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ORTHOGONAL DESIGNS CONSTRUCTED FROM CIRCULANT WEIGHTING MATRICES

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Abstract: Circulant weighting matrices are used to construct some new infinite families of short amicable sets and amicable sets. These give new infinite families of orthogonal designs in four, six and eight variables. In particular, from circulant weight matrices $CW(n,k)$ we construct orthogonal designs $OD(8nt;k+s,k+s,k+s,k+s,k+s,k+s)$, for $s=0,1,2,3$ and all $t\geq 1$.

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1. Introduction

An *orthogonal design* of order n and type $(s_1,s_2,\ldots,s_u)$ denoted $OD(n;s_1,s_2,\ldots,s_u)$ in the variables $x_1,x_2,\ldots,x_u$, is a matrix A of order n with entries in the set $\{0,\pm x_1,\pm x_2,\ldots,\pm x_u\}$ satisfying

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$$AA^T = \sum_{i=1}^{u} (s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n. Let B_i, $i = 1, 2, 3, 4$ be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$\sum_{i=1}^{4} B_i B_i^T = \sum_{i=1}^{u} (s_i x_i^2) I_n.$$

Then the Goethals-Seidel array

$$G = \begin{pmatrix} B_1 & B_2R & B_3R & B_4R \\ -B_2R & B_1 & B_4^TR & -B_3^TR \\ -B_3R & -B_4^TR & B_1 & B_2^TR \\ -B_4R & B_3^TR & -B_2^TR & B_1 \end{pmatrix},$$

where R is the back-diagonal identity matrix, is an $OD(4n; s_1, s_2, \ldots, s_u)$. See page 107 of [1] for details.

A pair of matrices A, B is said to be amicable (anti-amicable) if $AB^T - BA^T = 0$ ($AB^T + BA^T = 0$). Following [5] a set $\{A_1, A_2, \ldots, A_{2n}\}$ of square real matrices is said to be *amicable*, if

$$\sum_{i=1}^{n} \left(A_{\sigma(2i-1)} A_{\sigma(2i)}^T - A_{\sigma(2i)} A_{\sigma(2i-1)}^T \right) = 0 \tag{1}$$

for some permutation σ of the set $\{1, 2, \ldots, 2n\}$. For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. So

$$\sum_{i=1}^{n} \left(A_{2i-1} A_{2i}^T - A_{2i} A_{2i-1}^T \right) = 0. \tag{2}$$

Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. Throughout the paper R_k denotes the back diagonal identity matrix of order k.

A set of matrices $\{B_1, B_2, \ldots, B_n\}$ of order m with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ is said to satisfy an additive property of type $(s_1, s_2, \ldots, s_u)$ if

$$\sum_{i=1}^{n} B_i B_i^T = \sum_{i=1}^{u} (s_i x_i^2) I_m. \tag{3}$$

Let $\{A_i\}_{i=1}^8$ be an amicable set of circulant matrices (or group developed of type 1) of type $(s_1, s_2, \ldots, s_u)$ of order t. Then using the $\{A_i\}_{i=1}^8$ in the Kharaghani array [5]

$$H =$$

$$\begin{pmatrix} A_1 & A_2 & A_4R_n & A_3R_n & A_6R_n & A_5R_n & A_8R_n & A_7R_n \\ -A_2 & A_1 & A_3R_n & -A_4R_n & A_5R_n & -A_6R_n & A_7R_n & -A_8R_n \\ -A_4R_n & -A_3R_n & A_1 & A_2 & -A_8^TR_n & A_7^TR_n & A_6^TR_n & -A_5^TR_n \\ -A_3R_n & A_4R_n & -A_2 & A_1 & A_7^TR_n & A_8^TR_n & -A_5^TR_n & -A_6^TR_n \\ -A_6R_n & -A_5R_n & A_8^TR_n & -A_7^TR_n & A_1 & A_2 & -A_4^TR_n & A_3^TR_n \\ -A_5R_n & A_6R_n & -A_7^TR_n & -A_8^TR_n & -A_2 & A_1 & A_3^TR_n & A_4^TR_n \\ -A_8R_n & -A_7R_n & -A_6^TR_n & A_5^TR_n & A_4^TR_n & -A_3^TR_n & A_1 & A_2 \\ -A_7R_n & A_8R_n & A_5^TR_n & A_6^TR_n & -A_3^TR_n & -A_4^TR_n & -A_2 & A_1 \end{pmatrix}$$

gives an $OD(8m; s_1, s_2, \ldots, s_u)$.

The Kharaghani array has been used in a number of papers [2, 3, 4, 5, 6, 7, 8] to obtain infinitely many families of orthogonal designs.

A set $\{A_i\}_{i=1}^4$ is said to be a *short amicable set* of length m and type (u_1, u_2, u_3, u_4) if (2) and (3) are satisfied for $n = 4$ and $u \leq 4$. Short amicable sets can be used in either the Goethals-Seidel array or the *short Kharaghani array*

$$\begin{pmatrix} A & B & CR & DR \\ -B & A & DR & -CR \\ -CR & -DR & A & B \\ -DR & CR & -B & A \end{pmatrix}$$

to form an $OD(4m; u_1, u_2, u_3, u_4)$.

A matrix $W = circ(w_1, \ldots, w_n)$, $w_i \in \{0, \pm 1\}$ which satisfies $WW^T = kI_n$ is called a *circulant weighting matrix* of order n and weight k, or $CW(n, k)$. The circulant matrix $T = circ(0, 1, 0, \cdots, 0)$ of order n is called the *shift matrix*. A summary of the known $CW(n, k)$ is given in Remark 1 of [6].

2. 2-Short amicable sets

We define *2-short amicable sets* to be two circulant matrices of order n and type (u_1, u_2) with entries from $\{0, \pm x_1, \pm x_2\}$ which satisfy

$$A_1A_1^T + A_2A_2^T = (u_1x_1^2 + u_2x_2^2)I_n, \quad A_1A_2^T - A_2A_1^T = 0,$$

or of weight k with entries from $\{0, \pm 1\}$ which satisfy

$$A_1A_1^T + A_2A_2^T = kI_n, \quad A_1A_2^T - A_2A_1^T = 0.$$

(An analogous block circulant matrix can also be considered but here we concentrate on circulant matrices.)

A 2-short amicable set is said to be *disjoint* if $A_1 * A_2 = 0$.

Lemma 1. *Suppose A_1 and A_2 are disjoint 2-short amicable sets of order n and type (u_1, u_2) with entries from $\{0, \pm x_1, \pm x_2\}$. Then,*

$$B_1 = A_1 + A_2\,, \qquad B_2 = -A_1 + A_2\,,$$

give 2-short amicable sets of type $(2u_1, 2u_2)$ and order n. The resultant 2-short amicable sets are also circulant.

Proof. We have that

$$B_1B_1^T + B_2B_2^T = 2(A_1A_1^T + A_2A_2^T) = (2u_1x_1^2 + 2u_2x_2^2)I_n$$

and

$$\begin{aligned} B_1B_2^T - B_2B_1^T =& -A_1A_1^T + A_1A_2^T - A_2A_1^T + A_2A_2^T \\ & +A_1A_1^T + A_1A_2^T - A_2A_1^T - A_2A_2^T \\ =& 2(A_1A_2^T - A_2A_1^T) = 0\,. \end{aligned}$$

Thus B_1, B_2 are 2-short amicable sets of order n and type $(2u_1, 2u_2)$. □

Lemma 2. *Suppose A_1 and A_2 are disjoint 2-short amicable sets of order n and weight k with entries from $\{0, \pm 1\}$. Then,*

$$B_1 = x_1A_1 + x_2A_2\,, \qquad B_2 = -x_2A_1 + x_1A_2\,,$$

give 2-short amicable sets of type $(2k, 2k)$ and order n with entries from $\{0, \pm x_1, \pm x_2\}$. The resultant 2-short amicable sets are also circulant.

Proof. We have that

$$B_1B_1^T + B_2B_2^T = 2(x_1^2 + x_2^2) \cdot (A_1A_1^T + A_2A_2^T) = (2kx_1^2 + 2kx_2^2)I_n$$

and

$$\begin{aligned} B_1B_2^T - B_2B_1^T =& -x_1x_2A_1A_1^T + x_1^2A_1A_2^T - x_2^2A_2A_1^T + x_2x_1A_2A_2^T \\ & +x_2x_1A_1A_1^T + x_2^2A_1A_2^T - x_1^2A_2A_1^T - x_1x_2A_2A_2^T \\ =&(x_1^2 + x_2^2)(A_1A_2^T - A_2A_1^T) = 0\,. \end{aligned}$$

Thus B_1, B_2 are 2-short amicable sets of length n and type $(2u_1, 2u_2)$. □

Example 1. 2-short amicable sets exist for $(1, 1)$: $A_1 = a$, $A_2 = b$ for all orders $n \geq 1$.

Lemma 3. *If there exists a $W = CW(n, k)$, then there exist 2-short amicable sets of order n and type (k, k).*

Proof. $A_1 = x_1W$ and $A_2 = x_2W$ are the required sets. □

Example 2. Let a, b be commuting variables. Then A_1, A_2 are 2-short amicable sets of order 7 and type $(4, 4)$:

$$A_1 = circ(-a, a, a, 0, a, 0, 0) \quad A_2 = circ(-b, b, b, 0, b, 0, 0).$$

Corollary 1. *Let p be a prime power and $n = p^2 + p + 1$. Then there exist 2-short circulant amicable sets of type (p^2, p^2) and an $OD(2n; p^2, p^2)$.*

Remark 1. We use the following definition of interleaving sequences with some zeros. If we have a sequence of length n, $X = \{x_1, \ldots, x_n\}$ and 0_{q-1} is a sequence of $q-1$ zeros then by interleaving sequence X with $q-1$ zeros we mean the sequence $\{x_1, 0_{q-1}, x_2, 0_{q-1}, \ldots, x_n, 0_{q-1}\}$. It is easy to increase the order n of circulant matrices to a multiple order qn, q integer $q \geq 1$ by interleaving sequences, of $q-1$ zeros, of the first row of the original circulant matrices. For example the circulant matrices A_1 and A_2 of the previous example, of order 7 become

$$A_1' = circ(-a,0,0,a,0,0,a,0,0,0,0,0,a,0,0,0,0,0,0,0,0)$$

and

$$A_2' = circ(-b,0,0,b,0,0,b,0,0,0,0,0,b,0,0,0,0,0,0,0,0),$$

of order 21.

3. Short amicable sets

Theorem 1. *Suppose X,Y are two disjoint $(0,\pm1)$ circulant matrices of order n and weight k satisfying $XX^T+YY^T=kI$. Then there are two short circulant amicable sets of order n and type (k,k,k,k) satisfying the additive property.*

Proof. Suppose $\pm a, \pm b, \pm c, \pm d$ are commuting variables. Let

$$\begin{aligned} A_1 &= aX+bY \\ A_2 &= dX+cY \\ A_3 &= -bX+aY \\ A_4 &= cX-dY \end{aligned} \qquad \text{and} \qquad \begin{aligned} A_1 &= aX+bY \\ A_2 &= -dX+cY \\ A_3 &= -bX+aY \\ A_4 &= cX+dY. \end{aligned}$$

Then A_1, A_2, A_3, A_4 are the required short circulant amicable sets. □

Lemma 4. *Let X,Y be two $(0,\pm1)$ circulant matrices of order n and weight k satisfying $XX^T+YY^T=kI$. Then there are short circulant amicable sets of order (length) n and type (k,k) satisfying the additive property.*

Proof. Set $A=aX$; $B=bX$; $C=aY$; $D=bY$. Then $AA^T+BB^T+CC^T+DD^T=k(a^2+b^2)I_{4n}$ and $AB^T-BA^T+CD^T-DC^T=0$. □

Lemma 5. *Suppose there exists a $W = CW(n,k)$ with $W * I = 0$. Then there exist short circulant amicable sets of order n and types*

(i) $(1,1,k,k)$,

(ii) $(2,2,2k,2k)$,

(iii) $(1,1,4k,4k)$ *when* $W^T * W = W^T * I = 0$.

Proof. Let a,b,c,d be commuting variables. Then let for (i)

$$A_1 = aI, \quad A_2 = bI,$$
$$A_3 = cW, \quad A_4 = dW,$$

for (ii)

$$A_1 = bI + aW, \quad A_2 = cI + dW,$$
$$A_3 = bI - aW, \quad A_4 = cI - dW,$$

and for (iii)

$$A_1 = bI + aW - aW^T, \quad A_2 = cW + cW^T,$$
$$A_3 = aW + aW^T, \qquad A_4 = dI + cW - cW^T.$$

□

Example 3. Order 7 with $W = circ(0,-1,1,1,0,1,0)$ gives $OD(28;1,1,4,4)$ and $OD(28;2,2,8,8)$. Order 21 with

$$W = circ(0,-1,0,0,1,0,0,1,0,0,0,0,0,1,0,0,0,0,0,0,0)$$

gives $OD(84;1,1,4,4)$, $OD(84;2,2,8,8)$ and with $CW(21,16)$ gives $OD(84;1,1,16,16)$ and $OD(84;2,2,32,32)$ using (i) and (ii) and another $OD(84;1,1,16,16)$ using (iii).

Corollary 2. *Let p be a prime power and $n = p^2+p+1$. Then there exist*

(i) $OD(4n;1,1,p^2,p^2)$,

(ii) $OD(4n;2,2,2p^2,2p^2)$, *and*

(iii) $OD(4nq; 2, 2, 4p^2, 4p^2)$, q *odd,* $q \geq 3$,

constructed using the short Kharaghani array.

Proof. Use the $CW(p^2 + p + 1)$ and the circulant amicable sets described in Lemma 3 to obtain the four circulant matrices A, B, C, D from A_1, A_2, A_3, A_4 respectively. This gives the result. □

Lemma 6. ([6]) *If there exists a circulant weighting matrix* $W = circ(w_0, w_1, \ldots, w_{n-1})$ $W(n, k)$*, then there exists a circulant weighting matrix* $Q = circ(w_0, 0_{t-1}, w_1, 0_{t-1}, \ldots, w_{n-1}, 0_{t-1},)$ *of order* nt *and weight* k.

Theorem 2. *Suppose there exists a* $W = CW(n, k)$*. Then there are circulant amicable sets of order* n *and type*

(i) (k, k, k, k, k, k, k, k),

(ii) $(k + 1, k + 1, k + 1, k + 1, k + 1, k + 1)$, *if* $n - k \geq 1$,

(iii) $(k + 2, k + 2, k + 2, k + 2, k + 2, k + 2)$, *if* $n - k \geq 2$,

(iv) $(k + 3, k + 3, k + 3, k + 3, k + 3, k + 3)$, *if* $n - k \geq 3$.

Proof.

(i) Use the matrices $A_1 = aW$, $A_2 = bW$, $A_3 = cW$, $A_4 = dW$, $A_5 = eW$, $A_6 = fW$, $A_7 = gW$, $A_8 = hW$.

(ii)-(iv) Use the matrices

$$\begin{array}{rrrrcrrrr}
A_1 = & aW & +bB & +cC, & \qquad & A_2 = & dW & +eB & +fC, \\
A_3 = & -bW & +aB & -cA, & & A_4 = & eW & -dB & -fA, \\
A_5 = & -cW & +aC & +bA, & & A_6 = & fW & -dC & +eA, \\
A_7 = & cB & -bC & +aA, & & A_8 = & -fB & +eC & +dA.
\end{array}$$

Let $W = circ(w_0, w_1, \ldots, w_{n-1})$.

(ii) Suppose that one zero appears in position i. Then set $A = T^i$ and $B = C = \mathbf{0}$.

(iii) Suppose that two zeros appear in positions i and j. Then set $A = T^i$, $B = T^j$ and $C = \mathbf{0}$.

(iv) Suppose that three zeros appear in positions i, j and m. Then set $A = T^i$, $B = T^j$ and $C = T^m$.

□

Example 4. (a) For order 7 we use the circulant weighting matrix: $W = circ(-1, 1, 1, 0, 1, 0, 0)$.

(i) Using the matrices $A_1 = aW$, $A_2 = bW$, $A_3 = cW$, $A_4 = dW$, $A_5 = eW$, $A_6 = fW$, $A_7 = gW$, $A_8 = hW$ in Theorem 2(i), we obtain an orthogonal design of order 56 and of type $(4, 4, 4, 4, 4, 4, 4, 4)$.

(ii) Using the matrices $A = T^3$, $B = C = \mathbf{0}$ in Theorem 2(ii) we obtain an orthogonal design of order 56 and of type $(5, 5, 5, 5, 5, 5)$.

(iii) Using the matrices $A = T^3$, $B = T^5$ and $C = \mathbf{0}$ in Theorem 2(iii) we obtain an orthogonal design of order 56 and of type $(6, 6, 6, 6, 6, 6)$.

(iv) Using the matrices $A = T^3$, $B = T^5$ and $C = T^6$ in Theorem 2(iv) we obtain an orthogonal design of order 56 and of type $(7, 7, 7, 7, 7, 7)$.

(b) For order 13 using the circulant weighting matrix:

$$W = circ(0, 1, 0, 1, 1, 0, 0, -1, -1, 1, 1, -1, 1),$$

in Theorem 2 we obtain an orthogonal design

$$OD(104; 9, 9, 9, 9, 9, 9, 9, 9)$$

and $OD(104; 9 + s, 9 + s, 9 + s, 9 + s, 9 + s, 9 + s)$, for $s = 1, 2, 3$.

Corollary 3. *Let p be a prime power and $n = p^2 + p + 1$. Then there exist: an orthogonal design $OD(8nt; p^2, p^2, p^2, p^2, p^2, p^2, p^2, p^2)$ and orthogonal designs $OD(8nt; p^2+s, p^2+s, p^2+s, p^2+s, p^2+s, p^2+s)$, $s = 1, 2, 3$, for all $t \geq 1$.*

Proof. Use $CW(p^2 + p + 1, p^2)$ and the circulant amicable sets described in Theorem 2 in the Kharaghani array. □

Theorem 3. *Suppose there exist short (circulant or block) amicable sets, A_1, A_2, A_3, A_4 of order n and type (u_1, u_2, u_3, u_4). Then there exist short block amicable sets of type $(2u_1, 2u_2, 2u_3, 2u_4)$ for lengths $2n$.*

Proof. Now

$$A_1 A_2^T - A_2 A_1^T + A_3 A_4^T - A_4 A_3^T = 0\,.$$

Write

$$B_i = circ(A_j, A_k) = \begin{bmatrix} A_j & A_k \\ A_k & A_j \end{bmatrix}.$$

Then form

$$\begin{aligned} B_1 &= circ(A_1, \quad A_4)\,, & B_2 &= circ(A_2, \quad -A_3)\,, \\ B_3 &= circ(A_3, \quad A_2)\,, & B_4 &= circ(A_4, \quad -A_1)\,. \end{aligned}$$

Thus

$$B_1 B_2^T - B_2 B_1^T + B_3 B_4^T - B_4 B_3^T = 0\,.$$

Then, noting that the additive property for $\{A_i\}_{i=1}^4$ gives the additive property for $\{B_i\}_{i=1}^4$, we have the result. □

Remark 2. The matrices B_i of the previous theorem can be used with $R_{2n} = R_n \times R_2$ in the Kharaghani array to obtain orthogonal designs.

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