \sigma(A_4) = q^3.$

Let

$$\begin{aligned} B_1 &= T_1 \times A_1 + T_2 \times A_2 + T_3 \times A_3 + T_4 \times A_4, \\ B_2 &= T_1 \times A_2 - T_2 \times A_1 + T_3 \times A_4 - T_4 \times A_3, \\ B_3 &= T_1 \times A_3 + T_2 \times A_4 - T_3 \times A_1 - T_4 \times A_2, \\ B_4 &= T_1 \times A_4 - T_2 \times A_3 - T_3 \times A_2 + T_4 \times A_3, \end{aligned} \tag{12}$$

where \times is Kronecker product. It is obvious that $B_i B_j = B_j B_i, i \neq j, i, j = 1, 2, 3, 4,$ and

$$\sum_{i=1}^4 B_i B_i^T = \left(\sum_{i=1}^4 T_i T_i^T \right) \times \left(\sum_{i=1}^4 A_i^2 \right) = 4(tq)^2 I_{(tq)^2}.$$

Since $\sigma(T_i \times A_i) = \sigma(T_i) \sigma(A_i), i = 1, 2, 3, 4,$

$$\sigma(B_1) = \sigma(B_2) = \sigma(B_3) = \sigma(B_4) = (tq)^3.$$

Let

$$Q = R \times I_{q^2}, \quad (13)$$

where R is a monomial matrix of order t^2 that satisfies $R = R^T$, $R^2 = I$, and $(T_i R)^T = T_i R$, $i = 1, 2, 3, 4$. It is easy to show that Q is a permutation matrix and $(B_i Q)^T = B_i Q$, $i = 1, 2, 3, 4$. Let

$$H = \begin{pmatrix} B_1 & B_2 Q & -B_3 Q & B_4 Q \\ B_2 Q & -B_1 & B_4^T Q & B_3^T Q \\ B_3 Q & B_4^T Q & B_1 & -B_2^T Q \\ -B_4 Q & B_3^T Q & B_2^T Q & B_1 \end{pmatrix} \quad (14)$$

Then $HH^T = 4k^2 I_{4k^2}$, and $\sigma(H) = 2(\sum_{i=1}^4 \sigma(B_i)) = 8k^3$. In this case, the matrix H of order $4k^2$ defined from (12), (13), (14) has the maximum excess. There exist $SBIBD(4k^2, 2k^2 + k, k^2 + k)$, $k = tq$. The proof is complete. \square

Proposition 5 *When $k = q_1 q_2$, where $q_1 = 1 \pmod{4}$, $q_2 = 3 \pmod{8}$ are prime powers, there exist $SBIBD(4k^2, 2k^2 + k, k^2 + k)$.*

Proof. From [15] we know that there exist T -matrices T_1, T_2, T_3, T_4 satisfying theorem 3. This completes the proof. \square

Theorem 4 *Suppose*

1. *there exist T -matrices T_1, T_2, T_3, T_4 of order t^2 that satisfy*

$$\sigma(T_1) = t^3, \quad \sigma(T_2) = \sigma(T_3) = \sigma(T_4) = 0,$$

and

2. *there exist $(1, -1)$ matrices (type 1) A_1, A_2, A_3 and A_4 of order q^2 that satisfy*

$$(i) \quad \sum_{i=1}^4 A_i A_i^T = 4q^2 I_{q^2},$$

$$(ii) \quad A_1 A_2^T + A_2 A_1^T + A_3 A_4^T + A_4 A_3^T = 0,$$

$$(iii) \quad \sigma(A_1) = \sigma(A_2) = \sigma(A_3) = q^3 = -\sigma(A_4).$$

Then there exist $SBIBD(4k^2, 2k^2 + k, k^2 + k)$, $k = tq$.

Proof. Let

$$\begin{aligned} B_1 &= T_1 \times A_1 + T_2 \times A_2 + T_3 \times A_3 + T_4 \times A_4, \\ B_2 &= T_1 \times A_2 + T_2 \times A_1 + T_3 \times A_4 + T_4 \times A_3, \\ B_3 &= T_1 \times A_3^T + T_2 \times A_4^T - T_3 \times A_1 T - T_4 \times A_2^T, \\ B_4 &= -T_1 \times A_4^T - T_2 \times A_3^T + T_3 \times A_2^T + T_4 \times A_1^T, \end{aligned}$$

It is easy to verify that

$$\sum_{i=1}^4 B_i B_i^T = \left(\sum_{i=1}^4 T_i T_i^T \right) \times \left(\sum_{i=1}^4 A_i A_i^T \right) = 4k^2 I_{k^2},$$

and

$$\sigma(B_1) = \sigma(B_2) = \sigma(B_3) = \sigma(B_4) = k^3.$$

Set

$$H = \begin{pmatrix} B_1 & B_2 R & -B_3 R & B_4 R \\ B_2 R & -B_1 & B_4^T R & B_3^T R \\ B_3 R & B_4^T R & B_1 & -B_2^T R \\ -B_4 R & B_3^T R & B_2^T R & B_1 \end{pmatrix},$$

where $R = R_1 \times R_2$, R_1, R_2 are monomial matrices of order t^2 and q^2 , and $(T_i R_1)^T = T_i R_1$, $(A_i R_2)^T = A_i R_2$, $i = 1, 2, 3, 4$. In this case $HH^T = 4k^2 I_{4k^2}$, and $\sigma(H) = 2 \sum_{i=1}^4 \sigma(B_i) = 8k^3$. Then H is a *Hadamard* matrix with maximum excess, and $\frac{1}{2}(H + J)$ is a *SBIBD*($4k^2, 2k^2 + k, k^2 + k$). \square

Proposition 6 *Let $k = 2^{a_1} 3^{a_2} p_1^{a_3} p_2^{a_4} N^2$, $a_1, a_2, a_3, a_4 = 0$ or 1 , $p_1 = 5 \pmod{8}$, $p_2 = 3 \pmod{8}$ are primes and N be an arbitrary integer. Then there exist *SBIBD*($4k^2, 2k^2 + k, k^2 + k$).*

Proof. When $a_4 = 0$, from lemma 7, the result is true. When $a_4 = 1$, set $t = p_2$, $q = 2^{a_1} 3^{a_2} p_1^{a_3} N^2$. From lemma 5, proposition 4 and theorem 3, we can prove the result is correct. \square

Remark. Let $k = 2^{a_1} 3^{a_2} p_1^{a_3} p_2^{a_4} N^2$, where $a_1, a_2, a_3, a_4 \geq 0$, $p_1 \equiv 5 \pmod{8}$, $p_2 \equiv 3 \pmod{8}$, then proposition 6 is still true.

Let $a_i = 2s_i + r_i$, where $s_i \geq 0$, $0 \leq r_i \leq 1$, $i = 1, 2, 3, 4$. Then $k = 2^{r_1} 3^{r_2} p_1^{r_3} p_2^{r_4} (2^{s_1} 3^{s_2} p_1^{s_3} p_2^{s_4} N)^2$ satisfies the condition of proposition 6.

Proposition 7 *If $q \equiv 1 \pmod{4}$ is a prime power, there exist *SBIBD*($4(7q)^2, 2(7q)^2 + 7q, (7q)^2 + 7q$).*

Proposition 8 *When $p_2^{a_4}$ in proposition 6 is replaced by 7, the conclusion of proposition 6 is still true.*

Proof. Let $g = x \oplus 2$ be a generator of $GF(7^2)$, set

$$F_i = \{g^{16j+i} \pmod{x^2 \oplus 1, \pmod{7}} : j = 0, 1, 2\}, \quad i = 0, 1, \dots, 15.$$

$$\begin{aligned} E_1 &= \{0\} \cup F_{11} \cup F_{12} \cup F_{15}, & E_2 &= F_0 \cup F_{13}, & E_3 &= F_3 \cup F_6, & E_4 &= F_4 \cup F_{14}, \\ E_5 &= F_{10}, & E_6 &= F_1 \cup F_2, & E_7 &= F_7 \cup F_8, & E_8 &= F_5 \cup F_9. \end{aligned}$$

It is easy to verify that

$$\sum_{i=1}^8 \Delta E_i = 49\theta + \sum_{i=1}^4 \Delta(E_i, E_{i+4}).$$

Without loss of generality, let g_1, \dots, g_{49} be an arbitrary order on the elements of $GF(7^2)$. Set

$$T_i = \left(t_{jk}^{(i)} \right)_{1 \leq j, k \leq 49}, \quad t_{jk}^{(i)} = \begin{cases} 1, & \text{if } g_k \ominus g_j \in E_i, \\ -1, & \text{if } g_k \ominus g_j \in E_{i+4}, \\ 0, & \text{otherwise,} \end{cases} \quad i = 1, 2, 3, 4.$$

The matrices T_1, T_2, T_3, T_4 are T-matrices of order 49, and

$$\sigma(T_1) = 7^3, \quad \sigma(T_2) = \sigma(T_3) = \sigma(T_4) = 0.$$

From theorem 3 and theorem 4, we know the proposition 7 and 8 are all true. This completes the proof. \square

From proposition 8 we know, for any integer $r \geq 1$, there exist $SBIBD(4 \cdot 7^{2r}, 2 \cdot 7^{2r} + 7^r, 7^{2r} + 7^r)$. When r is even, $q = 7^r = 1 \pmod{4}$, from proposition 7 we know the conclusion is true. When r is odd, then $7^{r-1} = 1 \pmod{4}$. In this case let $q = 7^{r-1}$, then from proposition 7, the conclusion is true. For any integer $a, b \geq 1$, $p \equiv 5 \pmod{8}$ a prime, from proposition 8 we know there exist $SBIBD(4(7^a p^b)^2, 2(7^a p^b)^2 + 7^a p^b, (7^a p^b)^2 + 7^a p^b)$. From proposition 8 we conclude that for $a, b, c \geq 0$, $p \equiv 5 \pmod{8}$ a prime, there exist $SBIBD(4(3^a 7^b p^c)^2, 2(3^a 7^b p^c)^2 + 3^a 7^b p^c, (3^a 7^b p^c)^2 + 3^a 7^b p^c)$.

Lemma 8 *There exist $4 - \{23^2; 23 \cdot 11, 23 \cdot 21\}$ SDS of order 23^2 .*

Proof. Let $g = x + 2$ be a generator on $GF(23)^2$. Set

$$E_i = \{g^{48j+i} \pmod{x^2 + 1, \pmod{23}} : j = 0, 1, \dots, 10\}, \quad i = 0, \dots, 47.$$

Put

$$\begin{aligned} A_1 &= \{0\} \cup E_9 \cup E_{12} \cup E_{13} \cup E_{28} \cup E_{41} \cup E_{44} \cup E_{45}, \\ A_2 &= E_0 \cup E_{16} \cup E_{17} \cup E_{29} \cup E_{32} \cup E_{33}, \\ A_3 &= E_2 \cup E_4 \cup E_{18} \cup E_{20} \cup E_{34} \cup E_{36}, \\ A_4 &= E_3 \cup E_8 \cup E_{19} \cup E_{24} \cup E_{35} \cup E_{40}, \\ A_5 &= E_1 \cup E_5 \cup E_6 \cup E_{22} \cup E_{38}, \\ A_6 &= E_{10} \cup E_{21} \cup E_{25} \cup E_{26} \cup E_{37} \cup E_{42}, \\ A_7 &= E_7 \cup E_{11} \cup E_{23} \cup E_{27} \cup E_{39} \cup E_{43}, \\ A_8 &= E_{14} \cup E_{15} \cup E_{30} \cup E_{31} \cup E_{46} \cup E_{47}. \end{aligned} \tag{15}$$

Let g_1, \dots, g_{23^2} be an arbitrary order on the elements of $GF(23)^2$. Set matrix

$$T_i = (t_{jk}^{(i)})_{1 \leq j, k \leq 23^2}, \quad t_{jk}^{(i)} = \begin{cases} 1, & \text{if } g_k - g_j \in A_i, \\ -1, & \text{if } g_k - g_j \in A_{i+4}, \\ 0, & \text{otherwise.} \end{cases} \quad i = 1, 2, 3, 4, \quad (16)$$

Then T_1, T_2, T_3 and T_4 defined in (16) are T -matrices of order 23^2 and

$$\sigma(T_1) = 23^3, \quad \sigma(T_2) = \sigma(T_3) = \sigma(T_4) = 0.$$

In this case the set $\{A_i\}_{i=1}^8$ defined in (15) is the C -Partitions (see [15] for details). The set

$$\begin{aligned} D_1 &= A_5 \cup A_2 \cup A_3 \cup A_4, & D_2 &= A_5 \cup A_2 \cup A_7 \cup A_8, \\ D_3 &= A_5 \cup A_3 \cup A_6 \cup A_8, & D_4 &= A_5 \cup A_4 \cup A_6 \cup A_7. \end{aligned} \quad (17)$$

is the $4 - \{23^2; 23 \cdot 11, 23 \cdot 21\}$ SDS. This complete the proof. \square

Proposition 9 *There exist $SBIBD(4 \cdot 23^2; 2 \cdot 23^2 + 23, 23^2 + 23)$.*

Proposition 9 follows easily from Theorem 1.

Proposition 10 *When 7 in proposition 7 is replaced by 23, there exist $SBIBD(4 \cdot (23q)^2, 2 \cdot (23q)^2 + 23q, (23q)^2 + 23q)$.*

Proposition 11 *There exist $SBIBD(4 \cdot 23^{2r}, 2 \cdot 23^{2r} + 23^r, 23^{2r} + 23^r)$.*

Proof. For any integer $r \geq 1$, when r is even, $q = 23^r \equiv 1 \pmod{4}$, from proposition 7, there exist $SBIBD(4 \cdot 23^{2r}, 2 \cdot 23^{2r} + 23^r, 23^{2r} + 23^r)$. When r is odd, then $q = 23^{r-1} \equiv 1 \pmod{4}$, in this case the conclusion is again true. \square

Remark. For any integer $a, b \geq 1$, $p \equiv 5 \pmod{8}$ be a prime, there exist $SBIBD(4 \cdot (23^a p^b)^2, 2 \cdot (23^a p^b)^2 + 23^a p^b, (23^a p^b)^2 + 23^a p^b)$.

For any $a, b, c \geq 0$, $p \equiv 5 \pmod{8}$ a prime, there exist $SBIBD(4 \cdot (3^a 23^b p^c)^2, 2 \cdot (3^a 23^b p^c)^2 + 3^a 23^b p^c, (3^a 23^b p^c)^2 + 3^a 23^b p^c)$.

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