Short amicable sets and Kharaghani type orthogonal designs

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Dedicated to George Szekeres on his 90th Birthday

Abstract

Short amicable sets were introduced recently and have many applications. The construction of short amicable sets had lead to the construction of many orthogonal designs, weighing matrices and Hadamard matrices. In this paper we give some constructions for short amicable sets as well as some multiplication theorems. We also present a table of the short amicable sets known to exist and we construct some infinite families of short amicable sets and orthogonal designs.

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1 Introduction

An *orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$ denoted $OD(n; s_1, s_2, \ldots, s_u)$ in the variables $x_1, x_2, \ldots, x_u$, is a matrix A of order n with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$AA^T = \sum_{i=1}^{u} (s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n. Let B_i, $i = 1, 2, 3, 4$ be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$\sum_{i=1}^{4} B_i B_i^T = \sum_{i=1}^{u} (s_i x_i^2) I_n.$$

Then the Goethals-Seidel array

$$G = \begin{pmatrix} B_1 & B_2R & B_3R & B_4R \\ -B_2R & B_1 & B_4^TR & -B_3^TR \\ -B_3R & -B_4^TR & B_1 & B_2^TR \\ -B_4R & B_3^TR & -B_2^TR & B_1 \end{pmatrix}$$

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where R is the back-diagonal identity matrix, is an $OD(4n; s_1, s_2, \ldots, s_u)$. See page 107 of [2] for details.

A pair of matrices A, B is said to be amicable (anti-amicable) if $AB^T - BA^T = 0$ ($AB^T + BA^T = 0$). To be consistent in the notation of this paper we will also denote these as $2-SAS(n; s_1, s_2; G)$, where the group G is described below. Following [4] a set $\{A_1, A_2, \ldots, A_{2n}\}$ of square real matrices is said to be *amicable* if

$$\sum_{i=1}^{n} \left(A_{\sigma(2i-1)} A_{\sigma(2i)}^T - A_{\sigma(2i)} A_{\sigma(2i-1)}^T \right) = 0 \quad (1)$$

for some permutation σ of the set $\{1, 2, \ldots, 2n\}$. For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. So

$$\sum_{i=1}^{n} \left(A_{2i-1} A_{2i}^T - A_{2i} A_{2i-1}^T \right) = 0. \quad (2)$$

Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. Throughout the paper R_k denotes the back diagonal identity matrix of order k.

A set of matrices $\{B_1, B_2, \ldots, B_n\}$ of order m with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ is said to satisfy an additive property of type $(s_1, s_2, \ldots, s_u)$ if

$$\sum_{i=1}^{n} B_i B_i^T = \sum_{i=1}^{u} (s_i x_i^2) I_m. \quad (3)$$

Let $\{A_i\}_{i=1}^{8}$ be an amicable set of circulant matrices (or group developed or type 1) of type $(s_1, s_2, \ldots, s_u)$ and order t. We denote these by $8-AS(t; s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8; Z_t)$ (or $8-AS(t; s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8; G)$ for group developed or type 1). In all cases, the group G of the matrix is such that the extension by Seberry and Whiteman [9] of the group from circulant to type 1 allows the same extension to R. Then the Kharaghani array [4]

$$H = \begin{pmatrix}
A_1 & A_2 & A_4R_n & A_3R_n & A_6R_n & A_5R_n & A_8R_n & A_7R_n \\
-A_2 & A_1 & A_3R_n & -A_4R_n & A_5R_n & -A_6R_n & A_7R_n & -A_8R_n \\
-A_4R_n & -A_3R_n & A_1 & A_2 & -A_8^TR_n & A_7^TR_n & A_6^TR_n & -A_5^TR_n \\
-A_3R_n & A_4R_n & -A_2 & A_1 & A_7^TR_n & A_8^TR_n & -A_5^TR_n & -A_6^TR_n \\
-A_6R_n & -A_5R_n & A_8^TR_n & -A_7^TR_n & A_1 & A_2 & -A_4^TR_n & A_3^TR_n \\
-A_5R_n & A_6R_n & -A_7^TR_n & -A_8^TR_n & -A_2 & A_1 & A_3^TR_n & A_4^TR_n \\
-A_8R_n & -A_7R_n & -A_6^TR_n & A_5^TR_n & A_4^TR_n & -A_3^TR_n & A_1 & A_2 \\
-A_7R_n & A_8R_n & A_5^TR_n & A_6^TR_n & -A_3^TR_n & -A_4^TR_n & -A_2 & A_1
\end{pmatrix}$$

is an $OD(8t; s_1, s_2, \ldots, s_u)$.

The Kharaghani array has been used in a number of papers [1, 3, 4] to obtain infinitely many families of orthogonal designs. Research has yet to be initiated to explore the algebraic restrictions imposed an amicable set by the required constraints.

Short amicable set were defined in [1] as a set of matrices $\{A_i\}_{i=1}^{4}$ of order m and type (u_1, u_2, u_3, u_4), abbreviated as $4-SAS(m; u_1, u_2, u_3, u_4; G)$, if (2) and (3) are satisfied for $n = 4$ and $u \leq 4$. $4-SAS(m; u_1, u_2, u_3, u_4; G)$ can be used in either the Goethals-Seidel array or the *short Kharaghani array*

$$\begin{bmatrix} A & B & CR & DR \\ -B & A & DR & -CR \\ -CR & -DR & A & B \\ -DR & CR & -B & A \end{bmatrix}$$

to form an $OD(4m; u_1, u_2, u_3, u_4)$. In all cases, the group G of the matrices in the *amicable set* is such that the extension by Seberry and Whiteman [9] of the group from circulant to type 1 allows the same extension to R.

In general a set of $2n$ matrices of order m and type $(s_1, s_2, \ldots, s_u)$ that satisfy equations (2) and (3) will be denoted as $2n - SAS(m; s_1, s_2, \ldots, s_u; G)$. Moreover if these matrices are circulant they will be denoted as $2n - SCAS(m; s_1, s_2, \ldots, s_u; Z_m)$.

In [1] where all this was first defined was mentioned that:

Remark 1

1. If there exists a $2 - SAS(n; s_1, s_2; G)$ and a $2 - SAS(n; s_3, s_4; G)$ then there exists a $4 - SAS(n; s_1, s_2, s_3, s_4; G)$.
2. If there exists a $2 - SAS(n; s_1, s_2; G)$, $2 - SAS(n; s_3, s_4; G)$, $2 - SAS(n; s_5, s_6; G)$ and a $2 - SAS(n; s_7, s_8; G)$ there exists an $8 - AS(n; s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8; G)$.
3. If there exists a $4 - SAS(n; s_1, s_2, s_3, s_4; G)$ and a $4 - SAS(n; s_5, s_6, s_7, s_8; G)$ there exists an $8 - AS(n; s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8; G)$.

Thus we can obtain many classes of $4 - SAS(n; s_1, s_2, s_3, s_4; G)$ combining together two pairs of the given $2 - SAS(n; s_1, s_2; G)$ and $2 - SAS(n; s_3, s_4; G)$. Moreover, in Table 2, we give some $4 - SAS(m; u_1, u_2, u_3, u_4; Z_m)$ that can not be constructed by this method.

Generally, unless we have other information regarding the structure, we are unable to ensure that the matrix R with the desired properties for the Kharaghani, Goethals-Seidel or short Kharaghani arrays exists unless the amicable sets have been group generated (circulant or type 1) or constructed from blocks of these kinds. Thus is we have the required matrix R_i for the group G_i, $i = 1, 2$ then $R_G = R_1 \times R_2$ will be the required matrix for $G = G_1 \times G_2$, (see [9]).

Let A_1 and A_2 be matrices of order m. We define $circ(A_1, A_2) = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}$. Amicable sets made from $2n$ such block circulant matrices will be called *block amicable sets*, *short block amicable sets* or *2-short block amicable sets*, $2n - SBAS(2m; s_1, s_2, \ldots, s_u; G)$, $n = 1, 2, 4$, where, using R_t for the back-diagonal matrix of order t, $G = Z_2 \times Z_m$ and $R_G = R_2 \times R_m$. Here, if A_1 and A_2 are circulant, then we use the backdiagonal matrix of the same order for R ensuring $A_i(A_jR)^T = A_jRA_i^T$. The required $R_G = R_2 \times R$.

A $(1, -1)$ matrix of order n is called a *Hadamard* matrix if $HH^T = H^TH = nI_n$, where H^T is the transpose of H and I_n is the identity matrix of order n. A $(1, -1)$ matrix A of order n is said to be of *skew* type if $A - I_n$ is skew-symmetric.

A matrix $W = circ(w_1, \ldots, w_n)$, $w_i \in \{0, \pm 1\}$ which satisfies $WW^T = kI_n$ is called a *circulant weighing matrix* of order n and weight k or $CW(n, k)$.

We denote the product $Z_p \times Z_p \times \cdots \times Z_p (r\ times)$ by $EA(p^r)$ the Elementary Abelian group. Moreover $-a$ is denoted by $\bar{a}$.

Throughought this paper we use the symbol 0_m to denote the sequence of length m with all elements zero and the symbol O_t to denote the $t \times t$ matrix with all entries zero.

For the undefined terms we refer the reader to the book by Geramita and Seberry [2].

Suppose $C = circ(c_0, c_1, \ldots, c_{n-1})$ is a circulant matrix of order n.

Let

$$T_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

of order n, be the shift matrix. Then we can write $C = c_0 I + c_1 T_n + \ldots + c_{n-1} T_n^{n-1}$. Note that $T_n^n = I$ the identity matrix of order n. We say the Hall polynomial of C is $\sum_{i=0}^{n-1} c_i x^i$. The Hall polynomial of C^T is $\sum_{i=0}^{n-1} c_i x^{n-i}$.

Given a set of ℓ sequences $A_j = \{a_{j1}, a_{j2}, \ldots, a_{jn}\}$, $j = 1, \ldots, \ell$, of length n the *non-periodic autocorrelation function*, denoted $NPAF$, $N_A(s)$ is defined as

$$N_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n-s} a_{ji} a_{j,i+s}, \quad s = 0, 1, \ldots, n-1, \tag{4}$$

If $A_j(z) = a_{j1} + a_{j2} z + \ldots + a_{jn} z^{n-1}$ is the associated polynomial of the sequence A_j, then

$$A(z)A(z^{-1}) = \sum_{j=1}^{\ell} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ji} a_{jk} z^{i-k} = N_A(0) + \sum_{j=1}^{\ell} \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}), z \neq 0. \tag{5}$$

Given A_ℓ, as above, of length n the *periodic autocorrelation function*, denoted PAF, $P_A(s)$ is defined, reducing $i + s$ modulo n, as

$$P_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n} a_{ji} a_{j,i+s}, \quad s = 0, 1, \ldots, n-1. \tag{6}$$

We note NPAF sequences imply PAF sequences exist, the NPAF sequences being padded at the end with sufficient zeros to make longer lengths. Hence NPAF sequences can give more general results. If two NPAF sequences have differing lengths then sufficient zeros are added to the end of each to make all the sequences the same length. In all cases NPAF and PAF sequences can be used to make circulant matrices satisfying the additive property (see [3, 4]); if NPAF sequences of lengths n_1 and n_2 are used, then by padding, circulant matrices for all orders $n \geq max(n_1, n_2)$ will exist; if PAF sequences of lengths n are used, then circulant matrices of order n exist.

2 Constructions

Theorem 1 *Write* 0_s *for the sequence of* s *zeros, and let* a, b, c *and* d *be commuting variables. Use the matrices* A_1, A_2, A_3 *and* A_4 *given by*

$$A_1 = \mathrm{circ}(0_\mathrm{s}\mathrm{ba}\bar{\mathrm{b}}0_\mathrm{s}), \qquad A_2 = \mathrm{circ}(0_\mathrm{s}\mathrm{c}0\mathrm{c}0_\mathrm{s}),$$
$$A_3 = \mathrm{circ}(0_\mathrm{s}\bar{\mathrm{c}}\bar{\mathrm{d}}\mathrm{c}0_\mathrm{s}), \qquad A_4 = \mathrm{circ}(0_\mathrm{s}\mathrm{b}0\mathrm{b}0_\mathrm{s}),$$

can be used in the Goethals-Seidel array to obtain an $OD(8s + 12; 1, 1, 4, 4)$.

Proof. Observe that

$$A_1 A_1^T + A_2 A_2^T + A_3 A_3^T + A_4 A_4^T = (a^2 + d^2 + 4b^2 + 4d^2) I_n$$

and

$$A_1 A_1^T - A_2 A_2^T + A_3 A_3^T - A_4 A_4^T = 0.$$

Thus A_2, A_2, A_3, A_4 are a short amicable set and satisfy the additive property (2) so they can be used in the Goethals-Seidel array to obtain an $OD(8s + 12; 1, 1, 4, 4)$. □

The Melding Construction

Suppose the matrices A_1, A_2, A_3 and A_4 are are short amicable sets, on the set of commuting variables $\{0, \pm x_1, \pm x_2, \cdots, \pm x_u\}$ or from $\{0, \pm 1\}$, and satisfy the additive property

$$\sum_{i=1}^{4} \left(A_i A_i^T \right) = \sum_{j=1}^{u} p_j x_j^2 I_n, \tag{7}$$

and the matrices A_5, A_6, A_7 and A_8 are also short amicable sets, on the set of commuting variables $\{0, \pm y_1, \pm y_2, \cdots, \pm y_v\}$ or from $\{0, \pm 1\}$, and satisfy the additive property

$$\sum_{i=5}^{8} \left(A_i A_i^T \right) = \sum_{j=1}^{v} q_j y_j^2 I_n. \tag{8}$$

Then the eight matrices will form an amicable set so we can use the two together in the Kharaghani array to obtain an $OD(8n; p_1, p_2, \cdots, p_u, q_1, q_2, \cdots, q_v)$. □

order	type	group	order	type	group	order	type	group	order	type	group
n	$1,1$	Z_n	$6n$	$4,4$	Z_{6n}	$10n$	$4,4$	Z_{10n}	$14n$	$8,8$	Z_{14n}
$2n$	$2,2$	Z_{2n}	$6n$	$5,5$	Z_{6n}	$10n$	$9,9$	Z_{10n}	$14n$	$10,10$	Z_{14n}
$4n$	$1,4$	Z_{4n}	$7n$	$4,4$	Z_{7n}	$12n$	$8,8$	Z_{12n}	$14n$	$13,13$	Z_{14n}
$4n$	$4,4$	Z_{4n}	$8n$	$8,8$	Z_{8n}	$13n$	$9,9$	Z_{13n}			

Table 1: Order and type for small 2-short amicable sets for all $n \geq 1$.

Using table 2, remark 1 and the above Melding Construction we obtain many 4-short amicable sets and 8-amicable sets.

Type	A_1 A_2	A_3 A_4	ZERO
(1,1,1,1)	a c	b d	NPAF n
(1,1,1,4)	0 -d a d 0 d 0 d	0 b 0 0 0 c 0 0	NPAF $4n$
(1,1,2,2)	a 0 b 0	c d c -d	NPAF $2n$
(1,1,2,8)	0 -c a c 0 c b c	0 -c b -c 0 -c d c	NPAF $4n$
(1,1,4,4)	a b -a c 0 c	a 0 a c d -c	NPAF $3n$
(1,1,5)	-a a a c 0 0	a 0 a 0 b 0	NPAF $4n$
(1,1,5,5)	-c a c 0 c -d c 0	-d b d 0 d c d 0	NPAF $4n$
(1,1,8,8)	0 -c -d a d c 0 c d 0 d c	0 c -d 0 -d c 0 -c d b -d c	NPAF $6n$
(1,2,2,4)	0 -d a d 0 d 0 d	c 0 b 0 c 0 -b 0	NPAF $4n$
(1,4,4,4)	0 -b a b 0 b 0 b	d c -d c -c d c d	NPAF $4n$
(2,2,2,2)	a b c d	a -b c -d	NPAF $2n$
(2,2,4,4)	a 0 b 0 a 0 -b 0	d c -d c -c d c d	NPAF $4n$
(2,2,5,5)	0 a 0 0 b 0 0 a 0 0 -b 0	c -d 0 -d c d d c 0 c d -c	NPAF $6n$
(2,2,8,8)	-d c a c d 0 -d -c a -c d 0	d -c b c d 0 -d -c b c -d 0	NPAF $6n$
(3,3)	a b a 0	b-a b 0	NPAF $2n$
(4,4,4,4)	a a b-b d d-c c	b b-a a c c d-d	NPAF $4n$
(4,4,8,8)	d a -c c a -d -d -b c c b -d	d b c -c b -d d -a c c a d	NPAF $6n$
(5,5)	a a -a b b -b	a 0 a b 0 b	NPAF $3n$
(5,5,5,5)	-a b a 0 a b b a -b 0 -b a	-c d c 0 c d d c -d 0 -d c	NPAF $6n$
(6,6)	a -b a b a b	a a -a b b -b	NPAF $3n$
(6,6,12)	c a c b-c a -c b-c-a c b	c a c-a c-a -c b c-b-c-b	NPAF $6n$
(8,8)	a a a-a b b b-b	b b-b b a a-a a	NPAF $4n$

Table 2: Short amicable sets.

Type	A_1 A_2	A_3 A_4	ZERO
(8,8,8,8)	a a a-a b b-b b c c c-c d d-d d	b b b-b a a-a a d d d-d c c-c c	NPAF $8n$
(10,10,10,10)	disjoint	from Golay	NPAF $n \geq 10$
(13,13)	c 0 -c c -c 0 0 c c g 0 -g g -g 0 0 g g	c c -c c c c 0 0 -c g g -g g g g 0 0 -g	NPAF $9n$
(13,13,13,13)	from disjoint sequences of length 18 and weight 13		NPAF $n \geq 18$
(16,16,16,16)	disjoint	from Golay	NPAF $n \geq 16$
(17,17,17,17)	from disjoint sequences of length 26 and weight 17		NPAF $n \geq 26$
(20,20,20,20)	disjoint	from Golay	NPAF $n \geq 20$
(25,25,25,25)	disjoint sequences of length 36 and weight 25		NPAF $n \geq 36$
(26,26,26,26)	disjoint	from Golay	NPAF $n \geq 26$
(14,14)	a b -b -b b a a b -a a a -a b b	-b a -b a -b b b a b a b a -a -a	NPAF $7n$
(17,17)	a -a a a a a -a a 0 c -c c c c c -c c 0	c -c -c c c c c -c -c a -a -a a a a a -a -a	PAF $9n$

Table 2: (continued).

3 Some general results

We now consider the use of sequences with zero non-periodic autocorrelation function to make an amicable set of matrices. We refer the reader to [7, 8] for any undefined terms.

Theorem 2 (General construction) *Let* X, Y *be two disjoint* $(0, \pm 1)$ *sequences with zero non-periodic autocorrelation function of length* n *and weight* k*, Let* a, b, c, d *be commuting variables and write* aV, bW *for the circulant (type 1) matrices of order* n *formed by using the first rows with the elements of* X *multiplied by* a *and the elements of* Y *multiplied by* b *respectively.*

Let A_i *be the circulant matrices of order* n *given by*

$$A_1 = aV + bW \quad A_2 = cV + dW \quad A_3 = dV - cW \quad A_4 = bV - aW \tag{9}$$

then $\{A_i\}_{i=1}^4$ *is a short amicable set satisfying*

$$\sum_{i=1}^{2} \left(A_{2i-1} A_{2i}^T - A_{2i} A_{2i-1}^T \right) = 0, \tag{10}$$

and the additive property

$$\sum_{i=1}^{4} \left(A_i A_i^T \right) = k(a^2 + b^2 + c^2 + d^2)) I_n. \tag{11}$$

Proof. Now $A_1 = aV + bW$, where V, W are disjoint $(0, \pm 1)$ circulant (type 1 or group developed also suffice) matrices of order n which satisfy $VV^T + WW^T = kI_n$, and similarly for the other A_j, $j = 2, 3, 4$.

Then

$$A_1A_1^T = (aV + bW)(aV^T + bW^T) = a^2VV^T + b^2WW^T + ab(VW^T + WV^T).$$

Hence

$$\begin{aligned} \sum_{i=1}^4 \left(A_iA_i^T\right) &= (a^2 + b^2 + c^2 + d^2)(VV^T + WW^T) \\ &= k(a^2 + b^2 + c^2 + d^2)I_n, \end{aligned}$$

Now

$$\begin{aligned} A_1A_2^T - A_2A_1^T &= (aV + bW)(cV^T + dW^T) - (cV + dW)(aV^T + bW^T) \\ &= (ad - bc)VW^T + (-ad + bc)WV^T, \end{aligned}$$

and

$$\begin{aligned} A_3A_4^T - A_4A_3^T &= (dV - cW)(bV^T - aW^T) - (bV - aW)(dV^T - cW^T) \\ &= (-ad + cb)VW^T + (ad - cb)WV^T. \end{aligned}$$

Thus summing over the four A_i we see they form a short amicable set satisfying the additive property.

□

Corollary 1 *Let X, Y be a pair of disjoint $(0, \pm 1)$ sequences with zero non-periodic autocorrelation function of length n and weight k. Then there exists a short amicable set which can be used to form an $OD(4n; k, k, k, k)$.*

Proof. Use the sequences as in the theorem to form an amicable set with the additive property. Then use this set in the Goethals-Seidel array to obtain the result. □

For $\alpha, \beta, \gamma, \delta, \epsilon, \phi, \psi, \mu, \nu$ non-negative integers, Koukouvinos and Seberry [5, p. 160] show that there exist two disjoint $(0, \pm 1)$ sequences, with zero non-periodic autocorrelation function, of length $\geq n$, $n \in N = \{2 \times 2^\alpha 6^\beta 10^\gamma 9^\delta 14^\epsilon 18^\phi 26^\psi 24^\mu 34^\nu\}$ and weight k, $k \in K = \{2^\alpha 5^\beta 10^\gamma 13^\delta 17^\epsilon 25^\phi 26^\psi 34^\mu 50^\nu\}$. These give the results presented in Table 3.

References

[1] S. Georgiou, C. Koukouvinos and J.Seberry, *Short amicable sets*, (submitted).

[2] A.V.Geramita, and J.Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York-Basel, 1979.

[3] W.H. Holzmann, and H. Kharaghani, On the Plotkin arrays, *Australas. J. Combin.*, 22 (2000), 287–299.

[4] H. Kharaghani, Arrays for orthogonal designs, *J. Combin. Designs*, 8 (2000), 166–173.

Type	ZERO
(1,1,1,1)	NPAF $n \geq 1$
(2,2,2,2)	NPAF $n \geq 2$
(4,4,4,4)	NPAF $n \geq 4$
(5,5,5,5)	NPAF $n \geq 6$
(8,8,8,8)	NPAF $n \geq 8$
(10,10,10,10)	NPAF $n \geq 10$
(13,13,13,13)	NPAF $n \geq 18$
(16,16,16,16)	NPAF $n \geq 16$
(17,17,17,17)	NPAF $n \geq 26$
(20,20,20,20)	NPAF $n \geq 20$
(25,25,25,25)	NPAF $n \geq 36$
(26,26,26,26)	NPAF $n \geq 26$

Table 3: Short amicable sets from corollary 1

[5] C. Koukouvinos and J. Seberry, New weighing matrices and orthogonal designs constructed using two sequences with zero autocorrelation function - a review, *J. Statist. Plann. Inference*, 81 (1999), 153–182.

[6] M. Plotkin, Decomposition of Hadamard matrices, *J. Combin. Theory, Ser. A*, 13 (1972), 127–130.

[7] J. Seberry and R. Craigen, Orthogonal designs, in *CRC Handbook of Combinatorial Designs*, C.J. Colbourn and J.H. Dinitz (Eds.), CRC Press, (1996), 400–406.

[8] J. Seberry and M. Yamada, Hadamard matrices, sequences and block designs, in *Contemporary Design Theory: A Collection of Surveys*, J.H. Dinitz and D.R. Stinson (Eds.), J. Wiley and Sons, New York, (1992), 431–560.

[9] J. Seberry and A.L. Whiteman, New Hadamard matrices and conference matrices obtained via Mathon's construction, *Graphs and Combinatorics*, 4 (1988), 355-377.

[10] D. J. Street, *Cyclotomy and Designs,* PhD Thesis, University of Sydney, 1981.