

An infinite family of Hadamard matrices with fourth last pivot $\frac{n}{2}$

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Abstract

We show that the equivalence class of Sylvester Hadamard matrices give an infinite family of Hadamard matrices in which the fourth last pivot is $\frac{n}{2}$. Analytical examples of Hadamard matrices of order n having as fourth last pivot $\frac{n}{2}$ are given for $n = 16$ and 32 . In each case this distinguished case with the fourth pivot $\frac{n}{2}$ arose in the equivalence class containing the Sylvester Hadamard matrix.

Key words and phrases: Gaussian elimination, pivot size, complete pivoting, Sylvester Hadamard matrices.

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1 The growth conjecture for Hadamard matrices

Let A be an $n \times n$ real matrix, and let \mathbf{b} be a real n -vector. In his fundamental work on backward error analysis Wilkinson [9] proved that when the linear system $A \cdot \mathbf{x} = \mathbf{b}$ is solved in floating point arithmetic by Gaussian elimination (GE) with either partial or complete pivoting the computed solution $\hat{\mathbf{x}}$ satisfies

$$(A + E) \cdot \hat{\mathbf{x}} = \mathbf{b}$$

where the norm of the perturbation matrix E can be bounded from above as follows

$$\|E\|_{\infty} \leq g(n, A) \cdot f(n) \cdot u \|A\|_{\infty}$$

where u is the unit roundoff, $f(n)$ is a cubic polynomial of n , and $g(n, A)$ is the growth factor defined by

$$g(n, A) = \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{|a_{11}^{(0)}|}$$

where $a_{ij}^{(k)}$, $k = 1, 2, \dots, n-1$ denotes the (i, j) th element that occurs at the k -th step of elimination. The elements $a_{ii}^{(n-1)}$ are called pivots. We say that a matrix A is completely

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pivoted (CP) if the rows and columns have been permuted so that Gaussian elimination with no pivoting satisfies the requirements for complete pivoting.

Let $g(n, A)$ denote the growth associated with Gaussian elimination on a CP $n \times n$ matrix A and $g(n) = \sup\{g(n, A)\}$. The problem of determining $g(n)$ for various values of n is called the *growth problem*. The determination of $g(n)$ remains a challenging problem. Wilkinson in [9],[10] noted that there were no known examples of matrices for which $g(n) > n$. In [1] Cryer conjectured that “ $g(n, A) \leq n$, with equality if and only if A is a Hadamard matrix”. This was proved to be false in [5].

An Hadamard matrix H of order n is an $n \times n$ matrix with elements ± 1 and $HH^T = nI$. These matrices were first studied by Sylvester [8] (see also [7]) who observed that if H is an Hadamard matrix of order n , then

$$\begin{bmatrix} H & H \\ H & -H \end{bmatrix} \quad (1)$$

is also an Hadamard matrix of order $2n$. Indeed, using the matrix of order 2, we have:

Lemma 1 (Sylvester [8]) *There is an Hadamard matrix of order 2^t for all positive integers t .*

We call matrices of order 2^t formed by Sylvester’s construction (1) Sylvester Hadamard matrices. Any Hadamard matrix which can be obtained from a Sylvester Hadamard matrix by rearrangement of the rows and/or columns and multiplying rows and columns by -1 is said to be in the equivalence class of Sylvester Hadamard matrices. Alternatively we say A and B are equivalent if there exist monomial matrices P and Q so that $B = PAQ$.

Since Wilkinson’s initial conjecture seems to be connected with Hadamard matrices the following conjecture was posed (see [1],[2]):

Let A be an $n \times n$ CP Hadamard matrix. Reduce A by GE. Then

- (i) $g(n, A) = n$.
- (ii) The four last pivots are equal to $\frac{n}{2}$ or $\frac{n}{4}, \frac{n}{2}, \frac{n}{2}, n$.

The equality in (i) above has been proved for a certain class of $n \times n$ Hadamard matrices [2]. Cryer [1] have shown (ii) for the pivots $\frac{n}{2}, \frac{n}{2}$ and n . Day and Peterson [2] has shown that the values $\frac{n}{2}$ or $\frac{n}{4}$ appear in the fourth pivot when Gaussian Elimination not necessarily with complete pivoting is applied to a Hadamard matrix. They posed the conjecture that when Gaussian elimination with complete pivoting is done on a Hadamard matrix the value of $\frac{n}{2}$ is impossible. In [3] an Hadamard matrix of order 16 is given which has fourth last pivot $\frac{n}{2}$. It was not known how to categorize this matrix. In the present paper we give ten more matrices of order 16 having fourth last pivot 8. All these matrices and the one in [3] arose in the equivalence class containing the Sylvester Hadamard matrix. Furthermore a 32×32 CP Hadamard matrix in the equivalence class containing the Sylvester Hadamard matrix is found, with fourth last pivot 16. This leads us to pose the following conjecture for the fourth last pivot:

Conjecture for the fourth last pivot

Let A be an $n \times n$ CP Hadamard matrix. Reduce A by GE. Then the fourth last pivot can take the value $\frac{n}{2}$ only for Hadamard matrices in the equivalence class containing the Sylvester Hadamard matrix.

Interesting results in the size of pivots appears when GE is applied to CP skew-Hadamard and weighing matrices of order n and weight $n - 1$. In these matrices, the growth is also large, and experimentally, we have been led to believe it equals $n - 1$ and special structure appears for the first few and last few pivots [6].

2 An infinite class with fourth last pivot $\frac{n}{2}$

Lemma 2 *Suppose that the application of GE gives a CP Hadamard matrix of order n with fourth last pivot $\frac{n}{2}$. Then there is a CP Hadamard matrix of order $2n$ with fourth last pivot n .*

Proof. Suppose Q and P are the matrices of order n which effect the required GE on H and $D = PHQ$ is the resultant diagonal matrix of pivots. Then $|\det Q| \cdot |\det P| = 1$ and $|\det H| = |\det D|$. Now consider

$$E = \begin{bmatrix} P & 0 \\ P & -P \end{bmatrix} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} \begin{bmatrix} Q & -Q \\ 0 & Q \end{bmatrix} = \begin{bmatrix} PH & PH \\ 0 & 2PH \end{bmatrix} \begin{bmatrix} Q & -Q \\ 0 & Q \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & 2D \end{bmatrix}.$$

If the fourth last pivot of D was $\frac{n}{2}$ then the fourth last pivot of E is n . We note that E is in the equivalence class of the Sylvester Hadamard matrix but is not a Sylvester Hadamard matrix. Multiplying out the matrices shows that the CP structure has been retained. \square

Corollary 1 *There exists an Hadamard matrix in the equivalence class of Sylvester Hadamard matrices of order 2^t with fourth last pivot 2^{t-1} for all $t \geq 4$.*

Proof. Let $H_{2^1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $H_{2^s} = \begin{bmatrix} H_{2^{s-1}} & H_{2^{s-1}} \\ H_{2^{s-1}} & -H_{2^{s-1}} \end{bmatrix}$.

In this paper we have shown there exist Hadamard matrices of orders $2^4 = 16$ and $2^5 = 32$ with fourth last pivot 2^3 and 2^4 respectively. Hence by the lemma and induction the statement of the corollary holds and we have an infinite number of Sylvester Hadamard matrices of order 2^t with fourth last pivot 2^{t-1} for all $t \geq 4$. \square

Numerical Examples

(i) $n = 16$

For Hadamard matrices of order 16 it is proved in [4] that there are 5 equivalence classes and examples of each are given.

In our subsequent experiments we took 40000 cases from each of the five equivalence classes and applied GECP to each. The following ten matrices are CP Hadamard matrices, where $+$ stands for 1 and $-$ stands for -1 . When Gaussian Elimination is applied to them they give the following pivot structure

$$(1, 2, 2, 4, 3, \frac{8}{3}, 2, 4, 4, 4, 4, 8, 8, 8, 8, 16).$$

Thus they have their fourth last pivot equal to $\frac{16}{2}$. All of them belong to Class I. The matrix in [3] which also gives as fourth last pivot 8 and attains the above pivot structure, also belongs to Class I. Class I is the equivalence class containing the Sylvester Hadamard matrix. Our experiments did not yield a single example from classes II, III, IV, V with fourth last pivot $\frac{16}{2}$.

$$\begin{bmatrix} + + + + + - + - + + + + - \\ + - + - + - - - - - + - - \\ + + - - + + - - + - - + + \\ - - + - - - - - - + + + \\ + + + - + - + - + - + + \\ - - + - + - + - + - + - \\ + - - + + + + + - + - - \\ - - - + + + + - - - + + + \\ + + + + - + - + - + + + \\ + - + - + - + - + - + + \\ - + - - + - - + + + + - \\ + - + + + - - + + + + - \\ - + + + + + - + + + + + \\ + - - - + + + - + + + + \end{bmatrix}$$

$$\begin{bmatrix} + - + + + + - - + + - + + \\ - - - + + + - + + + + + \\ + - - - + + + - - - - + + \\ - - + - - + - - - - - + - \\ + - - + - + - + - + + + \\ - + + - + + + - + + + + \\ + + - + - + + + + - + + \\ + + + - + - + - + - + - \\ + + + + + - + - + - + - \\ + - + + - - - - - + + + \\ - - - - + - - + + + - + - \\ - + - + + + - - + - + - \\ + - + + + - - - - + + + \\ + + - - + - - - - + + + \\ + - - - - - + - + + - + \end{bmatrix}$$

$$\begin{bmatrix} + - + + - - - + + + - + - \\ - - + + + - + + + + + + \\ - + - - - + + + + + + + \\ - + + + + + - + + - + - \\ - + - + - - - + - + - + \\ + - + + + + - + - + - + \\ + + + + - - - + + + + + \\ - + + + - + + - + + + + \\ - + - + + + + - + + + - \\ + - - + + + - + + + + - \\ + + + - + - + - + + + + \\ + + + + + + + - + + + + \\ + - + + + + + - + + + + \\ + + + - + + + + + + + \\ - + + + + + + + + + + \end{bmatrix}$$

$$\begin{bmatrix} + + + + - + + - + + + - + \\ + - + - + - + + - + + + \\ + - - + + - - - - - + + + \\ + + - - + + + - - - - + + \\ - - + + + + + + - - - + - \\ + - + + + + + - + - - - \\ + - - + + - - + + + + + \\ + + - - + - + - - - + - \\ - - - - + + + + + + - - \\ + + + - + + + - - - + + + \\ - + - + + + - + - + + + \\ - + - + + + - + + + + - \\ + + + - - - - + + - + - \\ + - + + + - - + + + - + \\ - + + + + - - + + + + + \end{bmatrix}$$

$$\begin{bmatrix} - - + + - + + + - - - + - \\ - + - + - + - - + + + + \\ + + + + - + - + - + + + \\ + - - + - - - + + + - + \\ + + + - + - - + - - + + \\ + + + + + + - + - + + + \\ + - - + - + + + - - + - \\ + - - + + + + + - + + + \\ + - + - - + + + - + + + \\ + - - - + + + - + + + + \\ + - + - - + + + - + + + \\ + - + + + - + + + - + + \\ + + + + + + - + + + - \\ + - + + + - - + + + - \\ - + + + - - - + + + - \end{bmatrix}$$

$$\begin{bmatrix} + + + - + - + - + + + + + \\ + - + + + - - - + + + - + \\ + + - + - + + - - - + - + \\ + - - + - + - - - - - + \\ - + - + + + + + + - - + \\ + - + - + + + + + + - - \\ + + - - - - + + + - - - \\ + - - + - + + - + + + + \\ + - + + + + + - - + - - \\ + + + - + - - + - + + + \\ - - + - + + + - + - + + \\ + - - + + + + + - - + + \\ + + - + + + + + - - + + \\ + - - + + + - + + - + + \\ - - - + + - - + + + - - \\ + - - - - + - + + + - + \end{bmatrix}$$

$$\begin{bmatrix} + + + + + + + + + - + - \\ + - + - - + - + - - + + \\ - + + - + + + - - - + + \\ - - + + + + - - - + + + \\ + - - - + - + - + + + - \\ + - - - + - + - + + + - \\ + + + + - - - + - - + + \\ + - + + + + - - - - + + \\ + + - + + + - - + - + - \\ + - + + + - + + - + - \\ + + + - + - + + - + - \\ - + - + - + + - + - + - \\ + - + + + - + - + + + + \\ - - + + + - + + + - + + \\ - + + + - - - + + + - \\ + - + + - - - + + + + + \\ + + - + + - - - - + + + \\ + + - + + - - - - + + + \end{bmatrix}$$

$$\begin{bmatrix} + + - - - + - - + - + + + \\ + - - + - - - + - + + + \\ - - - + - - - + - - + + \\ + - + + + - - - - + - + \\ + + - + + + + - - + - + \\ - - + + + + - + - + + + \\ + + + - - + + - + + + + \\ + - + + + - + - + + + + \\ + - + + - + + + + - + + \\ + + + - + + + - - - + + \\ + + + - + + + - + - + - \\ + - - + + - + - + + + + \\ + - + + - + - + + - + - \\ - + - + - + - + + - + - \\ - + + + - - - + + + - + \\ - + + + - - - + + + - + \\ - + + + + - - + + + - + \end{bmatrix}$$

$$\begin{bmatrix} - & + & + & + & + & + & + & - & + & + & + & + & - & - & - \\ - & - & + & - & + & + & + & - & + & - & - & - & + & + \\ - & + & - & - & - & + & + & + & + & + & + & + & + & + \\ + & + & - & - & + & + & - & + & - & + & - & + & - & + \\ - & - & + & + & - & - & - & - & - & + & + & + & + & + \\ + & + & - & + & + & - & - & + & - & + & - & + & - & + \\ + & - & - & + & - & - & + & + & + & + & + & + & - & - \\ + & + & + & - & + & - & + & + & - & - & + & - & - & + \\ + & - & + & + & - & + & + & + & + & + & + & + & - & - \\ - & + & - & - & + & + & - & + & - & + & + & + & + & + \\ - & + & - & + & - & + & - & - & + & - & + & + & + & + \\ - & - & + & - & - & - & + & - & - & + & - & - & - & - \\ - & + & + & - & - & - & + & - & - & + & - & - & + & - \\ - & - & - & - & + & + & + & - & - & + & + & + & - & - \end{bmatrix} \quad \begin{bmatrix} + & - & - & + & + & + & + & + & - & - & + & - & - & - \\ + & + & + & - & - & + & + & + & - & - & + & - & - & + \\ - & - & + & - & - & + & - & - & + & + & + & + & - & - \\ - & + & - & - & - & + & - & - & + & - & - & - & - & + \\ + & - & - & - & + & - & + & - & - & + & + & + & - & + \\ + & - & - & - & + & - & - & - & + & - & + & - & + & + \\ - & + & - & - & + & - & - & - & + & - & + & - & + & + \\ - & + & - & - & + & - & - & - & + & - & + & - & + & + \\ + & - & - & - & + & - & - & - & + & - & + & - & + & + \\ + & - & - & - & + & - & - & - & + & - & + & - & + & + \\ - & + & - & - & + & - & - & - & + & - & + & - & + & + \\ - & + & - & - & + & - & - & - & + & - & + & - & + & + \\ - & - & + & - & - & - & + & - & - & + & - & - & + & + \\ + & - & + & + & - & - & + & - & - & + & - & - & + & + \\ + & + & + & - & - & - & + & - & - & + & - & - & + & + \end{bmatrix}$$

(ii) $n = 32$

Let us consider a modified Sylvester's construction of the form

$$\begin{bmatrix} H & H \\ PHQ & -PHQ \end{bmatrix} \quad (2)$$

where H an Hadamard matrix and P, Q are monomial permutation matrices of $+1$'s and -1 's. By this we mean that P and Q have exactly one nonzero entry in every row and in every column, and this nonzero entry is $+1$ or -1 . P gives the permutation and change of sign of rows; Q of columns. If we choose as H the following 16×16 Hadamard matrix, not necessarily CP, which has as fourth last pivot 8 and as PHQ the following equivalent to H matrix

$$H = \begin{bmatrix} + & + & + & + & + & + & + & + & - & - & + & - & - & - & + \\ + & - & - & + & + & - & - & - & - & - & + & - & - & - & + \\ - & + & + & + & + & - & - & - & - & - & + & + & - & - & + \\ + & + & - & + & - & - & - & - & - & - & + & - & - & - & + \\ + & - & - & + & - & - & - & - & - & - & + & - & - & - & + \\ - & - & + & + & - & - & - & - & - & - & + & - & - & - & + \\ + & - & - & + & - & - & - & - & - & - & + & - & - & - & + \\ + & + & - & - & + & - & - & - & - & - & + & - & - & - & + \\ + & - & - & + & - & - & - & - & - & - & + & - & - & - & + \\ + & - & - & + & - & - & - & - & - & - & + & - & - & - & + \\ - & + & - & - & + & - & - & - & - & - & + & - & - & - & + \\ - & + & - & - & + & - & - & - & - & - & + & - & - & - & + \\ + & - & - & + & - & - & - & - & - & - & + & - & - & - & + \\ - & - & - & + & - & - & - & - & - & - & + & - & - & - & + \\ + & + & - & - & + & - & - & - & - & - & + & - & - & - & + \\ + & + & - & - & + & - & - & - & - & - & + & - & - & - & + \end{bmatrix} \quad PHQ = \begin{bmatrix} + & - & + & + & + & - & - & - & + & - & - & - & + & + \\ + & + & - & + & + & - & - & - & + & - & - & - & + & + \\ - & + & + & - & - & + & + & - & - & + & - & - & + & + \\ - & + & - & + & - & - & - & - & + & - & - & + & + \\ + & + & + & - & + & + & - & - & + & - & - & + & + \\ + & - & - & + & - & - & - & - & + & - & - & + & + \\ + & + & + & - & - & + & - & - & + & - & - & + & + \\ + & + & + & - & - & + & - & - & + & - & - & + & + \\ + & + & + & - & - & + & - & - & + & - & - & + & + \\ + & + & + & - & - & + & - & - & + & - & - & + & + \\ + & + & + & - & - & + & - & - & + & - & - & + & + \\ + & + & + & - & - & + & - & - & + & - & - & + & + \\ + & + & + & - & - & + & - & - & + & - & - & + & + \\ + & + & + & - & - & + & - & - & + & - & - & + & + \\ + & + & + & - & - & + & - & - & + & - & - & + & + \end{bmatrix}$$

then, the 32×32 Hadamard matrix of construction (2) has as fourth last pivot 16.

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