SHORT AMICABLE SETS

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Abstract: A pair of matrices X and Y are said to be amicable if $XY^T = YX^T$. In this paper, if X and Y are orthogonal designs, group generated or circulant on the group G, these will be denoted $2-SAS(n;u_1,u_2;G)$. Recently Kharaghani, in "Arrays for orthogonal designs", *J. Combin. Designs*, 8 (2000), 166–173, extended this concept to an amicable set, $\{A_i\}_{i=1}^{2n}$, of $2n$ circulant matrices, which satisfy

$$\sum_{i=1}^{n}\left(A_{\sigma(2i-1)}A_{\sigma(2i)}^T - A_{\sigma(2i)}A_{\sigma(2i-1)}^T\right) = 0.$$

In this paper we concentrate on constructing short amicable sets, which satisfy the same equation but contain four, called short, or two, called 2-short, matrices. We give a method of multiplying the order of 2-short circulant amicable sets and thus we obtain many infinite classes of 2-short circulant amicable sets. We give some constructions for infinite families of circulant amicable sets. We then contrast by comparing with short block amicable sets which are block circulant matrices and defined on a group $G_1 \times G_2$.

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1 Introduction

An *orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$ denoted $OD(n; s_1, s_2, \ldots, s_u)$ in the variables $x_1, x_2, \ldots, x_u$, is a matrix A of order n with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$AA^T = \sum_{i=1}^{u}(s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n. Let B_i, $i = 1, 2, 3, 4$ be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$\sum_{i=1}^{4} B_i B_i^T = \sum_{i=1}^{u}(s_i x_i^2) I_n.$$

Then the Goethals-Seidel array

$$G = \begin{pmatrix} B_1 & B_2 R & B_3 R & B_4 R \\ -B_2 R & B_1 & B_4^T R & -B_3^T R \\ -B_3 R & -B_4^T R & B_1 & B_2^T R \\ -B_4 R & B_3^T R & -B_2^T R & B_1 \end{pmatrix}$$

where R is the back-diagonal identity matrix, is an $OD(4n; s_1, s_2, \ldots, s_u)$. See page 107 of [3] for details.

A pair of matrices A, B is said to be amicable (anti-amicable) if $AB^T - BA^T = 0$ ($AB^T + BA^T = 0$). To be consistent in the notation of this paper we will also denote these as $2 - SAS(n; s_1, s_2; G)$, where the group G is described below. Following [9] a set $\{A_1, A_2, \ldots, A_{2n}\}$ of square real matrices is said to be *amicable* if

$$\sum_{i=1}^{n} \left(A_{\sigma(2i-1)} A_{\sigma(2i)}^T - A_{\sigma(2i)} A_{\sigma(2i-1)}^T \right) = 0 \tag{1}$$

for some permutation σ of the set $\{1, 2, \ldots, 2n\}$. For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. So

$$\sum_{i=1}^{n} \left(A_{2i-1} A_{2i}^T - A_{2i} A_{2i-1}^T \right) = 0. \tag{2}$$

Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. Throughout the paper R_k denotes the back diagonal identity matrix of order k.

A set of matrices $\{B_1, B_2, \ldots, B_n\}$ of order m with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ is said to satisfy an additive property of type $(s_1, s_2, \ldots, s_u)$ if

$$\sum_{i=1}^{n} B_i B_i^T = \sum_{i=1}^{u}(s_i x_i^2) I_m. \tag{3}$$

Let $\{A_i\}_{i=1}^8$ be an amicable set of circulant matrices (or group developed or type 1) of type $(s_1, s_2, \ldots, s_u)$ and order t. We denote these by $8-AS(t; s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8; Z_t)$ (or $8-AS(t; s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8; G)$ for group developed or type 1). In all cases, the group G of the matrix is such that the extension by Seberry and Whiteman [11] of the group from circulant to type 1 allows the same extension to R. Then the Kharaghani array [9]

$$H = \begin{pmatrix} A_1 & A_2 & A_4R_n & A_3R_n & A_6R_n & A_5R_n & A_8R_n & A_7R_n \\ -A_2 & A_1 & A_3R_n & -A_4R_n & A_5R_n & -A_6R_n & A_7R_n & -A_8R_n \\ -A_4R_n & -A_3R_n & A_1 & A_2 & -A_8^TR_n & A_7^TR_n & A_6^TR_n & -A_5^TR_n \\ -A_3R_n & A_4R_n & -A_2 & A_1 & A_7^TR_n & A_8^TR_n & -A_5^TR_n & -A_6^TR_n \\ -A_6R_n & -A_5R_n & A_8^TR_n & -A_7^TR_n & A_1 & A_2 & -A_4^TR_n & A_3^TR_n \\ -A_5R_n & A_6R_n & -A_7^TR_n & -A_8^TR_n & -A_2 & A_1 & A_3^TR_n & A_4^TR_n \\ -A_8R_n & -A_7R_n & -A_6^TR_n & A_5^TR_n & A_4^TR_n & -A_3^TR_n & A_1 & A_2 \\ -A_7R_n & A_8R_n & A_5^TR_n & A_6^TR_n & -A_3^TR_n & -A_4^TR_n & -A_2 & A_1 \end{pmatrix}$$

is an $OD(8t; s_1, s_2, \ldots, s_u)$.

The Kharaghani array has been used in a number of papers [5, 6, 7, 9, 10] to obtain infinitely many families of orthogonal designs. Research has yet to be initiated to explore the algebraic restrictions imposed an amicable set by the required constraints.

A set $\{A_i\}_{i=1}^4$ is said to be a *short amicable set* of order m and type (u_1, u_2, u_3, u_4), abbreviated as $4 - SAS(m; u_1, u_2, u_3, u_4; G)$, if (2) and (3) are satisfied for $n = 4$ and $u \leq 4$. $4 - SAS(m; u_1, u_2, u_3, u_4; G)$ can be used in either the Goethals-Seidel array or the *short Kharaghani array*

$$\begin{pmatrix} A & B & CR & DR \\ -B & A & DR & -CR \\ -CR & -DR & A & B \\ -DR & CR & -B & A \end{pmatrix}$$

to form an $OD(4m; u_1, u_2, u_3, u_4)$. In all cases, the group G of the matrices in the *amicable set* is such that the extension by Seberry and Whiteman [11] of the group from circulant to type 1 allows the same extension to R.

In general a set of $2n$ matrices of order m and type $(s_1, s_2, \ldots, s_u)$ that satisfy equations (2) and (3) will be denoted as $2n - SAS(m; s_1, s_2, \ldots, s_u; G)$. Moreover if these matrices are circulant they will be denoted as $2n - SCAS(m; s_1, s_2, \ldots, s_u; Z_m)$.

Remark 1 Clearly

1. If there exists a $2 - SAS(n; s_1, s_2; G)$ and a $2 - SAS(n; s_3, s_4; G)$ then there exists a $4 - SAS(n; s_1, s_2, s_3, s_4; G)$.

2. If there exists a $2-SAS(n; s_1, s_2; G)$, $2-SAS(n; s_3, s_4; G)$, $2-SAS(n; s_5, s_6; G)$ and a $2-SAS(n; s_7, s_8; G)$ there exists an $8-AS(n; s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8; G)$.

3. If there exists a $4-SAS(n;s_1,s_2,s_3,s_4;G)$ and a $4-SAS(n;s_5,s_6,s_7,s_8;G)$ there exists an $8-AS(n;s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8;G)$.

Thus we can obtain many classes of $4-SAS(n;s_1,s_2,s_3,s_4;G)$ combining together two pairs of the given $2-SAS(n;s_1,s_2;G)$ and $2-SAS(n;s_3,s_4;G)$. Moreover, in Table , we give some $4-SAS(m;u_1,u_2,u_3,u_4;Z_m)$ that can not be constructed by this method.

Generally, unless we have other information regarding the structure, we are unable to ensure that the matrix R with the desired properties for the Kharaghani, Goethals-Seidel or short Kharaghani arrays exists unless the amicable sets have been group generated (circulant or type 1) or constructed from blocks of these kinds. Thus if we have the required matrix R_i for the group G_i, $i=1,2$ then $R_G=R_1\times R_2$ will be the required matrix for $G=G_1\times G_2$, (see [11]).

Let A_1 and A_2 be matrices of order m. We define $circ(A_1,A_2)=\begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}$. Amicable sets made from $2n$ such block circulant matrices will be called *block amicable sets*, *short block amicable sets* or *2-short block amicable sets*, $2n-SBAS(2m;s_1,s_2,\ldots,s_u;G)$, $n=1,2,4$, where, using R_t for the back-diagonal matrix of order t, $G=Z_2\times Z_m$ and $R_G=R_2\times R_m$. Here, if A_1 and A_2 are circulant, then we use the backdiagonal matrix of the same order for R ensuring $A_i(A_jR)^T=A_jRA_i^T$. The required $R_G=R_2\times R$.

A $(1,-1)$ matrix of order n is called a *Hadamard* matrix if $HH^T=H^TH=nI_n$, where H^T is the transpose of H and I_n is the identity matrix of order n. A $(1,-1)$ matrix A of order n is said to be of *skew* type if $A-I_n$ is skew-symmetric.

A matrix $W=circ(w_1,\ldots,w_n)$, $w_i\in\{0,\pm1\}$ which satisfies $WW^T=kI_n$ is called a *circulant weighing matrix* of order n and weight k or $CW(n,k)$.

Four $\{\pm1\}$ circulant and symmetric matrices X_1,X_2,X_3,X_4 of order n are called four *Williamson matrices* if they satisfy

$$X_1^2+X_2^2+X_3^2+X_4^2=4nI_n.$$

Four $\{\pm1\}$ matrices X_1,X_2,X_3,X_4 of order n are called four *Williamson type matrices* if they are pairwise amicable and satisfy

$$X_1X_1^T+X_2X_2^T+X_3X_3^T+X_4X_4^T=4nI_n.$$

We denote the product $Z_p\times Z_p\times\cdots\times Z_p(r\ times)$ by $EA(p^r)$ the Elementary Abelian group. Moreover $-a$ is denoted by $\bar{a}$.

Throughought this paper we use the symbol 0_m to denote the sequence of length m with all elements zero and the symbol O_t to denote the $t\times t$ matrix with all entries zero.

For the undefined terms we refer the reader to the book by Geramita and Seberry [3].

Suppose $C = circ(c_0, c_1, \ldots, c_{n-1})$ is a circulant matrix of order n. Let

$$T_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

of order n, be the shift matrix. Then we can write $C = c_0 I + c_1 T_n + \ldots + c_{n-1}T_n^{n-1}$. Note that $T_n^n = I$ the identity matrix of order n. We say the Hall polynomial of C is $\sum_{i=0}^{n-1} c_i x^i$. The Hall polynomial of C^T is $\sum_{i=0}^{n-1} c_i x^{n-i}$.

Given a set of ℓ sequences $A_j = \{a_{j1}, a_{j2}, \ldots, a_{jn}\}$, $j = 1, \ldots, \ell$, of length n the *non-periodic autocorrelation function*, denoted $NPAF$, $N_A(s)$ is defined as

$$N_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n-s} a_{ji} a_{j,i+s}, \quad s = 0, 1, \ldots, n-1, \tag{4}$$

If $A_j(z) = a_{j1} + a_{j2}z + \ldots + a_{jn}z^{n-1}$ is the associated polynomial of the sequence A_j, then

$$A(z)A(z^{-1}) = \sum_{j=1}^{\ell} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ji}a_{jk}z^{i-k} = N_A(0) + \sum_{j=1}^{\ell} \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}), z \neq 0. \tag{5}$$

Given A_ℓ, as above, of length n the *periodic autocorrelation function*, denoted PAF, $P_A(s)$ is defined, reducing $i + s$ modulo n, as

$$P_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n} a_{ji} a_{j,i+s}, \quad s = 0, 1, \ldots, n-1. \tag{6}$$

We note NPAF sequences imply PAF sequences exist, the NPAF sequences being padded at the end with sufficient zeros to make longer lengths. Hence NPAF sequences can give more general results. If two NPAF sequences have differing lengths then sufficient zeros are added to the end of each to make all the sequences the same length. In all cases NPAF and PAF sequences can be used to make circulant matrices satisfying the additive property (see [5, 9]); if NPAF sequences of lengths n_1 and n_2 are used, then by padding, circulant matrices for all orders $n \geq max(n_1, n_2)$ will exist; if PAF sequences of lengths n are used, then circulant matrices of order n exist.

2 2-short circulant amicable sets

Definition 1 We define *2-short circulant amicable sets*, abbreviated as $2-SCAS(n; u_1, u_2; Z_n)$, to be two circulant matrices of order n with entries from $\{0, \pm x_1, \pm x_2\}$ which satisfy

$$A_1A_1^T + A_2A_2^T = (u_1x_1^2 + u_2x_2^2)I_n, \quad A_1A_2^T - A_2A_1^T = 0,$$

or with entries from $\{0, \pm 1\}$ which satisfy

$$A_1A_1^T + A_2A_2^T = kI_n, \quad A_1A_2^T - A_2A_1^T = 0.$$

If $A_1 * A_2 = 0$ then $x_1A_1 + x_2A_2$ form an $OD(n; u_1, u_2)$.

Example 1

1. $A_1 = circ(0, a, b, -a)$ and $A_2 = circ(0, a, 0, a)$ is a $2-SCAS(4; 1, 4; Z_4)$.
2. $A_1 = circ(a, -b, 0, -b, a, b)$ and $A_2 = circ(b, a, 0, a, b, -a)$ is $2-SCAS(6; 5, 5; Z_6)$.

Theorem 1 *Let X_1, X_2 be $\{0, \pm 1\}$ circulant matrices of order ℓ satisfying*

$$X_1X_1^T + X_2X_2^T = kI_\ell \tag{7}$$

$$X_1X_2^T - X_2X_1^T = 0 \tag{8}$$

$$X_1 * X_2 = 0 \tag{9}$$

then there exists a $2 - SCAS(\ell; k, k; Z_\ell)$.

Proof. Set

$$A = aX_1 + bX_2 \quad \text{and} \quad B = -bX_1 + aX_2.$$

A and B are both circulant matrices and a straightforward calculation shows

$$AA^T + BB^T = (ka^2 + kb^2)I_\ell,$$

and

$$AB^T - BA^T = 0.$$

Thus A and B is a $2 - SCAS(\ell; k, k; Z_\ell)$. □

Remark 2 Using the sequences given in Table 1 with theorem 1, the matrices given in example 1 and lemma 1, we have that there exist 2-short circulant amicable sets for orders and types which are described in Table 2.

Order $n \geq 1$	Weight	Sequences X_1 ; X_2	zero
n	1	1 ; 0	NPAF
$2n$	2	1,0 ; 0,1	NPAF
$4n$	4	0,1,0,1 ; 1,0,-1,0	NPAF
$6n$	4	0,0,1,0,0,1 ; 0,-1,0,0,1,0	NPAF
$6n$	5	0,1,0,-1,0,1 ; 0,0,1,0,1,0	NPAF
$7n$	4	0,0,1,0,1,1,-1 ; 0,0,0,0,0,0,0	PAF
$8n$	8	1,1,1,0,-1,1,-1,0 ; 0,0,0,1,0,0,0,1	NPAF
$10n$	9	0,1,0,1,0,-1,0,1,0,1 ; 0,0,1,0,-1,0,-1,0,1,0	PAF
$12n$	8	0,1,1,0,1,0,0,-1,1,0,-1,0 ; 0,0,0,0,0,1,0,0,0,0,0,1	PAF
$13n$	9	0,0,1,0,1,1,1,-1,-1,0,1,-1,1 ; 0,0,0,0,0,0,0,0,0,0,0,0,0	PAF
$14n$	8	0,0,1,0,1,0,-1,0,0,0,0,0,1,0 ; 0,0,0,0,0,1,0,0,0,1,0,1,0,-1	PAF
$14n$	10	1,0,0,0,0,0,0,0,1,0,0,0,1,0 ; 0,1,0,-1,0,1,0,1,0,-1,0,-1,0,-1	NPAF
$14n$	13	0,1,0,1,0,-1,0,1,0,-1,0,1,0,1 ; 0,0,1,0,1,0,-1,0,-1,0,1,0,1,0	PAF

Table 1: Disjoint amicable circulant matrices can be constructed from the above sequences.

order	type	order	type	order	type	order	type
n	$(1,1)$	$6n$	$(4,4)$	$10n$	$(4,4)$	$14n$	$(8,8)$
$2n$	$(2,2)$	$6n$	$(5,5)$	$10n$	$(9,9)$	$14n$	$(10,10)$
$4n$	$(1,4)$	$7n$	$(4,4)$	$12n$	$(8,8)$	$14n$	$(13,13)$
$4n$	$(4,4)$	$8n$	$(8,8)$	$13n$	$(9,9)$		

Table 2: Order and type for small 2-short circulant amicable sets for all $n \geq 1$.

Remark 3 We observe that although we carried out an exhaustive computer search for orders up to 15, we could not find appropriate sequences of orders $3, 5, 9, 11, 15$ satisfying equations (7), (8) and (9) and having weights greater than 1 (see table 1). We observe from Horton and Seberry [8] that the type $(4, 9)$ did not arise satisfying even equation (7) using circulant matrices and lengths 7, 9, 10, 11, 13, 15, 17, 23, or 25. However (4,9) satisfying (7) but not equations (8) and (9) exist for lengths 19 and 21. We conjecture that for odd lengths the only weights that will occur are squares and they will only exist for lengths for which there are circulant weighing matrices [1, 2].

Lemma 1 *Let* $A = circ(a_0, a_1, \ldots, a_{m-1})$ *and* $B = circ(b_0, b_1, \ldots, b_{m-1})$ *where* $a_i, b_i \in \{0, \pm a, \pm b\}$, $i = 0, 1, \ldots, m-1$ *be* $2-SCAS(m; u_1, u_2; Z_m)$ *(i.e.* $AA^T + BB^T = (u_1 a^2 + u_2 b^2) I_m$ *and* $AB^T - BA^T = 0$*). Then there exists a* $2-SCAS(\ell; u_1, u_2; Z_\ell)$ *where* $\ell \equiv 0 (mod\ m), \ell = mi$.

Proof: Let i be a constant integer. We map T_m^i used to define A and B to $S_\ell^k = T_{mi}^{ki}$ used to define A_1 and A_2 below. Set

$$A_1 = \sum_{j=0}^{m-1} a_j S_\ell^j \quad and \quad A_2 = \sum_{j=0}^{m-1} b_j S_\ell^j$$

Now

$$AA^T = \left(\sum_{j=0}^{m-1} a_j T_m^j\right)\left(\sum_{k=0}^{m-1} a_k T_m^{-k}\right) = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\left(a_j a_k T_m^{j-k}\right),$$

$$BB^T = \left(\sum_{j=0}^{m-1} b_j T_m^j\right)\left(\sum_{k=0}^{m-1} b_k T_m^{-k}\right) = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\left(b_j b_k T_m^{j-k}\right),$$

$$AA^T + BB^T = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\left((a_j a_k + b_j b_k) T_m^{j-k}\right) = (u_1 a^2 + u_2 b^2) I_m.$$

Thus we have that

(i) If m is odd then the coefficients of T_m^σ, $\sigma = -(m-1), \ldots, -1, 1, \ldots, m-1$ are zero, and the coefficient of T_m^0 is $u_1 a^2 + u_2 b^2$. That means

$$\sum_{\substack{j,k=0 \\ j-k=\sigma}}^{m-1} (a_j a_k + b_j b_k) = 0 \quad \text{and} \quad \sum_{j=0}^{m-1} (a_j^2 + b_j^2) = u_1 a^2 + u_2 b^2. \quad (10)$$

(ii) If m is even, $m = 2n$ then we have that $T_m^n = T_m^{-n}$ and so the coefficients of T_m^σ, $\sigma = -(2n-1), \ldots, -(n+1), -(n-1), \ldots, -1, 1, \ldots, n-1, n+1, \ldots, 2n-1$ are zero, the coefficient of T_m^n plus the coefficient of T_m^{-n} is zero and the coefficients of T_m^0 is $u_1 a^2 + u_2 b^2$.That means

$$\sum_{\substack{j,k=0 \\ j-k=\sigma \\ \sigma \neq \pm n}}^{m-1} (a_j a_k + b_j b_k) = 0, \qquad \sum_{\substack{j,k=0 \\ j-k=\pm n}}^{m-1} (a_j a_k + b_j b_k) = 0$$
$$\text{and} \sum_{j=0}^{m-1} (a_j^2 + b_j^2) = u_1 a^2 + u_2 b^2. \quad (11)$$

Also

$$AB^T = \left(\sum_{j=0}^{m-1} a_j T_m^j\right)\left(\sum_{k=0}^{m-1} b_k T_m^{-k}\right) = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\left(a_j b_k T_m^{j-k}\right),$$

$$BA^T = \left(\sum_{j=0}^{m-1} b_j T_m^j\right)\left(\sum_{k=0}^{m-1} a_k T_m^{-k}\right) = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\left(a_k b_j T_m^{j-k}\right),$$

$$AB^T - BA^T \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\left(a_k b_j T_m^{j-k}\right) = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\left((a_j b_k - a_k b_j) T_m^{j-k}\right) = 0.$$

Thus we have that

(i) if m odd, then the coefficients of T_m^σ, $\sigma = -(m-1), \ldots, m-1$ are zero. That means

$$\sum_{\substack{j,k=0 \\ j-k=\sigma}}^{m-1} (a_j b_k - a_k b_j) = 0 \tag{12}$$

(ii) if m even, $m = 2n$ then the coefficients of T_m^σ, $\sigma = -(2n-1), \ldots, -(n+1), -(n-1), \ldots, n-1, n+1, \ldots, 2n-1$ are zero and the coefficient of T_m^n plus the coefficient of T_m^{-n} is zero. That means

$$\sum_{\substack{j,k=0 \\ j-k=\sigma \\ \sigma \neq \pm n}}^{m-1} (a_j b_k - a_k b_j) = 0 \quad \text{and} \quad \sum_{\substack{j,k=0 \\ j-k=\pm n}}^{m-1} (a_j b_k - a_k b_j) = 0 \tag{13}$$

Now

$$A_1 A_1^T = \left(\sum_{j=0}^{m-1} a_j S_\ell^j\right)\left(\sum_{k=0}^{m-1} a_k S_\ell^{-k}\right) = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\left(a_j a_k S_\ell^{j-k}\right),$$

$$A_2 A_2^T = \left(\sum_{j=0}^{m-1} b_j S_\ell^j\right)\left(\sum_{k=0}^{m-1} b_k S_\ell^{-k}\right) = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\left(b_j b_k S_\ell^{j-k}\right),$$

$$A_1 A_1^T + A_2 A_2^T = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\left((a_j a_k + b_j b_k) S_\ell^{j-k}\right).$$

We have that the coefficients of S_ℓ^σ are equal to the coefficients of T_m^σ for all $\sigma = -(m-1), \ldots, m-1$, and so using equations (10) or (11) we obtain

$$A_1 A_1^T + A_2 A_2^T = (u_1 a^2 + u_2 b^2) I_{mi} = (u_1 a^2 + u_2 b^2) I_\ell. \tag{14}$$

Further

$$A_1 A_2^T = \left(\sum_{j=0}^{m-1} a_j S_\ell^j\right)\left(\sum_{k=0}^{m-1} b_k S_\ell^{-k}\right) = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\left(a_j b_k S_\ell^{j-k}\right),$$

$$A_2 A_1^T = \left(\sum_{j=0}^{m-1} b_j S_\ell^j\right)\left(\sum_{k=0}^{m-1} a_k S_\ell^{-k}\right) = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\left(a_k b_j S_\ell^{j-k}\right),$$

$$A_1 A_2^T - A_2 A_1^T = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}\left(a_k b_j S_\ell^{j-k}\right) = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}(a_j b_k - a_k b_j) S_\ell^{j-k}.$$

We have that the coefficients of S_ℓ^σ are equal to the coefficients of T_m^σ for all $\sigma = -(m-1), \ldots, m-1$ and so using equations (12) or equations (13) we obtain

$$A_1 A_2^T - A_1 A_2^T = 0. \tag{15}$$

Equations (14) and (15) show that A_1 and A_2 are $2 - SCAS(\ell; u_1, u_2; Z_\ell)$ where $\ell \equiv 0(mod\ m),\ \ \ell = mi,\ \ i = 1, 2, \ldots$. □

Example 2 There exist $2 - SCAS(\ell; 1, 4; Z_\ell))$ where $\ell \equiv 0(\text{mod } 4)$ (i.e. $\ell = 4i$). Using case 1 of example 1 we set $A_1 = 0I_\ell + aT_\ell^i + bT_\ell^{2i} - aT_\ell^{3i} = aT_\ell^i + bT_\ell^{2i} - aT_\ell^{3i}$, $A_2 = 0I_\ell + aT_\ell^i + 0T_\ell^{2i} + aT_\ell^{3i} = aT_\ell^i + aT_\ell^{3i}$. Then

$$A_1 A_2^T = a^2 I_\ell + abT_\ell^i - a^2 T_\ell^{2i} + a^2 T_\ell^{-2i} + abT_\ell^{-i} - a^2 I_\ell,$$

$$A_2 A_1^T = a^2 I_\ell + abT_\ell^{-i} - a^2 T_\ell^{-2i} + a^2 T_\ell^{2i} + abT_\ell^i - a^2 I_\ell.$$

Now since $\ell = mi = 2n$ then $T_\ell^{-n} = T_\ell^n$ we have that $A_1 A_2^T - A_2 A_1^T = 0$. So we obtain our $2 - SCAS(\ell; 1, 4; Z_\ell))$ where $\ell = 4i,\ \ i = 1, 2, \ldots$. □

Example 3 There exist $2 - SCAS(\ell; 5, 5; Z_\ell))$ where $\ell \equiv 0(\text{mod } 6)$ (i.e. $\ell = 6i$). Using case 2 of example 1 we set

$$A_1 = aI_\ell - bT_\ell^i + 0T_\ell^{2i} - bT_\ell^{3i} + aT_\ell^{4i} + bT_\ell^{5i} = aI_\ell - bT_\ell^i - bT_\ell^{3i} + aT_\ell^{4i} + bT_\ell^{5i},$$

$$A_2 = bI_\ell + aT_\ell^i + 0T_\ell^{2i} + aT_\ell^{3i} + bT_\ell^{4i} - aT_\ell^{5i} = bI_\ell + aT_\ell^i + aT_\ell^{3i} + bT_\ell^{4i} - aT_\ell^{5i}.$$

Then by simple calculation and the fact that $\ell = 2n$ is even we have $A_1 A_2^T - A_2 A_1^T = 0$. Thus we have our $2 - SCAS(\ell; 5, 5; Z_\ell))$ where $\ell = 6i,\ \ i = 1, 2, \ldots$. □

3 Short circulant amicable sets

Theorem 2 *Suppose X, Y are two disjoint $\{0, \pm 1\}$ sequences of length n and weight k with zero PAF (or zero NPAF). Then there are two different constructions of $4 - SCAS(n; k, k, k, k; Z_n)$ (or $4 - SCAS(s \geq n; k, k, k, k; Z_s)$).*

Proof. Suppose $\pm a, \pm b, \pm c, \pm d$ are commuting variables. Let

$$\begin{array}{ll} A_1 = aX + bY & (or\ A_1 = \{aX + bY, 0_{s-n}\}) \\ A_2 = dX + cY & (or\ A_2 = \{dX + cY, 0_{s-n}\}) \\ A_3 = -bX + aY & (or\ A_3 = \{-bX + aY, 0_{s-n}\}) \\ A_4 = cX - dY & (or\ A_4 = \{cX - dY, 0_{s-n}\}) \end{array}$$

and

$$\begin{array}{ll} A_1 = aX + bY & (or\ A_1 = \{aX + bY, 0_{s-n}\}) \\ A_2 = -dX + cY & (or\ A_2 = \{-dX + cY, 0_{s-n}\}) \\ A_3 = -bX + aY & (or\ A_3 = \{-bX + aY, 0_{s-n}\}) \\ A_4 = cX + dY & (or\ A_4 = \{cX + dY, 0_{s-n}\}). \end{array}$$

Then A_1, A_2, A_3, A_4 are the required $4-SCAS(n; k, k, k, k; Z_n)$ (or $4-SCAS(s \geq n; k, k, k, k; Z_s)$). □

Example 4 Let $X_1 = \{1, 0, 0, 0, 0, 0, 1, 0, -1, 0, -1\}$ and $X_2 = \{0, 1, 0, 1, 1, 0, 0, 1, 0, -1, 0\}$ be two disjoint sequences of length 11, weight 9 and zero PAF. Using these sequences set A, B, C, D as in theorem 2 to obtain $4-SCAS(11; 9, 9, 9, 9; Z_{11})$.

Corollary 1 *Suppose* $C = \{c_1, c_2, \ldots, c_n\}, D = \{d_1, d_2, \ldots, d_n\}$ *are two* $\{0, \pm 1\}$ *sequences of length* n *and weight* k *with zero NPAF. Then there exists a* $4 - SCAS(s; k, k, k, k; Z_s),\ s \geq 2n$.

Proof.
Set $X_1 = circ\{c_1, c_2, \ldots, c_n, 0_{s-n}\}$ and $X_2 = \{0_{s-n}, d_1, d_2, \ldots, d_n\}$ and use theorem 2. □

Example 5 Let $C = \{1, 0, 1\}$ and $D = \{1, 1, -1\}$ be two sequences of length 3 and weight 5. Using these sequences set in corollary 1 we obtain $4-SCAS(s; 5, 5, 5, 5; Z_s),\ s \geq 6$.

Corollary 2 *Suppose* C, D *are two* $\{0, \pm 1\}$ *sequences of length* n *and weight* k *with zero PAF. Then there exists a* $4 - SCAS(s; k, k, k, k; Z_s),\ s = (m + 1)n,\ m \geq 1$.

Proof.
Set $X_1 = \{C|0_m\} = \{c_1, 0_m, c_2, 0_m, \ldots, c_n, 0_m\}$ and $X_2 = \{0_m|D\} = \{0_m, d_1, 0_m, d_2, \ldots, 0_m, d_n\}$ and use theorem 2. □

Example 6 Let $C = \{1, -1, -1, 1, 1, 1, 1, -1, -1\}$ and $D = \{0, -1, 1, -1, -1, -1, -1, 1, -1\}$ be two $\{0, \pm 1\}$ sequences of length 9 and weight 17. Using corollary 2 we obtain $4 - SCAS(9(m + 1); 17, 17, 17, 17; Z_{9(m+1)}),\ m \geq 1$.

Lemma 2 *Let* $p \equiv 1 (\mathrm{mod}\ 4)$ *be a prime power. Then there exists a* $4 - SCAS(s\frac{1}{2}(p + 1); p + 1, p + 1, p + 1, p + 1; Z_s)$ $s = (m + 1)\frac{1}{2}(p + 1),\ m \geq 1$.

Proof. Use the circulant symmetric (± 1), matrices of order $\frac{1}{2}(p+1)$, P and S found by Goethals and Seidel [4] in corollary 2. □

Theorem 3 *Let A_1, A_2, A_3, A_4 be four circulant matrices of order n with elements from $\{0, \pm 1\}$ satisfying*

$$A_1A_1^T + A_2A_2^T + A_3A_3^T + A_4A_4^T = 4kI_n. \tag{16}$$

$$A_1A_2^T = A_2A_1^T \quad \text{and} \quad A_3A_4^T = A_4A_3^T. \tag{17}$$

$$\frac{A_1 \pm A_2}{2} \quad \textit{and} \quad \frac{A_3 \pm A_4}{2} \quad \textit{are also } \{0, \pm 1\} \textit{ circulant matrices.} \tag{18}$$

Then there exists a $4 - SCAS(n; 2k, 2k; Z_n)$.

Proof. Set

$$X_1 = \frac{A_1 + A_2}{2}, \quad X_2 = \frac{A_1 - A_2}{2}, \quad X_3 = \frac{A_3 + A_4}{2}, \quad X_4 = \frac{A_3 - A_4}{2}.$$

Then $X_1X_1^T + X_2X_2^T + X_3X_3^T + X_4X_4^T = 2kI_n$. Furthermore $X_1 * X_2 = 0$ and $X_1X_2^T = X_2X_1^T$ and $X_3 * X_4 = 0$ and $X_3X_4^T = X_4X_3^T$.
Let

$$B_1 = aX_1 + bX_2, \quad B_2 = aX_2 - bX_1, \quad B_3 = aX_3 + bX_4, \quad B_4 = aX_4 - bX_3.$$

Then

$$\sum_{i=1}^{4} B_iB_i^T = 2k(a^2 + b^2)I_n \quad \textit{and} \quad \sum_{i=1}^{2}(B_{2i-1}B_{2i}^T - B_{2i}B_{2i-1}^T) = 0.$$

Thus B_1, B_2, B_3, B_4 is a $4 - SCAS(n; 2k, 2k; Z_n)$. □

Corollary 3 *Let A_1, A_2, A_3, A_4 be circulant Williamson matrices of order n which satisfy $A_i^T = A_i$, $i = 1, 2, 3, 4$. Then there exists a $4-SCAS(n; 2n, 2n; Z_n)$.*

Proof. Observe that A_1, A_2, A_3, A_4 satisfy the equations (16), (17) and (18) with $k = n$ and so from theorem 3 we have the result. □

Example 7 Let

$$A_1 = circ(-1, 1, 1, 1, 1), \ A_2 = circ(-1, 1, 1, 1, 1),$$

$$A_3 = circ(1, 1, -1, -1, 1), \ A_4 = circ(1, -1, 1, 1, -1)$$

be circulant Williamson matrices of order 5. Then

$$X_1 = circ(-1, 1, 1, 1, 1), \ X_2 = circ(0, 0, 0, 0, 0),$$

$$X_3 = circ(1, 0, 0, 0, 0), \ X_4 = circ(0, 1, -1, -1, 1)$$

and thus

$$B_1 = circ(\ \bar{a}, a, a, a, a), \quad B_2 = circ(b, \bar{b}, \bar{b}, \bar{b}, \bar{b}),$$

$$B_3 = circ(a, b, \bar{b}, \bar{b}, b), \quad B_4 = circ(\bar{b}, a, \bar{a}, \bar{a}, a)$$

is a $4 - SCAS(5; 10, 10; Z_5)$.

Type	A_1 A_2	A_3 A_4	Zero Order
(1,1,2,8)	$(0,\bar{c},a,c)$ $(0,c,b,c)$	$(0,\bar{c},b,\bar{c})$ $(0,\bar{c},d,c)$	NPAF $4n$
(1,1,4,4)	$(a,b,\bar{a})$ $(c,0,c)$	$(a,0,a)$ $(c,d,\bar{c})$	NPAF $3n$
(1,1,5,5)	$(\bar{c},a,c,0)$ $(c,\bar{d},c,0)$	$(\bar{d},b,d,0)$ $(d,c,d,0)$	NPAF $4n$
(1,1,8,8)	$(0,\bar{c},\bar{d},a,d,c)$ $(0,c,d,0,d,c)$	$(0,c,\bar{d},0,\bar{d},c)$ $(0,\bar{c},d,b,\bar{d},c)$	NPAF $6n$
(2,2,8,8)	$(d,a,\bar{d},\bar{c},b,c)$ $(c,0,c,d,0,d)$	$(d,a,\bar{d},c,\bar{b},\bar{c})$ $(c,0,c,\bar{d},0,\bar{d})$	NPAF $6n$
(1,1,5)	$(\bar{a},a,a)$ $(c,0,0)$	$(a,0,a)$ $(0,b,0)$	NPAF $3n$
(3,3)	(a,b) $(a,0)$	$(b,\bar{a})$ $(b,0)$	NPAF $2n$
(5,5)	$(a,a,\bar{a})$ $(b,b,\bar{b})$	$(a,0,a)$ $(b,0,b)$	NPAF $3n$
(6,6)	$(a,\bar{b},a)$ (b,a,b)	$(a,a,\bar{a})$ $(b,b,\bar{b})$	NPAF $3n$
(6,6,12)	$(c,a,c,b,\bar{c},a)$ $(\bar{c},b,\bar{c},\bar{a},c,b)$	$(c,a,c,\bar{a},c,\bar{a})$ $(\bar{c},b,c,\bar{b},\bar{c},\bar{b})$	NPAF $6n$
(13,13)	$(c,0,\bar{c},c,\bar{c},0,0,c,c)$ $(d,0,\bar{d},d,\bar{d},0,0,d,d)$	$(c,c,\bar{c},c,c,c,0,0,\bar{c})$ $(d,d,\bar{d},d,d,d,0,0,\bar{d})$	NPAF $9n$
(14,14)	$(a,b,\bar{b},\bar{b},b,a,a)$ $(b,\bar{a},a,a,\bar{a},b,b)$	$(\bar{b},a,\bar{b},a,\bar{b},b,b)$ $(a,b,a,b,a,\bar{a},\bar{a})$	NPAF $7n$
(17,17)	$(a,\bar{a},a,a,a,a,\bar{a},a,0)$ $(c,\bar{c},c,c,c,c,\bar{c},c,0)$	$(c,\bar{c},\bar{c},c,c,c,c,\bar{c},\bar{c})$ $(a,\bar{a},\bar{a},a,a,a,a,\bar{a},\bar{a})$	PAF $9n$

Table 3: $4-SCAS(n;u_1,u_2,u_3,u_4;Z_n)$, n small, $n\geq 1$.

4 Some constructions for short block and circulant amicable sets

Theorem 4 (Doubling the order and the type) *Let* $\{A_i\}_{i=1}^{2m}$ *be a* $2m-SAS(t; s_1, s_2, \ldots, s_u; G)$. *Then there exist a* $2m-SBAS(2t; 2s_1, 2s_2, \ldots, 2s_u; Z_2\times G)$.

Proof. Set

$$B_{2i-1} = circ(A_{2i-1}, A_{2i}) = \begin{pmatrix} A_{2i-1} & A_{2i} \\ A_{2i} & A_{2i-1} \end{pmatrix},$$
$$B_{2i} = circ(-A_{2i-1}, A_{2i}) = \begin{pmatrix} -A_{2i-1} & A_{2i} \\ A_{2i} & -A_{2i-1} \end{pmatrix},\ i = 1, \ldots, m.$$

Now $\sum_{i=1}^{2m} B_i B_i^T = \sum_{i=1}^{m} \left\{ B_{2i-1}B_{2i-1}^T + B_{2i}B_{2i}^T \right\} =$

$$= \sum_{i=1}^{m} \left\{ \begin{pmatrix} A_{2i-1} & A_{2i} \\ A_{2i} & A_{2i-1} \end{pmatrix} \begin{pmatrix} A_{2i-1}^T & A_{2i}^T \\ A_{2i}^T & A_{2i-1}^T \end{pmatrix} + \begin{pmatrix} -A_{2i-1} & A_{2i} \\ A_{2i} & -A_{2i-1} \end{pmatrix} \begin{pmatrix} -A_{2i-1}^T & A_{2i}^T \\ A_{2i}^T & -A_{2i-1}^T \end{pmatrix} \right\} =$$

$$= \sum_{i=1}^{m} \left\{ \begin{pmatrix} 2(A_{2i-1}A_{2i-1}^T + A_{2i}A_{2i}^T) & 0 \\ 0 & 2(A_{2i-1}A_{2i-1}^T + A_{2i}A_{2i}^T) \end{pmatrix} \right\} = (2s_1x_1^2 + \ldots + 2s_ux_u^2)I_{2t}$$

Moreover $\sum_{i=1}^{m} \left\{ B_{2i-1}B_{2i}^T - B_{2i}B_{2i-1}^T \right\} =$

$$= \sum_{i=1}^{m} \left\{ \begin{pmatrix} A_{2i-1} & A_{2i} \\ A_{2i} & A_{2i-1} \end{pmatrix} \begin{pmatrix} -A_{2i-1}^T & A_{2i}^T \\ A_{2i}^T & -A_{2i-1}^T \end{pmatrix} - \begin{pmatrix} -A_{2i-1} & A_{2i} \\ A_{2i} & -A_{2i-1} \end{pmatrix} \begin{pmatrix} A_{2i-1}^T & A_{2i}^T \\ A_{2i}^T & A_{2i-1}^T \end{pmatrix} \right\} = 0.$$

Thus $\{B_i\}_{i=1}^{2m}$ is a $2m - SAS(2t; 2s_1, 2s_2, \ldots, 2s_u; Z_2 \times G)$. □

Corollary 4 *Suppose there exists a* $2 - SAS(t; s_1, s_2; G)$ *then there exists a* $2 - SBAS(2t; 2s_1,\ 2s_2; Z_2 \times G)$.

Proof. We use Theorem 4 with $m = 1$ and $u = 2$. □

Corollary 5 *Suppose there exists a* $2 - SAS(t; s_1, s_2; G)$ *then there exists a* $2 - SBAS(2^st; 2^ss_1,\ 2^ss_2; EA(2^s) \times G)$ *for* $s = 0, 1, 2, \ldots$.

Proof. We use s times Corollary 4. □

Example 8 (i) $A_1 = [a]$ and $A_2 = [b]$ is a $2 - SCAS(1; 1, 1; Z_1)$. Using corollary 5 we obtain $2 - SBAS(2^s; 2^s, 2^s; EA(2^s) \times Z_1)$ for $s = 0, 1, 2 \ldots$.

(ii) $A_1 = circ(0, \bar{b}, a, b)$ and $A_2 = circ(0, b, 0, b)$ is a $2-SCAS(4; 1, 4; Z_4)$. Using corollary 5 we obtain $2-SBAS(2^{s+2}; 2^s, 2^{s+2}; EA(2^s) \times Z_4)$ for all $s = 0, 1, 2 \dots$.

(iii) $A_1 = circ(a, \bar{b}, 0, \bar{b}, a, b)$ and $A_2 = circ(b, a, 0, a, b, \bar{a})$ is a $2-SCAS(6; 5, 5; Z_6)$. Using corollary 5 we obtain $2-SBAS(2^{s+1}{\cdot}3; 2^s{\cdot}5, 2^s{\cdot}5; EA(2^s) \times Z_6)$ for all $s = 0, 1, 2 \dots$

Corollary 6 *If A_1, A_2, A_3, A_4 is a $4-SBAS(t; s_1, s_2, s_3, s_4; G)$ then there exists a $4-SBAS(2t;\ 2s_1, 2s_2, 2s_3, 2s_4; Z_2 \times G)$.*

Proof. We use Theorem 4 with $m = 2$ and $u = 4$. □

Corollary 7 *If A_1, A_2, A_3, A_4 is a $4-SAS(t; s_1, s_2, s_3, s_4; G)$ then there exists a $4-SBAS(2^s t;\ 2^s s_1, 2^s s_2, 2^s s_3, 2^s s_4; EA(2^s) \times G)$.*

Proof. We use s times Corollary 6. □

Example 9 **(i)** $A_1 = circ(0, \bar{c}, a, c)$, $A_2 = circ(0, c, b, c)$, $A_3 = circ(0, \bar{c},\ b, \bar{c})$ and $A_4 = circ(0, \bar{c}, d, c)$ is a $4-SCAS(4; 1, 1, 8, 8; Z_4)$. From corollary 7 we obtain $4-SBAS(2^{s+2}; 2^s, 2^s, 2^{s+3}, 2^{s+3}; EA(2^s) \times Z_4)$ for $s = 0, 1, 2 \dots$.

(ii) $A_1 = circ(\bar{a}, a, a)$, $A_2 = circ(c, 0, 0)$, $A_3 = circ(a, 0, a)$ and $A_4 = circ(0, b, 0)$ is a $4-SCAS(3; 1, 1, 5; Z_3)$. From corollary 7 we obtain $4-SBAS(2^s \cdot 3; 2^s, 2^s, 2^s \cdot 5; EA(2^s) \times Z_3)$ for $s = 0, 1, 2 \dots$.

Corollary 8 *If $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8$ is a $8-SBAS(t; s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8; G)$ then there exists a $8-SBAS(2t; 2s_1, 2s_2, 2s_3, 2s_4, 2s_5, 2s_6, 2s_7, 2s_8; Z_2 \times G)$.*

Proof. We use Theorem 4 with $m = 4$ and $u = 8$. □

Corollary 9 *If $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8$ is a $8-SBAS(t; s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8; G)$ then there exists a $8-SBAS(2^s t; 2^s s_1, 2^s s_2, 2^s s_3, 2^s s_4, 2^s s_5, 2^s s_6, 2^s s_7, 2^s s_8; EA(2^s) \times G)$.*

Proof. We use s times Corollary 8. □

Example 10 $A_1 = circ(\bar{g}, g, g, 0, g, c, e)$, $A_2 = circ(\bar{h}, h, h, 0, h, d, f)$, $A_3 = circ(g, \bar{g}, \bar{g}, 0, \bar{g}, c, e)$, $A_4 = circ(h, \bar{h}, \bar{h},\ 0, \bar{h}, d, f)$, $A_5 = circ(f, b, \bar{f}, 0, 0, 0, 0)$, $A_6 = circ(e, \bar{c}, e, 0, 0, 0, 0)$, $A_7 = circ(f, \bar{d}, f, 0, 0, 0, 0)$ and $A_8 = circ(e, a, \bar{e}, 0, 0, 0, 0)$ is a $8-SCAS(7; 1, 1, 3, 3, 6, 6, 8, 8; Z_7)$. Corollary 9 gives $8-SBAS(2^s 7; 2^s, 2^s, 2^s 3, 2^s 3, 2^{s+1} 3, 2^{s+1} 3, 2^{s+3}, 2^{s+3}; EA(2^s) \times Z_7)$ for $s = 0, 1, 2 \dots$.

Theorem 5 (Double the order and the type) *Let* $\{A_i\}_{i=1}^{2m}$ *be a* $2m-SAS(t; s_1, s_2, \dots, s_u; G)$. *Suppose the matrix* R *which ensures* $A_i(A_jR)^T = A_jRA_i^T$ *exists. Then for* $m = 1, 2, 4$ *there exists an* $OD(4mt; 2s_1, 2s_2, \dots, 2s_u; Z_2 \times G)$.

Proof. Theorem 4 ensures the required $2m-SBAS(2t; 2s_1, 2s_2, \dots, 2s_u; Z_2 \times G)$ exists. We use these $SBAS$ with $R_{Z_2\times G} = R_2 \times R_G$ in the appropriate Kharaghani or Goethal-Seidel array to obtain the result.

Theorem 6 (Doubling the order but not the type) *Let* $\{A_i\}_{i=1}^{2m}$ *be a* $2m-SAS(t; s_1, s_2, \dots, s_u; G)$ *Then there exist a* $2m - SBAS(2t; s_1, s_2, \dots, s_u; Z_2 \times G)$. *Furthermore* $B_{2i} * B_{2i-1} = 0,\ i = 1, 2, \dots, m$, *where* $*$ *denotes the Hadamard product.*

Proof. Set

$$B_{2i-1} = circ(A_{2i-1}, O_t), \quad B_{2i} = circ(O_t, A_{2i}),\ i = 1, \dots, m$$

It is easy to see that $B_{2i-1} * B_{2i} = 0,\ i = 1, \dots, m$. Now

$$\sum_{i=1}^{2m} B_iB_i^T = \sum_{i=1}^{m} \left\{ B_{2i-1}B_{2i-1}^T + B_{2i}B_{2i}^T \right\} =$$

$$= \sum_{i=1}^{m} \left\{ \begin{pmatrix} A_{2i-1} & O_t \\ O_t & A_{2i-1} \end{pmatrix} \begin{pmatrix} A_{2i-1}^T & O_t \\ O_t & A_{2i-1}^T \end{pmatrix} + \begin{pmatrix} O_t & A_{2i} \\ A_{2i} & O_t \end{pmatrix} \begin{pmatrix} O_t & A_{2i}^T \\ A_{2i}^T & O_t \end{pmatrix} \right\} =$$

$$= \sum_{i=1}^{m} \left\{ \begin{pmatrix} (A_{2i-1}A_{2i-1}^T + A_{2i}A_{2i}^T) & O_t \\ O_t & (A_{2i-1}A_{2i-1}^T + A_{2i}A_{2i}^T) \end{pmatrix} \right\} = (s_1x_1^2 + \dots + s_ux_u^2)I_{2t}$$

Moreover $\sum_{i=1}^{m} \left\{ B_{2i-1}B_{2i}^T - B_{2i}B_{2i-1}^T \right\} =$

$$= \sum_{i=1}^{m} \left\{ \begin{pmatrix} A_{2i-1} & O_t \\ O_t & A_{2i-1} \end{pmatrix} \begin{pmatrix} O_t & A_{2i}^T \\ A_{2i}^T & O_t \end{pmatrix} - \begin{pmatrix} O_t & A_{2i} \\ A_{2i} & O_t \end{pmatrix} \begin{pmatrix} A_{2i-1}^T & O_t \\ O_t & A_{2i-1}^T \end{pmatrix} \right\} = 0.$$

Thus $\{B_i\}_{i=1}^{2m}$ is a $2m - SBAS(2t; s_1, s_2, \dots, s_u; Z_2 \times G)$. □

Corollary 10 *If* A_1, A_2 *is a* $2 - SAS(t; s_1, s_2; G)$ *then there exists a* $2 - SBAS(2t; s_1, s_2; Z_2 \times G)$.

Proof. We use Theorem 6 with $m = 1$ and $u = 2$. □

Corollary 11 *If* A_1, A_2 *is a* $2-SAS(n; u_1, u_2; G)$ *then there are* $2-SBAS(2^sn; u_1, u_2; EA(2^s) \times G)$ *for* $s = 0, 1, 2, \dots$.

Proof. We use s times Corollary 10. □

Example 11 **(i)** $A_1 = [a]$ and $A_2 = [b]$ is a $2-SCAS(1;1,1;Z_1)$. From corollary 11 we obtain $2-SBAS(2^s;1,1;EA(2^s))$ for $s = 0,1,2\ldots$.

(ii) $A_1 = circ(0,\bar{b},a,b)$ and $A_2 = circ(0,b,0,b)$ is a $2-SCAS(4;1,4;Z_4)$. From corollary 11 we obtain $2-SBAS(2^{s+2}1,4;EA(2^s)\times Z_4)$ for $s = 0,1,2\ldots$.

(iii) $A_1 = circ(a,\bar{b},0,\bar{b},a,b)$ and $A_2 = circ(b,a,0,a,b,\bar{a})$ is a $2-SCAS(6;5,5;Z_6)$. From corollary 11 we obtain $2-SBAS(2^{s+1}\cdot 3;5,5;EA(2^s)\times Z_6)$ for $s = 0,1,2\ldots$.

Corollary 12 *If A_1,A_2,A_3,A_4 is a $4-SAS(t;s_1,s_2,s_3,s_4;G)$ then there exists a $4-SBAS(2t;s_1,\ s_2,s_3,s_4;Z_2\times G)$.*

Proof. We use Theorem 6 with $m = 2$ and $u = 4$. □

Corollary 13 *If A_1,A_2,A_3,A_4 is $4-SAS(t;s_1,s_2,s_3,s_4;G)$ then there exists a $4-SBAS(2^st;s_1,\ s_2,s_3,s_4;EA(2^s)\times G)$.*

Proof. We use s times Corollary 12. □

Example 12 **(i)** $A_1 = circ(0,\bar{c},a,c)$, $A_2 = circ(0,c,b,c)$, $A_3 = circ(0,\bar{c},b,\bar{c})$ and $A_4 = circ(0,\bar{c},d,c)$ is a $4-SCAS(4;1,1,8,8;Z_4)$. From corollary 12 we obtain $4-SBAS(2^{s+2};1,\ 1,8,8;EA(2^s)\times Z_4)$ for $s = 0,1,2\ldots$.

(ii) $A_1 = circ(\bar{a},a,a)$, $A_2 = circ(c,0,0)$, $A_3 = circ(a,0,a)$ and $A_4 = circ(0,b,0)$ is a $4-SCAS(3;1,1,5;Z_3)$. From corollary 12 we obtain $4-SBAS(2^s\cdot 3;1,1,5;EA(2^s)\times Z_3)$ for $s = 0,1,2\ldots$.

Corollary 14 *If $A_1,A_2,A_3,A_4,A_5,A_6,A_7,A_8$ is a $8-SAS(t;s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8;G)$ then there exists a $8-SBAS(2t;s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8;Z_2\times G)$.*

Proof. We use Theorem 6 with $m = 4$ and $u = 8$. □

Corollary 15 *If $A_1,A_2,A_3,A_4,A_5,A_6,A_7,A_8$ is a $8-SAS(t;s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8;G)$ then there exists a $4-SBAS(2^st;s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8;EA(2^s)\times G)$.*

Proof. We use s times Corollary 14. □

Example 13 $A_1 = circ(\bar{g}, g, g, 0, g, c, e)$, $A_2 = circ(\bar{h}, h, h, 0, h, d, f)$, $A_3 = circ(g, \bar{g}, \bar{g}, 0, \bar{g}, c, e)$, $A_4 = circ(h, \bar{h}, \bar{h}, 0, \bar{h}, d, f)$, $A_5 = circ(f, b, \bar{f}, 0, 0, 0, 0)$, $A_6 = circ(e, \bar{c}, e, 0, 0, 0, 0)$, $A_7 = circ(f, \bar{d}, f, 0, 0, 0, 0)$ and $A_8 = circ(e, a, \bar{e}, 0, 0, 0, 0)$ is a $8 - SCAS(7; 1, 1, 3, 3, 6, 6, 8, 8; Z_7)$. From corollary 15 we obtain $8 - SBAS(2^s \cdot 7; 1, 1, 3, 3, 6, 6, 8, 8; EA(2^s) \times Z_7)$ for $s = 0, 1, 2 \ldots$.

Theorem 7 (Doubling the order but not the type) *Let* $\{A_i\}_{i=1}^{2m}$ *be a* $2m-SAS(t; s_1, s_2, \ldots, s_u; G)$*. Then for* $m = 1, 2, 4$ *there exists an* $OD(4mt; s_1, s_2, \ldots, s_u; Z_2 \times G)$*.*

Proof. Theorem 6 ensures the required $2m - SBAS(2t; s_1, s_2, \ldots, s_u; Z_2 \times G)$ exists. We use these $SBAS$ with $R_{Z_2 \times G} = R_2 \times R_G$ in the appropriate Kharaghani or Goethal-Seidel array to obtain the result.

Lemma 3 *Let* $C = X + iY$*, where* $i^2 = -1$*, be a complex Hadamard matrix of order* c*. Then* X *and* Y *is* $2 - SAS(c; c; G)$*.*

Proof. Since C is a complex Hadamard matrix $CC^* = cI_c = (X + iY)(X^T - iY^T) = XX^T + YY^T + i(YX^T - XY^T)$. Thus $XY^T - YX^T = 0$ and so X and Y are a disjoint set of $2 - SAS(c; c; G)$. □

Corollary 16 *Let* $C = X + iY$*, where* $i^2 = -1$*, be a complex Hadamard matrix of order* c*. Define*

$$A_{2j-1} = x_j X + y_j Y \quad \text{and} \quad A_{2j} = y_j X - x_j Y, \quad j = 1, 2, \ldots, n$$

where $x_j,\ y_j,\quad j = 1, 2, \ldots, n$ *are commuting variables. Then,*

$$\sum_{j=1}^{2n} A_j A_j^T = c \left(\sum_{i=1}^{n} (x_j^2 + y_j^2) \right) I_c$$

and

$$A_{2j-1} A_{2j}^T - A_{2j} A_{2j-1}^T = 0, \quad \text{for all} \quad j = 1, 2, \ldots, n$$

and thus $A_j,\ j = 1, 2, \ldots, 2n$ *is a* $2n - SAS(c; \underbrace{c, c, \ldots, c}_{2n}; G)$

Proof. Since C is a complex Hadamard matrix $XX^T + YY^T = cI_c$ and $XY^T - YX^T = 0$. Now simple arithmetic gives the result. □

Remark 4 Although we have established the existence of $2n - SAS$ we do not know any appropriate R is this case.

The next theorem is a generalization of theorem 1.

Theorem 8 *Let X_1, X_2 be $\{0, \pm 1\}$ matrices of order ℓ satisfying*

$$X_1 X_1^T + X_2 X_2^T = kI_\ell \tag{19}$$

$$X_1 X_2^T - X_2 X_1^T = 0 \tag{20}$$

$$X_1 * X_2 = 0 \tag{21}$$

(i.e. $2{-}SAS(\ell; k; G_1)$) and $Y_1, Y_2, \ldots, Y_{2n}$ be a $2n{-}SAS(m; s_1, s_2, \ldots, s_u; G_2)$. Then there exists a $2n - SBAS(\ell m; ks_1, ks_2, \ldots, ks_u; G_1 \times G_2)$.

Proof. Set

$$Z_{2i-1} = X_1 \times Y_{2i-1} + X_2 \times Y_{2i}, \quad Z_{2i} = -X_1 \times Y_{2i} + X_2 \times Y_{2i-1}, \quad i = 1, 2, \ldots, n.$$

Now

$$\begin{aligned} Z_{2i-1} Z_{2i-1}^T =& (X_1 \times Y_{2i-1} + X_2 \times Y_{2i})(X_1^T \times Y_{2i-1}^T + X_2^T \times Y_{2i}^T) = \\ =& X_1 X_1^T \times Y_{2i-1} Y_{2i-1}^T + X_1 X_2^T \times Y_{2i-1} Y_{2i}^T + X_2 X_1^T \times Y_{2i} Y_{2i-1}^T \\ &+ X_2 X_2^T \times Y_{2i} Y_{2i}^T \end{aligned}$$

$$\begin{aligned} Z_{2i} Z_{2i}^T =& (-X_1 \times Y_{2i} + X_2 \times Y_{2i-1})(-X_1^T \times Y_{2i}^T + X_2^T \times Y_{2i-1}^T) = \\ =& X_1 X_1^T \times Y_{2i} Y_{2i}^T - X_1 X_2^T \times Y_{2i} Y_{2i-1}^T - X_2 X_1^T \times Y_{2i-1} Y_{2i}^T \\ &+ X_2 X_2^T \times Y_{2i-1} Y_{2i-1}^T \end{aligned}$$

Thus

$$\begin{aligned} \sum_{i=1}^{n} (Z_{2i-1} Z_{2i-1}^T + Z_{2i} Z_{2i}^T) =& (X_1 X_1^T + X_2 X_2^T) \times \sum_{i=1}^{2n} Y_i Y_i^T = \\ =& kI_\ell \times \left(\sum_{i=1}^{u} s_i a_i^2 \right) I_m = \left(\sum_{i=1}^{u} k s_i a_i^2 \right) I_{\ell m} \end{aligned} \tag{22}$$

$$\begin{aligned} Z_{2i-1} Z_{2i}^T =& (X_1 \times Y_{2i-1} + X_2 \times Y_{2i})(-X_1^T \times Y_{2i}^T + X_2^T \times Y_{2i-1}^T) = \\ =& -X_1 X_1^T \times Y_{2i-1} Y_{2i}^T + X_1 X_2^T \times Y_{2i-1} Y_{2i-1}^T - X_2 X_1^T \times Y_{2i} Y_{2i}^T \\ &+ X_2 X_2^T \times Y_{2i} Y_{2i-1}^T \end{aligned}$$

$$\begin{aligned} Z_{2i} Z_{2i-1}^T =& (-X_1 \times Y_{2i} + X_2 \times Y_{2i-1})(X_1^T \times Y_{2i-1}^T + X_2^T \times Y_{2i}^T) = \\ =& -X_1 X_1^T \times Y_{2i} Y_{2i-1}^T + X_2 X_1^T \times Y_{2i-1} Y_{2i-1}^T - X_1 X_2^T \times Y_{2i} Y_{2i}^T \\ &+ X_2 X_2^T \times Y_{2i-1} Y_{2i}^T \end{aligned}$$

$$\begin{aligned} Z_{2i-1} Z_{2i}^T - Z_{2i} Z_{2i-1}^T =& -X_1 X_1^T \times (Y_{2i-1} Y_{2i}^T - Y_{2i} Y_{2i-1}^T) \\ &+ X_2 X_2^T \times (Y_{2i} Y_{2i-1}^T - Y_{2i-1} Y_{2i}^T) \end{aligned}$$

Now

$$\begin{aligned} \sum_{i=1}^{n} \left(Z_{2i-1} Z_{2i}^T - Z_{2i} Z_{2i-1}^T \right) =& -X_1 X_1^T \times \sum_{i=1}^{n} \left(Y_{2i-1} Y_{2i}^T - Y_{2i} Y_{2i-1}^T \right) + \\ &+ X_2 X_2^T \times \sum_{i=1}^{n} \left(Y_{2i} Y_{2i-1}^T - Y_{2i-1} Y_{2i}^T \right) = \\ =& -X_1 X_1^T \times 0 + X_2 X_2^T \times 0 = 0 \end{aligned} \tag{23}$$

Thus from equations (22) and (23) we have that the $\{Z_i,\ i = 1, 2, \ldots, 2n\}$ is a $2n - SAS(\ell m; ks_1,\ ks_2, \ldots, ks_u; G_1 \times G_2)$. □

Some matrices (circulant) satisfying conditions (19), (20) and (21) can be found in table 1.

Corollary 17 *Let X_1, X_2 be $\{0, \pm 1\}$ matrices of order ℓ satisfying equations (19), (20) and (21) on the group G_1 and Y_1, Y_2 be $2 - SAS(m; s_1, s_2; G_2)$. Then there exists a $2 - SBAS(\ell m; ks_1, ks_2;\ G_1 \times G_2)$*

Proof. Use theorem 8 with $n = 1$ and $u = 2$. □

Remark 5 Again, although we have established the existence of $2m - SBAS$ we do not know any appropriate R in this case. So at the moment we are not in a position to construct orthogonal designs using these matrices. This needs further investigation.

Remark 6 Using Hadamard, weighing and complex Hadamard matrices with the following $2 - SCAS(order; type; group)$ we obtain many infinite classes (from infinite classes of weighing, Hadamard and complex Hadamard matrices) of block $2 - SBAS(order; type; group)$ There exists $2 - SCAS(order; type; group)$ for orders, types and group which are described in Table 4.

order	type	group	order	type	group
n	$1,1$	Z_n	$6n$	$4,4$	Z_{6n}
$2n$	$2,2$	Z_{2n}	$6n$	$5,5$	Z_{6n}
$4n$	$1,4$	Z_{4n}	$7n$	$4,4$	Z_{7n}
$4n$	$4,4$	Z_{4n}	$8n$	$8,8$	Z_{8n}
order	type	group	order	type	group
$10n$	$4,4$	Z_{10n}	$14n$	$8,8$	Z_{14n}
$10n$	$9,9$	Z_{10n}	$14n$	$10,10$	Z_{14n}
$12n$	$8,8$	Z_{12n}	$14n$	$13,13$	Z_{14n}
$13n$	$9,9$	Z_{13n}			

Table 4: Order and type for small 2-short amicable sets for all $n \geq 1$.

Example 14 For the construction of weighing matrices which are used in this example, see [3]. Suppose there exists Y_1, Y_2 any $2 - SAS(m; s_1, s_2; G)$. Now $CW(13, 9)$, $W(15, 9)$ and $W(18, 17)$ all exist. Set

1. $X_1 = CW(13, 9)$ and $X_2 = O_9$,
2. $X_1 = W(15, 9)$ and $X_2 = O_{15}$ and

3. $X_1 = W(18,9)$ and $X_2 = O_{18}$.

Hence by corollary 17 there exists

1. $2 - SAS(13m; 9s_1, 9s_2; Z_{13} \times G)$,
2. $2 - SAS(15m; 9s_1, 9s_2; G_{15} \times G)$ and
3. $2 - SAS(18m; 17s_1, 17s_2; G_{18} \times G)$.

Corollary 18 *Let X_1, X_2 be $\{0, \pm 1\}$ matrices of order ℓ on group G_1 satisfying equations (7), (8) and (9) and Y_1, Y_2, Y_3, Y_4 be a $4 - SAS(m; s_1, s_2, s_3, s_4; G_2)$. Then there exists a $4 - SAS(\ell m; ks_1, ks_2, ks_3, ks_4; G_1 \times G_2)$.*

Proof. We use theorem 8 with $n = 2$ and $u = 4$. □

The next theorem is a modification of theorem 2 to be used for the construction of $SBAS$.

Theorem 9 *Suppose X, Y are two disjoint $\{0, \pm 1\}$ matrices of order n on the group G and weight k satisfying $XX^T + YY^T = kI_n$. Then there exists a $4 - SBAS(n; k, k, k, k; G)$.*

Proof. Suppose $\pm a, \pm b, \pm c, \pm d$ be commuting variables. Let

$$\begin{aligned} A_1 &= aX + bY \\ A_2 &= dX + cY \\ A_3 &= -bX + aY \\ A_4 &= cX - dY \end{aligned}$$

Then A_1, A_2, A_3, A_4 are the required $4 - SBAS(n; k, k, k, k; G)$. □

Corollary 19 *Let $D = x_1X_1 + x_2X_2$ be an $OD(n; u_1, u_2)$ on the group G. Then there exists a $4 - SBAS(n; u_1 + u_2, u_1 + u_2; G)$.*

Proof. Since $D = u_1X_1 + u_2X_2$, X_1, X_2 are two disjoint $\{0, \pm 1\}$ matrices of order n on the group G and weight $k = u_1 + u_2$ satisfying $X_1X_2^T + X_2X_2^T = kI_n$ and can be used in corollary 19 to obtain the result. □

Corollary 20 *Let $D = x_1X_1 + x_2X_2$ be an $OD(n; u_1, u_2)$ on the group G. Suppose there exists an R so that $X_1(X_2R)^T = (X_2R)X_1^T$. Then there exists an $OD(4n; u_1 + u_2, u_1 + u_2; G)$.*

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