A new algorithm for computer searches for orthogonal designs

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Abstract

We present a new algorithm for computer searches for orthogonal designs. Then we use this algorithm to find new sets of sequences with entries from $\{0, \pm a, \pm b, \pm c, \pm d\}$ on the commuting variables $a,\ b,\ c,\ d$ with zero autocorrelation function.

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1 Introduction

An *orthogonal design* A, of order n, and type $(s_1,\ s_2,\ \ldots,\ s_u)$, denoted $OD(n; s_1, s_2, \ldots, s_u)$, on the commuting variables $x_1, x_2, \ldots, x_u$, is a square matrix of order n with entries from $\{0, \pm x_1, \ldots, \pm x_u\}$ where for each k, $\pm x_k$ occurs s_k times in each row and column and such that the distinct rows are pairwise orthogonal.

In other words

$$AA^T = (s_1 x_1^2 + \cdots + s_u x_u^2) I_n$$

where I_n is the identity matrix.

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Some small orthogonal designs are given in the following example, see [6].

Example 1 Some small orthogonal designs. $OD(4;1,1,1,1)$ is the Williamson array.

$$\underset{OD(2;1,1)}{\begin{bmatrix} x & y \\ y & -x \end{bmatrix}}, \underset{OD(4;1,1,1,1)}{\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}}, \underset{OD(4;1,1,2)}{\begin{bmatrix} a & b & b & d \\ -b & a & d & -b \\ -b & -d & a & b \\ -d & b & -b & a \end{bmatrix}}, \underset{OD(4;1,1)}{\begin{bmatrix} a & 0 & -c & 0 \\ 0 & a & 0 & c \\ c & 0 & a & 0 \\ 0 & -c & 0 & a \end{bmatrix}}$$

□

It is known that the maximum number of variables in an orthogonal design is $\rho(n)$, the Radon number, defined by $\rho(n) = 8c + 2^d$, where $n = 2^a b$, b odd, and $a = 4c + d, 0 \leq d < 4$.

A weighing matrix $W = W(n,k)$ is a square matrix with entries $0, \pm 1$ having k non-zero entries per row and column and inner product of distinct rows zero. Hence W satisfies $WW^T = kI_n$, and W is equivalent to an orthogonal design $OD(n;k)$. The number k is called the *weight* of W. If $k = n$, that is, all the entries of W are ± 1 and $WW^T = nI_n$, then W is called an Hadamard matrix of order n. In this case $n = 1, 2$ or $n \equiv 0 (mod\ 4)$.

Given a set A of ℓ sequences, the sequences $A_j = \{a_{j1}, a_{j2}, ..., a_{jn}\}$, $j = 1, \ldots, \ell$, of length n the *non-periodic autocorrelation function* (abbreviated as NPAF) $N_A(s)$ is defined as

$$N_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n-s} a_{ji} a_{j,i+s}, \qquad s = 0, 1, ..., n-1. \tag{1}$$

If $A_j(z) = a_{j1} + a_{j2}z + \ldots + a_{jn}z^{n-1}$ is the associated polynomial of the sequence A_j, then

$$A(z)A(z^{-1}) = \sum_{j=1}^{\ell} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ji} a_{jk} z^{i-k} = N_A(0) + \sum_{j=1}^{\ell} \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}). \tag{2}$$

Given A_ℓ, as above, of length n the *periodic autocorrelation function* (abbreviated as PAF) $P_A(s)$ is defined, reducing $i + s$ modulo n, as

$$P_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n} a_{ji} a_{j,i+s}, \qquad s = 0, 1, ..., n-1. \tag{3}$$

The following theorem which uses four circulant matrices in the Gorthals-Seidel array is very useful in our construction for orthogonal designs.

Theorem 1 *[3, Theorem 4.49] Suppose there exist four circulant matrices* A, B, C, D *of order* n *satisfying*

$$AA^T + BB^T + CC^T + DD^T = fI_n$$

Let R *be the back diagonal matrix of order* n*, i.e.* $R = (r_{ij})$ *where* $r_{i,n-i+1} = 1$*, and* $r_{ij} = 0$ *if* $j \neq n - i + 1$*. Then*

$$GS = \begin{pmatrix} A & BR & CR & DR \\ -BR & A & D^TR & -C^TR \\ -CR & -D^TR & A & B^TR \\ -DR & C^TR & -B^TR & A \end{pmatrix}$$

is a $W(4n, f)$*, i.e a weighing matrix of order* $4n$ *and weight* f*, when* A, B, C, D *are* $(0, 1, -1)$ *matrices, and an orthogonal design* $OD(4n; s_1, s_2, \ldots, s_u)$ *on* $x_1, x_2, \ldots, x_u$ *when* A, B, C, D *have entries from* $\{0, \pm x_1, \ldots, \pm x_u\}$ *and* $f = \sum_{j=1}^{u}(s_j x_j^2)$*.* □

Corollary 1 *If there are four sequences* A, B, C, D *of length* n *with entries from* $\{0, \pm x_1, \pm x_2, \pm x_3, \pm x_4\}$ *with zero periodic or non-periodic autocorrelation function, then these sequences can be used as the first rows of circulant matrices which can be used in the Goethals-Seidel array to form an* $OD(4n; s_1, s_2, s_3, s_4)$*. We note that if there are sequences of length* n *with zero non-periodic autocorrelation function, then there are sequences of length* $n + m$ *for all* $m \geq 0$*.* □

2 The results

We now give the new orthogonal designs of order 44 constructed from four circulants found by the algorithm described in sections 3 and 4.

Tuple		Sequences										
(2,2,4,36)	NPAF=0	b	a	a	a	-a	a	a	-a	d	a	-a
		b	a	a	a	-a	a	-a	a	-d	-a	a
		b	-a	-a	-a	a	-a	-a	-a	-c	a	a
		b	-a	-a	-a	a	-a	a	a	c	-a	-a
(2,2,8,32)	NPAF=0	a	d	c	-d	d	-d	d	-d	-d	c	d
		a	-d	-c	d	-d	d	-d	d	d	-c	-d
		b	d	c	-d	d	d	d	d	d	-c	-d
		b	-d	-c	d	-d	-d	-d	-d	-d	c	d
(1,1,1,36)	PAF=0	a	-b	b	-b	-b	-b	b	b	b	-b	b
		b	b	b	0	b	b	0	b	0	0	0
		b	-b	b	b	b	-b	-b	-b	b	-b	c
		b	-b	b	b	b	-b	-b	-b	b	-b	d
(1,1,2,32)	NPAF=0	a	0	a	a	a	a	-c	-a	0	-a	a
		a	a	0	a	c	-a	a	-a	a	0	a
		a	-a	0	a	a	-b	-a	-a	0	a	-a
		a	a	0	-a	a	-d	-a	a	0	-a	-a
(1,1,8,32)	PAF=0	a	-a	-c	a	a	0	a	a	-c	-a	a
		a	a	c	a	-a	0	-a	a	c	a	a
		a	-a	c	a	a	-b	-a	-a	-c	a	-a
		a	a	-c	a	-a	-d	a	-a	c	-a	-a
(1,2,9,32)	PAF=0	a	a	-a	-a	c	-a	-a	d	-a	-a	-d
		a	-a	-d	-a	a	-a	-a	-c	-a	-a	d
		a	a	-b	-a	-a	-d	a	-a	a	-a	d
		a	a	-a	a	-d	-a	-a	-d	-a	a	-d
(2,2,8,32)	NPAF=0	b	b	-c	-b	-c	a	-b	b	b	-b	-b
		b	-b	-b	b	b	a	c	-b	c	b	-b
		b	b	-c	b	c	-d	b	-b	b	-b	b
		b	b	b	b	b	d	c	-b	-c	-b	b

Table 1: Four variable sequences of Length 11 which yield $OD(44; s_1, s_2, s_3, s_4)$.

Tuple		Sequences										
(1,3,38)	PAF=0	b	-b	b	b	b	a	-b	-b	-b	b	-b
		b	-b	b	-b	b	b	b	-c	-b	-b	b
		b	b	b	0	b	b	-b	-b	b	-c	-b
		b	0	b	b	-b	b	c	-b	b	b	b
(1,4,34)	PAF=0	a	a	-a	a	-a	a	-a	-a	c	0	c
		a	a	-a	-a	a	-a	a	a	-a	-a	-b
		a	-a	a	0	-a	-a	a	-a	-a	-a	0
		a	-c	0	-a	-a	-a	-a	-a	-a	0	c
(1,4,36)	PAF=0	a	-b	b	-b	-b	-b	b	b	b	-b	b
		b	-b	-b	-b	-b	c	-c	-b	-b	-b	0
		b	b	b	0	-b	b	-b	-b	-b	c	c
		b	-b	b	-b	b	-b	-b	b	b	-b	0
(1,5,36)	PAF=0	a	-b	-b	b	b	-b	b	-b	-b	b	b
		b	-b	b	b	-c	b	b	c	b	b	0
		b	b	-c	b	-b	b	-b	0	-b	-b	-c
		b	-b	-b	-b	b	-b	-b	b	b	b	c
(1,11,30)	PAF=0	a	a	-a	-a	-a	a	c	-a	c	c	0
		a	a	-a	-a	-a	c	-a	-c	-a	a	c
		a	a	a	a	-c	c	-c	a	c	c	0
		a	-a	-a	a	-a	a	-a	a	a	-a	-b
(1,2,38)	NPAF=0	b	b	b	-b	b	a	-b	b	-b	-b	-b
		b	-b	-c	b	-b	-b	-b	b	b	b	0
		b	b	b	-b	-b	b	-b	c	b	-b	0
		b	b	b	-b	b	0	b	-b	b	b	b
(2,11,31)	PAF=0	a	-b	-b	b	b	-c	b	c	b	b	-c
		a	b	-b	-b	-c	c	-b	-b	-b	b	c
		b	-b	b	-b	-c	-b	-b	b	-b	-b	c
		b	-c	b	-b	-b	-b	b	-b	-b	-c	-c

Table 2: Three variable sequences and Length 11 which yield $OD(44; s_1, s_2, s_3)$.

Tuple		Sequences
(3,6,32)	PAF=0	a a a a -a -a b -a -b -a 0
		a b -a a -a -a a a -a c b
		a -a -a 0 -a a -a -a -a c -b
		a -a -c -a a -a -a -a 0 -a b
(5,7,32)	PAF=0	a a a a -a -a -a c b -a c
		a -b a -a -a a a -a -b -a c
		a -a -a a -a -a -a -c b -a c
		a -a a -a a a b a b a -b
(1,10,14)	NPAF=0	a -a -a a b a b b b -b
		a -a -a -a b -a b -b -b b
		a c -a 0 0 0 0 0 0 0
		a 0 a 0 0 0 0 0 0 0
(1,8,24)	PAF=0	a b -b b b b -b -b -b b -b
		b -b 0 -b -b -b 0 0 0 -b 0
		b b -b b c c c -c 0 0 0
		b -b -b -b c -c c c 0 0 0
(1,10,16)	PAF=0	a b -b b b b -b -b -b b -b
		b -b 0 -b -b -b 0 0 0 -b 0
		c c -c c 0 c 0 0 0 0 0
		c c -c -c 0 -c 0 0 0 0 0
(1,5,21)	PAF=0	a b -b b b b -b -b -b b -b
		b -b 0 -b -b -b 0 0 0 -b 0
		b b -b c 0 c 0 0 0 0 0
		b 0 b c -c -c 0 0 0 0 0
(1,10,26)	PAF=0	a b -b b b b -b -b -b b -b
		b -b 0 -b -b -b 0 0 0 -b 0
		b -b -b b c b c c c -c 0
		b -b -b -b c -b c -c -c c 0

Table 2(cont.): Three variable sequences and Length 11 which yield $OD(44; s_1, s_2, s_3)$.

3 The method

Notation 1 For the remainder of this paper we use the following notations.

1. $\mathcal{N}$ denotes the set of non negative integers.
2. $\mathcal{N}^k$ denotes the vector space $\mathcal{N}^k = \underbrace{\mathcal{N} \times \mathcal{N} \times \cdots \times \mathcal{N}}_{k\ times}$ with elements
$$\mathbf{v} \in \mathcal{N}^k,\ \mathbf{v^T} = [v_1, v_2, \ldots, v_k],\ v_i \in \mathcal{N},\ i = 1, 2, \ldots, k.$$
3. $\mathcal{N}^{k\times\ell}$ will be the matrix space with dimension $k \times \ell$ and elements from $\mathcal{N}$. That is if $M \in \mathcal{N}^{k\times\ell}$ then
$$M = \begin{bmatrix} m_{11} & m_{12} & \ldots & m_{1\ell} \\ m_{21} & m_{22} & \ldots & m_{2\ell} \\ \vdots & \vdots & & \vdots \\ m_{k1} & m_{k2} & \ldots & m_{k\ell} \end{bmatrix} = \begin{bmatrix} \mathbf{m_1^T} \\ \mathbf{m_2^T} \\ \vdots \\ \mathbf{m_k^T} \end{bmatrix}$$
with $m_{ij} \in \mathcal{N}$, $\mathbf{m_i} \in \mathcal{N}^\ell$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, \ell$. □

In this paper we are interested in the construction of orthogonal designs using four circulant matrices in the Gorthals-Seidel array. Specifically, for positive integers $s_1, s_2, \ldots, s_u$ and odd n, the method searches for four circulant matrices A_1, A_2, A_3, A_4 or order n with entries from $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ such that

$$A_1A_1^T + A_2A_2^T + A_3A_3^T + A_4A_4^T = \left(\sum_{i=1}^{u} s_i x_i^2\right) I_n \tag{4}$$

Definition 1 If A_1, A_2, A_3, A_4 are $n \times n$ circulant matrices with entries from $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ and the first row of A_j has m_{ij} entries of the kind $\pm x_i$, then the $u \times 4$ matrix $M = (m_{ij})$ is called the *entry matrix* of (A_1, A_2, A_3, A_4). □

The elements of the entry matrices satisfy the following conditions.

(i) $\displaystyle\sum_{j=1}^{4} m_{ij} = s_i$ for $1 \le i \le u$

(ii) $\displaystyle\sum_{i=1}^{u} m_{ij} \le n$ for $1 \le j \le 4$

Thus the rows of the entry matrices refer to the variables x_i and the columns to the circulant matrices A_1, A_2, A_3, A_4 which are constructed from four sequences of length n as described in Corollary 2.

Definition 2 Suppose that the row sum of A_j is $\sum_{i=1}^{u} p_{ij} x_i$ for $1 \leq j \leq 4$. Then the $u \times 4$ integral matrix $P = (p_{ij})$ is called the *sum matrix* of (A_1, A_2, A_3, A_4). The *fill matrix* of (A_1, A_2, A_3, A_4) is $M - abs(P)$. The content of A_i is determined by the i-th columns of the sum and fill matrices. □

The following theorem may be used to find the sum matrix of a solution of 4.

Theorem 2 (Eades Sum Matrix Theorem) The sum matrix P of a solution of 4 satisfies $PP^T = diag(s_1, s_2, \dots, s_u)$. □

For more details about the construction of orthogonal designs which uses entry matrices, see [3].

Let D be an $OD\,(\,4n\,;\,u_1\,,\,u_2\,,\,\dots\,,\,u_t\,)$ with entries from the set $\{0, \pm x_1, \pm x_2, \dots, \pm x_t\}$ where $x_1, x_2, \dots, x_t$ are commuting variables. Using the terminology of [3], the symbols M_i represent the non isomorphic *entry matrices* of the orthogonal design.

Herein (because we use many non isomorphic entry matrices from different orthogonal designs) we will use the *type* of the orthogonal design in the symbol of the entry matrices, so that seeing the entry matrix we can tell from which orthogonal design it comes. For D we will write $M_{(u_1,u_2,\dots,u_t),i}$ for its non isomorphic entry matrices. Then we can write the entry matrices using their rows as follows

$$M_{(u_1,u_2,\dots,u_t),i} = \begin{bmatrix} \mathbf{v_1^T} \\ \mathbf{v_2^T} \\ \vdots \\ \mathbf{v_t^T} \end{bmatrix} \in \mathcal{N}^{t\times 4},\ \mathbf{v_j} \in \mathcal{N}^4,\ j = 1, 2, \dots, t.$$

Let $\mathcal{D}_{(u_1,u_2,\dots,u_t)}$ be the union of all non isomorphic entry matrices of the orthogonal design $OD(4n; u_1, u_2, \dots, u_t)$. We will write $M_{(u_1,u_2,\dots,u_t),i}|_{\mathcal{D}_{u_k,u_j}}$ for the entry matrix $M_{(u_1,u_2,\dots,u_t),i}$ after we eliminate all rows except from rows k and j. That is

$$M_{(u_1,u_2,\ldots,u_t),i}|_{\mathcal{D}_{u_k,u_j}} = \begin{bmatrix} \mathbf{v_k^T} \\ \\ \mathbf{v_j^T} \end{bmatrix} \in \mathcal{N}^{2\times 4}.$$

In order to illustrate the above notations and definitions we give the following example.

Example 2 Suppose we are searching for the $OD(4n; u_1, u_2, u_3, u_4) = OD(20; 2, 3, 6, 9)$. There are three entry matrices:

$$M_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 1 & 2 & 0 \\ 0 & 2 & 2 & 5 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 4 & 0 \\ 2 & 2 & 0 & 5 \end{bmatrix}, M_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}.$$

Using our terminology these are:

$$M_{(u_1,u_2,u_3,u_4),1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 1 & 2 & 0 \\ 0 & 2 & 2 & 5 \end{bmatrix}, \quad M_{(u_1,u_2,u_3,u_4),2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 4 & 0 \\ 2 & 2 & 0 & 5 \end{bmatrix},$$

$$M_{(u_1,u_2,u_3,u_4),3} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}.$$

With this terminology we can easily see that by setting the first variable equal to zero (i.e. eliminating the first row $\mathbf{v_1^T}$) in the above entry matrices, we obtain the following entry matrices of an orthogonal design $OD(20; 3, 6, 9)$:

$$M_{(u_2,u_3,u_4),1} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 1 & 2 & 0 \\ 0 & 2 & 2 & 5 \end{bmatrix}, \quad M_{(u_2,u_3,u_4),2} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 4 & 0 \\ 2 & 2 & 0 & 5 \end{bmatrix},$$

$$M_{(u_2,u_3,u_4),3} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}.$$

Similarly the entry matrices of an orthogonal design $OD(20; 5, 6, 9)$ obtained by setting first and second variable be the same symbol (i.e. replacing rows

$\mathbf{v_1^T}, \mathbf{v_2^T}$ by row $\mathbf{v_1^T} + \mathbf{v_2^T}$) are

$$M_{(u_1+u_2,u_3,u_4),1} = \begin{bmatrix} 2 & 2 & 1 & 0 \\ 3 & 1 & 2 & 0 \\ 0 & 2 & 2 & 5 \end{bmatrix}, \quad M_{(u_1+u_2,u_3,u_4),2} = \begin{bmatrix} 2 & 2 & 1 & 0 \\ 1 & 1 & 4 & 0 \\ 2 & 2 & 0 & 5 \end{bmatrix},$$

$$M_{(u_1+u_2,u_3,u_4),3} = \begin{bmatrix} 2 & 2 & 1 & 0 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}.$$

□

Now from [3] we have that from an orthogonal design over t variables we can obtain an orthogonal design over $t-1$ variables by "killing" one variable (i.e. setting one variable equal to zero) or "equating" two variables (i.e. setting two variables be the same symbol). If we do these many times we obtain the following lemma:

Lemma 1 *If an orthogonal design $OD(4n; u_1, u_2, \ldots, u_t)$ exist then the following orthogonal designs exist:*

i) All orthogonal designs $OD(4n; u_{i_1}, u_{i_2}, \ldots, u_{i_k})$ for all $k = 1, 2, \ldots, t$, over k variables and for all $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, t\}$.

ii) All orthogonal designs $OD\left(4n; \sum_{j=k_0=1}^{k_1} u_{i_j}, \sum_{j=k_1+1}^{k_2} u_{i_j}, \ldots, \sum_{j=k_{m-1}+1}^{k_m} u_{i_j}\right)$ over m variables where $1 \le m \le t$, $1 \le k_i \le t$, $\forall\ i = 1, 2, \ldots m$, $k_1 \le k_2 \le \ldots \le k_m$, $u_{i_j} \ne u_{i_\ell}$, $\forall\ j, \ell = 1, 2, \ldots, k_m$ and $i \ne \ell$, $\bigcup_{j=1}^{k_m} u_{i_j} \subseteq \{u_1, u_2, \ldots, u_t\}$.

Proof. By equating and killing variables we obtain the desirable result. □

¿From the above lemma it is obvious that

Corollary 2 *If there exist $k: \ 1 \le k \le t$ and $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, t\}$ such that an orthogonal design $OD(4n; u_{i_1}, u_{i_2}, \ldots, u_{i_k})$ does not exist then an orthogonal design $OD(4n; u_1, u_2, \ldots, u_t)$ can not exist.*

Our method relies on searching for $OD(4n; u_k, u_j)$, $1 \leq k, j \leq t$, in two variables, which is much faster, rather than using the matrix based algorithm, described in [3] for $OD(4n; u_1, u_2, \ldots, u_t)$, in t variables, which is much slower. Then we use the extension algorithm to construct the orthogonal design we want.

Moreover we do not have to check all non isomorphic entry matrices $M_{(u_k,u_j),i}$ but only a few of them. We also can select the k, j in such way that we minimize the set of $M_{(u_k,u_j),i}$ we have to search.

4 The algorithm

This algorithm relies on the two previously mentioned algorithms (the matrix based algorithm and the extension algorithm) given in [1, 3, 5] and in [2, 4] respectively.

Our new algorithm combines features of both algorithms with a new result given here for the first time to obtain a new, much faster, algorithm. It is an exhaustive search algorithm (i.e if the orthogonal design exists it will be found otherwise it does not exist constructed from four sequences).

Let D be the orthogonal design $OD(4n; u_1, u_2, \ldots, u_t)$. The steps of our algorithm are:

Step 0: Find all non-isomorphic entry matrices $M_{(u_1,u_2,\ldots,u_t),i}$ for D as it is described in [3].

Step 1: For $k, j \in \{1, 2, \ldots, t\}, k < j$ find all non-isomorphic entry matrices $M_{(u_k,u_j),i}$ for the orthogonal design $OD(4n; u_k, u_j)$.

Step 2: For all the above $\binom{t}{2}$ combinations check if $M_{(u_1,u_2,\ldots,u_t),i}|_{\mathcal{D}_{(u_k,u_j)}}$ is equal with any $M_{(u_k,u_j),\ell} \in \mathcal{D}_{(u_k,u_j)}$. Ignore similar matrices $M_{(u_1,u_2,\ldots,u_t),i}|_{\mathcal{D}_{(u_k,u_j)}}$ produced after using the two rows of $M_{(u_1,u_2,\ldots,u_t),i}$ and eliminate all others rows. These are the matrices that can be extended to $M_{(u_1,u_2,\ldots,u_t),i}$ and thus these might product the orthogonal design D.

Step 3: Select the k, j which give the smallest number of entry matrices $M_{(u_1,u_2,\ldots,u_t),i}|_{\mathcal{D}_{(u_k,u_j)}}$.

Step 4: Apply first algorithm (matrix based algorithm) to the selected entry matrices specified in Step 3, and find all $OD(4n; u_k, u_j)$.

Step 5: For each $OD(4n; u_k, u_j)$ found in Step 4, apply the second algorithm (extension algorithm), by replacing each zero by a unique variable x_p, $p = 1, 2, \ldots, 4n - (u_k + u_j)$.

Step 6: Exhaustively search all possibilities then if the solution exists, it will be found, otherwise an $OD(4n; u_1, u_2, \ldots, u_t)$ does not exist constructed by four sequences.

Example 3 We will apply our algorithm to search for an orthogonal design $D = OD(36; u_1, u_2, u_3) = OD(36; 6, 7, 21)$.

Step 0: The ten following matrices are all the non-isomorphic entry matrices $M_{(u_1,u_2,u_3),i}$ for D as it is described in [3]:

$$1)\begin{bmatrix} 3 & 1 & 2 & 0 \\ 3 & 1 & 1 & 2 \\ 2 & 6 & 6 & 7 \end{bmatrix}, \quad 2)\begin{bmatrix} 3 & 1 & 2 & 0 \\ 1 & 3 & 1 & 2 \\ 4 & 4 & 6 & 7 \end{bmatrix}, \quad 3)\begin{bmatrix} 3 & 1 & 2 & 0 \\ 1 & 1 & 1 & 4 \\ 4 & 6 & 6 & 5 \end{bmatrix},$$

$$4)\begin{bmatrix} 3 & 1 & 2 & 0 \\ 1 & 1 & 3 & 2 \\ 4 & 6 & 4 & 7 \end{bmatrix}, \quad 5)\begin{bmatrix} 1 & 1 & 4 & 0 \\ 3 & 1 & 1 & 2 \\ 4 & 6 & 4 & 7 \end{bmatrix}, \quad 6)\begin{bmatrix} 1 & 1 & 4 & 0 \\ 1 & 1 & 3 & 2 \\ 6 & 6 & 2 & 7 \end{bmatrix},$$

$$7)\begin{bmatrix} 1 & 1 & 4 & 0 \\ 1 & 1 & 1 & 4 \\ 6 & 6 & 4 & 5 \end{bmatrix}, \quad 8)\begin{bmatrix} 1 & 1 & 2 & 2 \\ 3 & 1 & 1 & 2 \\ 4 & 6 & 6 & 5 \end{bmatrix}, \quad 9)\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 3 & 2 \\ 6 & 6 & 4 & 5 \end{bmatrix},$$

$$10)\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 4 \\ 6 & 6 & 6 & 3 \end{bmatrix}$$

□

Step 1: We have that

$$|\mathcal{D}_{(u_1,u_2)}| = 10, \; |\mathcal{D}_{(u_1,u_3)}| = 53, \; |\mathcal{D}_{(u_2,u_3)}| = 21$$

Step 2: By setting the first variable equal to zero (i.e. eliminating the first row $\mathbf{v}_1^T$) we get only 5 non-isomorphic entry matrices $M_{(u_1,u_2,u_3),i}|_{\mathcal{D}_{(u_2,u_3)}}$

from the 21 entry matrices of the orthogonal design $OD(36;7,21)$. Those come from the matrices $M_{(u_1,u_2,u_3),i}$ numbered i=1,2,3,8, and 10 above by deleting the first row.

By setting the second variable equal to zero we get only 10 non-isomorphic entry matrices $M_{(u_1,u_2,u_3),i}|_{\mathcal{D}_{(u_1,u_3)}}$ from the 53 entry matrices of the orthogonal design $OD(36;6,21)$. Those come from the matrices $M_{(u_1,u_2,u_3),i}$ numbered $i=1,2,\ldots,10$ above by deleting the second row.

By setting the third variable equal to zero we get only 10 non-isomorphic entry matrices $M_{(u_1,u_2,u_3),i}|_{\mathcal{D}_{(u_1,u_2)}}$ from the 10 entry matrices of the orthogonal design $OD(36;6,7)$. Those come from the matrices $M_{(u_1,u_2,u_3),i}$ numbered $i=1,2,\ldots,10$ above by deleting the third row.

Step 3: Clearly in the case $k=2$ and $j=3$ we have fewer entry matrices to check than in any of the other cases, i.e five.

Step 4: Now we get all the quadruples of sequences with PAF=0 or NPAF=0, which can be used for the construction of $OD(36;7,21)$, via the Goethals-Seidel Array. This is applied to all five entry matrices described in steps 2 and 3.

Step 5: For each $OD(4n;u_k,u_j)=OD(36;7,21)$ found in Step 4, apply the second algorithm (extension algorithm), by replacing the zero of the sequences by the unique variables x_p, $p=1,2,\ldots,8$.

We want to make clear that if an $OD(36;6,7,21)$ existed it would have been found. We did not find any solutions by step 5 and thus, since our search is exhaustive for the orthogonal design $OD(36;6,7,21)$, this design does not exist using four sequences. □

Example 4 Applying our algorithm we try to find the $OD(36;6,8,19)$ and the $OD(36;7,8,19)$. There are 22 non-isomorphic entry matrices $M_{(6,8,19),i}$ corresponding to the orthogonal design $OD(36;u_1,u_2,u_3)=OD(36;6,8,19)$ and 22 for the second orthogonal design $OD(36;u_4,u_2,u_3)=OD(36;7,8,19)$.

By setting the first variable equal to zero we get only 17 non-isomorphic entry matrices $M_{(6,8,19),i}|_{\mathcal{D}_{(u_2,u_3)}}$ for the $OD(36;8,19)$.

We observe that the matrices $M_{(6,8,19),i}|_{\mathcal{D}_{(u_2,u_3)}}$ are exactly the same as the matrices $M_{(7,8,19),i}|_{\mathcal{D}_{(u_2,u_3)}}$ for the second orthogonal design.

Thus by searching those 17 non isomorphic entry matrices we can perform an exhaustive search for both orthogonal designs. Using the matrix based algorithm we would have had to check 44 entry matrices using three variables for both designs.

Applying our algorithm and following the same process as in the previous example we find, among others, the following solutions, which have PAF=0:

$$OD(36;6,8,19)$$

$$\begin{array}{ccccccccc} b & -c & 0 & b & b & b & a & c & -a \\ b & b & -b & b & c & -a & -b & c & a \\ c & b & -b & -b & -a & -b & b & -a & 0 \\ b & -b & -b & -c & b & -a & b & -a & 0 \end{array}$$

$$OD(36;7,8,19)$$

$$\begin{array}{ccccccccc} a & -b & -b & -b & c & -a & -c & -b & -c \\ b & -a & a & b & -c & -b & b & -b & -c \\ b & -b & a & a & b & b & -b & 0 & -c \\ a & -b & -b & -b & b & a & b & 0 & c \end{array}$$

□

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