Infinite Families of Orthogonal Designs : I

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Abstract

We generalize a method inspired by Kharaghani and Holzmann to obtain infinite families of 6-variables orthogonal designs, $OD(8t; k, k, k, k, k, k)$, and $OD(8t; k, k, k, k, 2k, 2k)$, for the first time for odd t.

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1 Preliminaries

An *orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$ denoted $OD(n; s_1, s_2, \ldots, s_u)$ in the variables $x_1, x_2, \ldots, x_u$, is a matrix A of order n with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$AA^T = \sum_{i=1}^{u} (s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n. Let B_i, $i = 1, 2, 3, 4$ be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying

$$\sum_{i=1}^{4} B_i B_i^T = \sum_{i=1}^{u} (s_i x_i^2) I_n.$$

Then the Goethals-Seidel array

$$G = \begin{pmatrix} B_1 & B_2R & B_3R & B_4R \\ -B_2R & B_1 & B_4^TR & -B_3^TR \\ -B_3R & -B_4^TR & B_1 & B_2^TR \\ -B_4R & B_3^TR & -B_2^TR & B_1 \end{pmatrix}$$

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where R is the back-diagonal identity matrix, is an $OD(4n; s_1, s_2, \dots, s_u)$. See page 107 of [1] for details.

Plotkin [5] showed that there is an Hadamard matrix of order $2t$, then there is an $OD(8t; t, t, t, t, t, t, t, t)$. In the same paper he also constructed an $OD(24; 3, 3, 3, 3, 3, 3, 3, 3)$. This OD has appeared in [1], [6] and in [7]. It is conjectured that there is an $OD(8n; n, n, n, n, n, n, n, n)$ for each odd integer n. Until recently, none except the original for $n = 3$ found by Plotkin, had been constructed in the ensuing twenty eight years. Holzmann and Kharaghani [2] using a new method constructed many new Plotkin ODs of order 24 and two new Plotkin ODs of order 40 and 56. Actually their construction provides many new orthogonal designs in 6, 7 and 8 variables which include the Plotkin ODs of order 40 and 56.

A pair of matrices A, B is said to be amicable (anti-amicable) if $AB^T - BA^T = 0$ ($AB^T + BA^T = 0$). Following [3] a set $\{A_1, A_2, \dots, A_{2n}\}$ of square real matrices is said to be *amicable* if

$$\sum_{i=1}^{n} \left(A_{\sigma(2i-1)} A_{\sigma(2i)}^T - A_{\sigma(2i)} A_{\sigma(2i-1)}^T \right) = 0 \tag{1}$$

for some permutation σ of the set $\{1, 2, \dots, 2n\}$. For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. So

$$\sum_{i=1}^{n} \left(A_{2i-1} A_{2i}^T - A_{2i} A_{2i-1}^T \right) = 0. \tag{2}$$

Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. Throughout the paper R_k denotes the back diagonal identity matrix of order k.

A set of matrices $\{B_1, B_2, \dots, B_n\}$ of order m with entries in $\{0, \pm x_1, \pm x_2, \dots, \pm x_u\}$ is said to be *amicable plug-in*, $AP(m; s_1, s_2, \dots, s_u)$, in the variables $x_1, x_2, \dots, x_u$ if it satisfies an additive property

$$\sum_{i=1}^{n} B_i B_i^T = \sum_{i=1}^{u} (s_i x_i^2) I_m. \tag{3}$$

First we need the following array from [3].

Let $\{A_i\}_{i=1}^{8}$ be an amicable set of circulant matrices (or group developed of type 1) of type $(s_1, s_2, \dots, s_u)$ of order t. Then the Kharaghani array

$$H = \begin{pmatrix}
A_1 & A_2 & A_4R_n & A_3R_n & A_6R_n & A_5R_n & A_8R_n & A_7R_n \\
-A_2 & A_1 & A_3R_n & -A_4R_n & A_5R_n & -A_6R_n & A_7R_n & -A_8R_n \\
-A_4R_n & -A_3R_n & A_1 & A_2 & -A_8^TR_n & A_7^TR_n & A_6^TR_n & -A_5^TR_n \\
-A_3R_n & A_4R_n & -A_2 & A_1 & A_7^TR_n & A_8^TR_n & -A_5^TR_n & -A_6^TR_n \\
-A_6R_n & -A_5R_n & A_8^TR_n & -A_7^TR_n & A_1 & A_2 & -A_4^TR_n & A_3^TR_n \\
-A_5R_n & A_6R_n & -A_7^TR_n & -A_8^TR_n & -A_2 & A_1 & A_3^TR_n & A_4^TR_n \\
-A_8R_n & -A_7R_n & -A_6^TR_n & A_5^TR_n & A_4^TR_n & -A_3^TR_n & A_1 & A_2 \\
-A_7R_n & A_8R_n & A_5^TR_n & A_6^TR_n & -A_3^TR_n & -A_4^TR_n & -A_2 & A_1
\end{pmatrix}$$

is an $OD(8m; s_1, s_2, \ldots, s_u)$.

We extend the construction of Holzmann and Kharaghani [2] for an $OD(56; 7, 7, 7, 7, 7, 7, 7, 7)$ to find an infinite family of 6-variables orthogonal designs, $OD(8t; k, k, k, k, k, k)$, and $OD(8t; k, k, k, k, 2k, 2k)$, for odd t.

2 The constructions

First we give the following definition.

Definition 1 Define K-matrices, K_1, K_2, K_3, K_4 to be four circulant $(0, \pm 1)$ matrices of order t satisfying

(i) $K_i * K_j = 0$, $i \neq j$

(ii) $\displaystyle\sum_{i=1}^{4} K_i K_i^T = kI_t$

where $*$ denotes the Hadamard product. We say k is the weight of these K-matrices.

From definition 1 we observe that T-matrices (see Seberry and Yamada [7] for more details) of order t are K-matrices with $k = t$.

Then we have

Theorem 1 *Suppose K_1, K_2, K_3, K_4 are K-matrices of order t and weight k. Then there exists a 6-variables ortohognal designs $OD(8t; k, k, k, k, k, k)$.*

Proof: Use

$$
\begin{array}{llllllllllll}
A_1 = & aK_1 & +bK_2 & +cK_3, & & \quad & A_2 = & dK_1 & +eK_2 & +fK_3, & \\
A_3 = & -bK_1 & +aK_2 & & -cK_4, & & A_4 = & eK_1 & -dK_2 & & -fK_4, \\
A_5 = & -cK_1 & & +aK_3 & +bK_4, & & A_6 = & fK_1 & & -dK_3 & +eK_4, \\
A_7 = & & +cK_2 & -bK_3 & +aK_4, & & A_8 = & & -fK_2 & +eK_3 & +dK_4.
\end{array}
$$

Formally multiply out the expression on the left hand side of (2). This is formally zero and we have (4).

$$
A_1A_2^T - A_2A_1^T + A_3A_4^T - A_4A_3^T + A_5A_6^T - A_6A_5^T + A_7A_8^T - A_8A_7^T = 0 \tag{4}
$$

These matrices also satisfy (3) and so these eight matrices may be used in the Kharaghani array to obtain the result. □

Example 1 (i) The following T-matrices of order 3 are K-matrices with $k = 3$, $K_1 = circ\,(100)$, $K_2 = circ\,(010)$, $K_3 = circ\,(001)$, $K_4 = circ\,(000)$. Then we have

$$
\begin{array}{ll}
A_1 = circ\,(a, b, c), & A_2 = circ\,(d, e, f) \\
A_3 = circ\,(-b, a, 0), & A_4 = circ\,(e, -d, 0) \\
A_5 = circ\,(-c, 0, a), & A_6 = circ\,(f, 0, -d) \\
A_7 = circ\,(0, c, -b), & A_8 = circ\,(0, -f, e)
\end{array}
$$

We observe that the relations (2) and (3) hold, and thus we can use these eight matrices in the Kharaghani array to obtain a 6-variables orthogonal design $OD(24; 3,3,3,3,3,3)$. The $OD(24; 3,3,3,3,3,3,3,3)$ was originally given in Plotkin [5], however our formulation allows for infinitely many 6-variables orthogonal designs.

(ii) The following are K-matrices of order 7 with $k=7$, $K_1 = circ\,(-110100)$, $K_2 = circ\,(0001000)$, $K_3 = circ\,(0000010)$, $K_4 = circ\,(0000001)$. Then we have

$$\begin{array}{ll} A_1 = circ\,(-a,a,a,b,a,c,0), & A_2 = circ\,(-d,d,d,e,d,f,0), \\ A_3 = circ\,(b,-b,-b,a,-b,0,-c), & A_4 = circ\,(-e,e,e,-d,e,0,-f), \\ A_5 = circ\,(c,-c,-c,0,-c,a,b), & A_6 = circ\,(-f,f,f,0,f,-d,e), \\ A_7 = circ\,(0,0,0,c,0,-b,a), & A_8 = circ\,(0,0,0,-f,0,e,d). \end{array}$$

We observe that the relations (2) and (3) hold, and thus we can use these eight matrices in the Kharaghani array to obtain a 6-variables orthogonal design $OD(56; 7,7,7,7,7,7)$.

(iii) The following are K-matrices of order 7 with $k=5$, $K_1 = circ\,(-110100)$, $K_2 = circ\,(0001000)$, $K_3 = K_4 = circ\,(0000000)$. Then we have

$$\begin{array}{ll} A_1 = circ\,(-a,a,a,b,a,0,0), & A_2 = circ\,(-d,d,d,e,d,0,0), \\ A_3 = circ\,(b,-b,-b,a,-b,0,0), & A_4 = circ\,(-e,e,e,-d,e,0,0), \\ A_5 = circ\,(c,-c,-c,0,-c,0,0), & A_6 = circ\,(-f,f,f,0,f,0,0), \\ A_7 = circ\,(0,0,0,c,0,0,0), & A_8 = circ\,(0,0,0,-f,0,0,0). \end{array}$$

We observe that the relations (2) and (3) hold, and thus we can use these eight matrices in the Kharaghani array to obtain a 6-variables orthogonal design OD(56;5,5,5,5,5,5). □

Theorem 2 *Suppose K_1, K_2, K_3, K_4 are K-matrices of order t and weight k. Then there exists a 6-variables orthogonal design $OD(8t; k,k,k,k,2k,2k)$.*

Proof: Use

$$\begin{array}{lllllllllll} A_1 = & aK_1 & +bK_2 & +cK_3 & +cK_4 & & A_2 = & dK_1 & +eK_2 & +fK_3 & +fK_4, \\ A_3 = & -bK_1 & +aK_2 & +cK_3 & -cK_4, & & A_4 = & eK_1 & -dK_2 & +fK_3 & -fK_4, \\ A_5 = & cK_1 & +cK_2 & -aK_3 & -bK_4, & & A_6 = & -fK_1 & -fK_2 & +eK_3 & +dK_4, \\ A_7 = & cK_1 & -cK_2 & +bK_3 & -aK_4, & & A_8 = & -fK_1 & +fK_2 & +dK_3 & -eK_4. \end{array}$$

Formally multiply out the expression on the left hand side of (2). This is formally zero and we have (4). These matrices also satisfy (3) and so these eight matrices may be used in the Kharaghani array to obtain the result. □

From Seberry and Yamada [7], where all undefined terms may be found, we have T-matrices of order $g, g+1, 2g+1$, $t(odd) = 1, 3, \ldots, 71, 75$, $t = y(2m+1)$, where y is a Yang number and $m+1, m+1, m, m$ the lengths of base sequences, and $G = \{g : g = 2^a 10^b 26^c, a, b, c$ non-negative integers$\}$.

Corollary 1 *Suppose there exist T-matrices of order t. Then there exists an $OD(8t;t,t,t,t,t,t)$ and an $OD(8t;t,t,t,t,2t,2t)$.* □

Remark 1 This gives infinite families of 6-variables orthogonal designs. In particular it gives $OD(8t;t,t,t,t,t,t)$ and $OD(8t;t,t,t,t,2t,2t)$ for the following $t < 196$: $1,3,\ldots,71,75,77,81,85,87,91,93,95,99,101,105,111,115,\ldots,125,129,\ldots,135,141,\ldots,147,$ $153,155,159,\ldots,165,169,171,175,177,187,189,195$. □

We also note that while T-matrices have full weight t theorem 1 does not require the weight to equal t. In fact from Koukouvinos and Seberry [4, p. 160] we have two $(0,\pm1)$ sequences, with zero non-periodic autocorrelation function, of length $n \in N$, $N = \{2^\alpha 6^\beta 10^\gamma 9^\delta 13^\epsilon 18^\phi 26^\psi 24^\mu 34^\nu\}$ and weight $k \in K$, $K = \{2^\alpha 5^\beta 10^\gamma 13^\delta 17^\epsilon 25^\phi 26^\psi 34^\mu 50^\nu\}$ where $\alpha,\beta,\gamma,\delta,\epsilon,\phi,\psi,\ \mu,\nu$ are non-negative integers. Thus for $a,b,\gamma,\delta,\epsilon,\phi,\psi\ \mu,\nu$ non-negative integers we have two $(0,\pm1)$ disjoint sequences with zero non-periodic autocorrelation function of length $2n$, $n \in N$ and weight $k \in K$.

Corollary 2 *Suppose there exist two $(0,\pm1)$ sequences, say A and B, with zero non-periodic autocorrelation function, of length m and weight k_1 and two $(0,\pm1)$ disjoint sequences, say C and D with zero non-periodic autocorrelation function, of length n and weight k_2. Set $t = 2m+n$ and $k = k_1 + k_2$. Then there exist an $OD(8s;k,k,k,k,k,k)$, and an $OD(8s;k,k,k,k,2k,2k)$, $s \geq t \geq k$.*

Proof. Set $K_1 = circ(A0_m0_n)$, $K_2 = circ(0_mB0_n)$, $K_3 = circ(0_{2m}C)$, and $K_4 = circ(0_{2m}D)$. These are K-matrices of order $t = 2m+n$ and weight $k = k_1 + k_2$ and thus can be used in Theorem 1 to obtain an orthogonal design $OD(8s;k,k,k,k,k,k)$, and in Theorem 2 to obtain an orthogonal design $OD(8s;k,k,k,k,2k,2k)$, $s \geq t \geq k$. □

Example 2 For $\alpha = \beta = \gamma = \delta = \phi = \psi = \mu = \nu = 0$ and $\epsilon = 1$ we have the following sequences $A = \{1,-1,1,0,-1,0,0,0,1,1,1,0,1\}$, $B = \{-1,0,-1,0,1,1,0,-1,0,1,1,-1,1\}$ of length $m = 13$ and weight $k_1 = 17$. $C = \{1,1,0,0\}$, $D = \{0,0,1,-1\}$ are two disjoint sequences with zero non-periodic autocorrelation function, of length $n = 4$ and weight $k_2 = 4$. Set $t = 2m+n = 30$ and $k = k_1 + k_2 = 21$. Then using these sequences in corollary 2 we obtain an $OD(8s;21,21,21,21,21,21)$, and an $OD(8s;21,21,21,21,42,42)$ $s \geq 30$. □

So far we have only considered sequences with zero non-periodic autocorrelation function, or NPAF $= 0$, however, as we now see, provided the sequences are all of the same lengths, sequences with zero periodic autocorrelation function can also be used greatly extending the scope of the construction. Thus

Corollary 3 *Suppose there exist two $(0,\pm1)$ sequences, say A and B, with zero non-periodic autocorrelation function, of length m and weight k_1 and two $(0,\pm1)$ disjoint sequences, say C and D with zero periodic autocorrelation function, of length $2m+n$ and weight k_2 of the form $0_{2m}X$ and $0_{2m}Y$ where X,Y are $(0,\pm1)$ sequences. Set $t = 2m+n$ and $k = k_1+k_2$. Then there exist $OD(8s;k,k,k,k,k,k)$, and $OD(8s;k,k,k,k,2k,2k)$, $s = t$.*

Proof. Set $K_1 = circ(A0_m0_n)$, $K_2 = circ(0_mB0_n)$, $K_3 = circ(C)$, and $K_4 = circ(D)$. These are K-matrices of order $t = 2m + n$ and weight $k = k_1 + k_2$ and thus can be used in theorem 1 and in theorem 2 to obtain an orthogonal design $OD(8t; k, k, k, k, k, k)$ and an orthogonal design $OD(8t; k, k, k, k, 2k, 2k)$. □

Example 3 We use the two disjoint sequences $C = \{0, 0, 1, 0, 0, 0, 0, -1, 0, -1, 1\}$ and $D = \{0, 0, 0, 1, 1, 1, -1, 0, 1, 0, 0\}$ with zero periodic autocorrelation function, of length $2m + n = 11$ and weight $k_2 = 9$. Let $A = \{1\}$, and $B = \{1\}$ be two sequences with zero non-periodic autocorrelation function, of length $m = 1$ and weight $k_1 = 2$. Set $t = 2m + n = 11$ and $k = k_1 + k_2 = 11$. Then using these sequences in corollary 3 we obtain an $OD(8 \cdot 11; 11, 11, 11, 11, 11, 11)$ and an $OD(8 \cdot 11; 11, 11, 11, 11, 22, 22)$.

□

Corollary 4 *Suppose there exist two pairs of $(0, \pm 1)$ sequences, with zero periodic autocorrelation function, of length m say A and B, of weight k_1 and, C and D also of length m with weight k_2. Further suppose all of A, B, C and D are mutually disjoint. Then there exist $OD(8m; k, k, k, k, k, k)$, $k = k_1 + k_2$.*

Proof. Set $K_1 = circ(A)$, $K_2 = circ(B)$, $K_3 = circ(C)$, and $K_4 = circ(D)$. These are K-matrices of order m and weight $k = k_1 + k_2$ and thus can be used in theorem 1 to obtain an orthogonal design $OD(8t; k, k, k, k, k, k)$. □

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