An Algorithm to find Formulae and Values of Minors for Hadamard Matrices

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Abstract

We give an algorithm to obtain formulae and values for minors of Hadamard matrices. One step in our algorithm allows the $(n-j)\times(n-j)$ minors of an Hadamard matrix to be given in terms of the minors of a $2^{j-1}\times 2^{j-1}$ matrix. In particular we illustrate our algorithm by finding explicitly all the $(n-4)\times(n-4)$ minors of an Hadamard matrix.

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1 Introduction

An Hadamard matrix H of order n is an $n\times n$ matrix with elements ± 1 and $HH^T = nI$. For more details and construction methods of Hadamard matrices we refer the interested reader to the books [1] and [2]. Hadamard matrices of order n have determinant of absolute value $n^{\frac{n}{2}}$. Sharpe [3] observed that all the $(n-1)\times(n-1)$ minors of an Hadamard matrix of order n are zero or $n^{\frac{n}{2}-1}$, and that all the $(n-2)\times(n-2)$ minors are zero or $2n^{\frac{n}{2}-2}$, and all the $(n-3)\times(n-3)$ minors are zero or $4n^{\frac{n}{2}-3}$. We note that the maximum determinant corresponds to having the submatrix

$$\begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & - \\ 1 & - & 1 \end{bmatrix}$$

in the upper lefthand corner of the Hadamard matrix for $n-2$ and $n-3$ respectively. We give a useful method for finding the $n-3$ and $n-4$ minors which points the way to finding other minors such as the $n-j$ minor.

Notation 1. We write

$$J_{b_1,b_2,\cdots,b_z}$$

for the all ones matrix with diagonal blocks of sizes $b_1\times b_1,\ b_2\times b_2\ \cdots\ b_z\times b_z$. Write

$$a_{ij}J_{b_1,b_2,\cdots,b_z}$$

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for the matrix for which the elements of the block with corners $(i+b_1+b_2+\cdots+b_{j-1}, i+b_1+b_2+\cdots+b_{i-1})$, $(i+b_1+b_2+\cdots+b_{j-1}, b_1+b_2+\cdots+b_i)$, $(b_1+b_2+\cdots+b_j, i+b_1+b_2+\cdots+b_{i-1})$, $(b_1+b_2+\cdots+b_j, b_1+b_2+\cdots+b_i)$ are a_{ij} an integer.

Write

$$(k-a_{ii})I_{b_1,b_2,\cdots,b_z}$$

for the direct sum $(k-a_{11})I_{b_1}+(k-a_{22})I_{b_2}+\cdots+(k-a_{zz})I_{b_z}$.

See the Appendix for examples.

Notation 2. We use $-$ for -1 in matrices in this paper. Also, when we say the determinants of a matrix we mean the absolute values of the determinants.

Remark: We note that the determinant of any matrix $A=(a_{ij})$ is the same as the determinant obtained by finding the element of maximum absolute value in each row i, say $b_i = max\ a_{ij}$, and dividing each element of row i by b_i so every element in that row has absolute value ≤ 1, and then multiplying the determinant by b_i. That is $det\ A = b_1 b_2 \cdots b_n det\ A'$ where every element of A' has absolute value ≤ 1. Call det A' the *determinant on the unit disk.*

2 Prelimimary Results

We first note a very useful lemma as it allows us to obtain bounds on the column structure of submatrices of an Hadamard matrix.

Lemma 1 *(The Distribution Lemma) Let H be any Hadamard matrix, of order $n>2$. Then for every triple of rows of H there are precisely $\frac{n}{4}$ columns which are*

- *(a)* $(1,1,1)^T$ *or* $(-,-,-)^T$
- *(b)* $(1,1,-)^T$ *or* $(-,-,1)^T$
- *(c)* $(1,-,1)^T$ *or* $(-,1,-)^T$
- *(a)* $(1,-,-)^T$ *or* $(-,1,1)^T$

Proof. Let the following rows represent three rows of an Hadamard matrix H of order n.

$$\begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\ 1...1 & 1...1 & 1...1 & 1...1 & -...- & -...- & -...- & -...- \\ 1...1 & 1...1 & -...- & -...- & 1...1 & 1...1 & -...- & -...- \\ 1...1 & -...- & 1...1 & -...- & 1...1 & -...- & 1...1 & -...- \end{array}$$

where $u_1, u_2, \ldots, u_8$ are the numbers of columns of each type. Then from the order and the inner product of rows we have

$$\begin{aligned} u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8 &= n \\ u_1+u_2-u_3-u_4-u_5-u_6+u_7+u_8 &= 0 \\ u_1-u_2+u_3-u_4-u_5+u_6-u_7+u_8 &= 0 \\ u_1-u_2-u_3+u_4+u_5-u_6-u_7+u_8 &= 0 \end{aligned} \qquad (1)$$

Solving we have $u_1+u_8=u_2+u_7=u_3+u_6=u_4+u_5=\frac{n}{4}$. □

We now note that without any loss of generality the rows and columns of an Hadamard matrix, H, may be multiplied through by -1. Hence the first row can be chosen to be all ones. Further, also without loss of generality, the columns of the Hadamard matrix can be sorted until they are in lexicographical order.

If we are considering the $(n-j)\times(n-j)$ minors, then the first j rows, ignoring the upper lefthand $j\times j$ matrix, have 2^{j-1} potentially different first j elements in each column. Let $\underline{x}^T_{\beta+1}$ the vectors containing the binary representation of each integer $\beta+2^{j-1}$ for $\beta=0,\ldots,2^{j-1}-1$. Replace all zero entries of $\underline{x}^T_{\beta+1}$ by -1 and define the $j\times 1$ vectors

$$\underline{u}_k=\underline{x}_{2^{j-1}-k+1},\quad k=1,\ldots,2^{j-1}$$

Let u_k indicate the number of columns beginning with the vectors $\underline{u}_k$, $k=1,\ldots,2^{j-1}$.
We note

$$\sum_{i=1}^{2^{j-1}} u_i = n-j. \tag{2}$$

We write U_j for all the $j\times(n-j)$ matrix in which $\underline{u}_k$ occurs u_k times. So

$$U_j = \begin{matrix} \overbrace{1...1}^{u_1} & \overbrace{1...1}^{u_2} & \ldots & \overbrace{1...1}^{u_{2^{j-1}-1}} & \overbrace{1...1}^{u_{2^{j-1}}} \\ 1...1 & 1...1 & \ldots & 1...1 & 1...1 \\ 1...1 & 1...1 & \ldots & -...- & -...- \\ . & . & \ldots & . & . \\ . & . & \ldots & . & . \\ 1...1 & 1...1 & \ldots & 1...1 & -...- \\ 1...1 & -...- & \ldots & -...- & -...- \end{matrix} \tag{3}$$

Example 1 For example for $j=3$. Here there are four numbers from 3 to 0. Their binary representations give columns beginning

$$(1,\ 1,\ 1)^T,\quad (1,\ 1,\ -1)^T,\quad (1,\ -1,\ 1)^T,\quad (1,\ -1,\ -1)^T$$

So we consider the matrix with first three rows in the form

$$U_3 = \begin{matrix} \overbrace{1}^{u_1} & \overbrace{1}^{u_2} & \overbrace{1}^{u_3} & \overbrace{1}^{u_4} \\ 1 & 1 & - & - \\ 1 & - & 1 & - \end{matrix}$$

where $\sum_{i=1}^{4} u_i = n-3$. We note that for $j\le 3$, we can solve uniquely for each u_i. Similarly for $j=4$, there are eight numbers from 7 to 0. Their binary representations, lead us to consider the matrix with four rows in the form

$$U_4 = \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & - & 1 & - & 1 & - & 1 & - \end{matrix}$$

where $\sum_{i=1}^{8} u_i = n-4$. For $j \geq 4$, we cannot solve uniquely for each i but we know $u_i \geq 0$ and many inequalities such as $u_1 + u_2 \leq \frac{n}{4}$, $u_3 + u_4 \leq \frac{n}{4}$ can be deduced from the orthogonality of the Hadamard matrix.

□

In fact we note if $n = 4t$, the orthogonality of the rows of the Hadamard matrix gives

$$t - j \leq u_1 + u_2 + \ldots + u_{2^{j-3}} \leq t, \quad t - j \leq u_{2^{j-3}+1} + \ldots + u_{2^{j-2}} \leq t$$

$$t - j \leq u_{2^{j-2}+1} + \ldots + u_{2^{j-2}+2^{j-3}} \leq t, \quad t - j \leq u_{2^{j-2}+2^{j-3}+1} + \ldots + u_{2^{j-1}} \leq t$$

Each of these equations can be rewritten so the constraints become

$$\begin{array}{c} 0 \leq t - u_1 - u_2 - \ldots - u_{2^{j-3}} \leq j, \quad 0 \leq t - u_{2^{j-3}+1} - \ldots - u_{2^{j-2}} \leq j \\ 0 \leq t - u_{2^{j-2}+1} - \ldots - u_{2^{j-2}+2^{j-3}} \leq j, \quad 0 \leq t - u_{2^{j-2}+2^{j-3}+1} - \ldots - u_{2^{j-1}} \leq j \end{array} \tag{4}$$

In practice we expect each $u_i \approx \left[\frac{t}{2^{j-3}}\right]$, where [ ] means the integer part.

3 The matrix D

We write

$$H = \begin{bmatrix} M & U_j \\ & C \end{bmatrix} \tag{5}$$

for the Hadamard matrix of order n. The coefficients in the $(n-j) \times (n-j)$ matrix CC^T obtained by removing the first j rows and columns of the Hadamard matrix can be permuted to appear in the form

$$CC^T = (n-j)I_{u_1,u_2,\cdots,u_z} + a_{ik}J_{u_1,u_2,\cdots,u_z}, \quad z = 2^{j-1}$$

where $(a_{ik}) = (-\underline{u}_i \cdot \underline{u}_k)$, with $\cdot$ the inner product. By the determinant simplification theorem (see appendix)

$$det\ CC^T = n^{n-2^{j-1}-j} det\ D$$

where D, of order 2^{j-1} is given by

$$D = \begin{bmatrix} n - ju_1 & u_2 a_{12} & u_3 a_{13} & \cdots & u_z a_{1z} \\ u_1 a_{21} & n - ju_2 & u_3 a_{23} & \cdots & u_z a_{2z} \\ \vdots & \vdots & \vdots & & \vdots \\ u_1 a_{z1} & u_2 a_{z2} & u_3 a_{z2} & \cdots & n - ju_z \end{bmatrix}$$

If the number of columns with the vector $\underline{u}_\ell$ is $u_\ell = 0$, then the symmetry of CC^T will force D to have its order reduced by 1 for each $u_\ell = 0$.

Next we apply a mathematical package such as Mathematica, Matlab or Maple to obtain an explicit formula for det$D$, and thus detCC^T, in terms of $n, t, j, u_1, \ldots, u_{2^{j-1}}$.

We gave five constraints on these variables above. We note we have in fact far more constraints as the orthogonality of pairs of rows of H, plus the order of H means we have $\binom{j}{2}+1$ constraints in total from the rows plus the 2^{j-1} constraints, $0 \le u_i \le t$, for all $i = 1, 2, \ldots, 2^{j-1}$.

Thus we wish to find the values of the det$D$, and especially the maximum value of the det$D$, the object function, subject to $\frac{1}{2}(j^2 - j + 2) + 2^{j-1}$ equations or inequalities. This is of course a straightforward linear programming problem. The nature of the constraints, obtained from orthogonality and simple counting, is such that the constraints form a convex hull. Now the simplex method for solving a linear programming problem, noting that the extremal values of the object function occur at a vertex of the convex hull, moves from one vertex to another, seeking to optimize the object function. All other values of the object function occur at interior lattice points of the convex hull.

We note each variable u_i can take at most $t+1$ values and there at most 2^{j-1} such u_i. Hence there are $(t+1)2^{j-1}$ possible vertices. We test each of these possible vertices against the constraints to see which possible vertices are actual vertices. We then test each of these actual vertices in the object function to see which real square integers occur as solutions, det$D = s^2$.

Then det$C = n^{\frac{1}{2}(n-j-2^{j-1})}s$.

Let A_j be the $2^{j-1} \times j$ matrix of entries ± 1 obtained by writing the binary numbers 0 to $2^{j-1} - 1$ as rows and then replacing all the zeros by 1 and ones by -1.

Example 2

$$j = 2, \quad A_j = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}$$

$$j = 3, \quad A_j = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & - \\ 1 & - & 1 \\ 1 & - & - \end{bmatrix}$$

□

We now give a lemma which allows to describe the distribution of coefficients in D.

Lemma 2 *Let $B = A_j A_j^T = (b_{i\ell})$. Then the element j occurs once in each row and column of B and $j - 2i$, $i = 0, \ldots, j-1$ occur in each row and column of B $\binom{j-1}{i}$ times.*

Proof. For $j = 2$, 2 occurs once and 0 occurs once in each row and column. For $j = 3$, 3 occurs once, 1 occurs twice and -1 occurs once in each row and column. For $j = 4$, 4 occurs once, 2 occurs three times, 0 occurs three times and -2 occurs once in each row and column.

We now proceed by induction we have the lemma is true for small values of j. We assume the lemma is true for all $j \le k$. We now proceed to prove the lemma is true for case $k+1$.

Now let the rows of A_k be the vectors (in $1, -1$ notation) $\underline{\beta}_1, \underline{\beta}_2, \cdots, \underline{\beta}_{2^{k-1}}$. We note that the rows of A_{k+1} are the vectors $\gamma_i = (1, \underline{\beta}_i)$, and $\gamma_{2^{k-1}+i} = (1, -\underline{\beta}_{2^{k-1}-i+1})$, $i = 1, \ldots, 2^{k-1}$.

Since $A_k A_k^T$ is symmetric with the same distribution of elements in each row and column it suffices to consider the inner product of the first row of A_{k+1} with each of the other rows. We first note the inner product of the first row with itself is $k+1$ and with the last row is $1-k$ as required by the lemma.

In the first row of A_k a coefficient, $k-2i$, $i \neq 0$, occured $\binom{k-1}{i}$ times meaning exactly $\binom{k-1}{i}$ rows in A_k had inner product $k-2i$ with the first row. These same rows will have inner product $1+k-2i$ if the row is in the first 2^{k-1} rows of A_{k+1} and $1-k+2i$ if the rows were in the second 2^{k-1} rows of A_{k+1}. That means $1+k-2i$ occurs $\binom{k-1}{i}$ from the first 2^{k-1} rows and $\binom{k-1}{k-i}$ times between the first row and rows of the second half of A_{k+1}. Thus the coefficients $k+1-2i$ occurs $\binom{k-1}{i}+\binom{k-1}{k-i}=\frac{k(k-1)!}{i!(k-i)!}=\binom{k}{i}$ as required.

Hence the lemma was true for small k, and being true for $j=k$ meant it was true for $j=k+1$. Hence we have the proof by induction. □

However our definitions of the coefficients a_{ik} of the matrix D described above gives that the coefficients $(a_{2k}, a_{3k}, \ldots, a_{zk})$, $z=2^{j-1}$ are, as given in the last lemma $(j-2i)$ occuring $\binom{j-1}{i}$, $i=1,\ldots,j-1$ times as the coefficient j occurs down the diagonal. Thus the sum of the squares of the elements in column i of D are

$$(n-ju_i)^2 + u_i^2 \sum_{i=1}^{j-1} \binom{j-1}{i} (j-2i)^2.$$

We illustrate the algorithm for $j=3$ and $j=4$.

4 $(n-3)\times(n-3)$ minors of Hadamard matrices

Theorem 1 *The $(n-3)\times(n-3)$ minors of an Hadamard matrix of order n are zero or $4n^{\frac{n}{2}-3}$.*

Proof. Assume the first row and column of the Hadamard matrix have been normalized to be all ones. This does not alter the determinant.

Assume the first three rows of the Hadamard matrix, ignoring the upper lefthand 3×3 matrix for which we are calculating the minor, are:

$$\begin{array}{cccc}
\overbrace{1,\ldots,1}^{u_1} & \overbrace{1,\ldots,1}^{u_2} & \overbrace{1,\ldots,1}^{u_3} & \overbrace{1,\ldots,1}^{u_4} \\
1,\ldots,1 & 1,\ldots,1 & -,\ldots,- & -,\ldots,- \\
1,\ldots,1 & -,\ldots,- & 1,\ldots,1 & -,\ldots,-.
\end{array}$$

Further assume that the rows have been permuted so that the first three columns contain first u_1' rows with $(1,1,1)$, then u_2' rows with $(1,1,-)$, then u_3' rows with $(1,-,1)$ and last u_4' rows with $(1,-,-)$.

We now remove the first three rows and columns from the original, permuted Hadamard matrix, and the remaining $(n-3)\times(n-3)$ matrix is called C. We write k for $n-3$ and $\bar{3}$ for

-3. In this case CC^T is

$$CC^T = \left[\begin{array}{cccc}
\overbrace{\begin{matrix} k & \bar{3} & \cdots & \bar{3} \\ \bar{3} & k & \cdots & \bar{3} \\ \vdots & \vdots & & \vdots \\ \bar{3} & \bar{3} & \cdots & k \end{matrix}}^{u_1} &
\overbrace{\begin{matrix} \bar{1} & \bar{1} & \cdots & \bar{1} \\ \bar{1} & \bar{1} & \cdots & \bar{1} \\ \vdots & \vdots & & \vdots \\ \bar{1} & \bar{1} & \cdots & \bar{1} \end{matrix}}^{u_2} &
\overbrace{\begin{matrix} \bar{1} & \bar{1} & \cdots & \bar{1} \\ \bar{1} & \bar{1} & \cdots & \bar{1} \\ \vdots & \vdots & & \vdots \\ \bar{1} & \bar{1} & \cdots & \bar{1} \end{matrix}}^{u_3} &
\overbrace{\begin{matrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{matrix}}^{u_4} \\
\begin{matrix} \bar{1} & \bar{1} & \cdots & \bar{1} \\ \bar{1} & \bar{1} & \cdots & \bar{1} \\ \vdots & \vdots & & \vdots \\ \bar{1} & \bar{1} & \cdots & \bar{1} \end{matrix} &
\begin{matrix} k & \bar{3} & \cdots & \bar{3} \\ \bar{3} & k & \cdots & \bar{3} \\ \vdots & \vdots & & \vdots \\ \bar{3} & \bar{3} & \cdots & k \end{matrix} &
\begin{matrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{matrix} &
\begin{matrix} \bar{1} & \bar{1} & \cdots & \bar{1} \\ \bar{1} & \bar{1} & \cdots & \bar{1} \\ \vdots & \vdots & & \vdots \\ \bar{1} & \bar{1} & \cdots & \bar{1} \end{matrix} \\
\begin{matrix} \bar{1} & \bar{1} & \cdots & \bar{1} \\ \bar{1} & \bar{1} & \cdots & \bar{1} \\ \vdots & \vdots & & \vdots \\ \bar{1} & \bar{1} & \cdots & \bar{1} \end{matrix} &
\begin{matrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{matrix} &
\begin{matrix} k & \bar{3} & \cdots & \bar{3} \\ \bar{3} & k & \cdots & \bar{3} \\ \vdots & \vdots & & \vdots \\ \bar{3} & \bar{3} & \cdots & k \end{matrix} &
\begin{matrix} \bar{1} & \bar{1} & \cdots & \bar{1} \\ \bar{1} & \bar{1} & \cdots & \bar{1} \\ \vdots & \vdots & & \vdots \\ \bar{1} & \bar{1} & \cdots & \bar{1} \end{matrix} \\
\begin{matrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{matrix} &
\begin{matrix} \bar{1} & \bar{1} & \cdots & \bar{1} \\ \bar{1} & \bar{1} & \cdots & \bar{1} \\ \vdots & \vdots & & \vdots \\ \bar{1} & \bar{1} & \cdots & \bar{1} \end{matrix} &
\begin{matrix} \bar{1} & \bar{1} & \cdots & \bar{1} \\ \bar{1} & \bar{1} & \cdots & \bar{1} \\ \vdots & \vdots & & \vdots \\ \bar{1} & \bar{1} & \cdots & \bar{1} \end{matrix} &
\begin{matrix} k & \bar{3} & \cdots & \bar{3} \\ \bar{3} & k & \cdots & \bar{3} \\ \vdots & \vdots & & \vdots \\ \bar{3} & \bar{3} & \cdots & k \end{matrix}
\end{array}\right].$$

Then, we have $det\ CC^T = n^{n-7} det\ D$, where

$$D = \begin{bmatrix} n-3u_1 & -u_2 & -u_3 & u_4 \\ -u_1 & n-3u_2 & u_3 & -u_4 \\ -u_1 & u_2 & n-3u_3 & -u_4 \\ u_1 & -u_2 & -u_3 & n-3u_4 \end{bmatrix}$$

where $n = 4t, \quad u_1+u_2+u_3+u_4 = 4t-3, \ 0 \le t-u_i \le 3, \quad i = 1,2,3,4$ and

$$\begin{aligned} det\ D = \ & u_1(n-4u_2)(n-4u_3)(n-4u_4) + u_2(n-4u_1)(n-4u_3)(n-4u_4) \\ & + u_3(n-4u_1)(n-4u_2)(n-4u_4) + u_4(n-4u_1)(n-4u_2)(n-4u_3) \\ & + (n-4u_1)(n-4u_2)(n-4u_3)(n-4u_4). \end{aligned}$$

or

$$det\ D = \sum_{i=1}^{4} \{u_i \prod_{k=1,k\neq i}^{4} (n-4u_k)\} + \prod_{i=1}^{4}(n-4u_i)$$

We now choose all possible choices for $u_1,\ u_2,\ u_3$ and u_4 so $u_1,\ u_2,\ u_3,\ u_4 \in \{t,\ t,\ t,\ t-3\}$ or $\in \{t,\ t,\ t-1,\ t-2\}$ or $\in \{t,\ t-1,\ t-1,\ t-1\}$. All choices from the first two sets give zero determinant. The third set gives $det\ D = 16n$.

The minors are $4n^{\frac{n}{2}-3}$ and 0 as $det\ CC^T = (det\ C)^2$. $\square$

5 $(n-4)\times(n-4)$ minors of Hadamard matrices

Theorem 2 *The $(n-4)\times(n-4)$ minors of an Hadamard matrix of order n are zero, $8n^{\frac{n}{2}-4}$ or $16n^{\frac{n}{2}-4}$.*

Proof. We proceed as in the previous proof but now we use the first four rows and columns and ignore the upper lefthand 4×4 matrix for which we are calculating the minor.

Let $\Sigma_{i=1}^{8}\ u_i = n-4$ where $u_1,\ u_2,\cdots,\ u_8$ correspond to the number of columns with entries

u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
1	1	1	1	1	1	1	1
1	1	1	1	−	−	−	−
1	1	−	−	1	1	−	−
1	−	1	−	1	−	1	−

in the first four rows respectively, after the 4×4 principal submatrix has been removed. Also above, we assume the rows have also been permuted so the elements of the first four columns are grouped similarly.

We now remove the first four rows and columns from the original, permuted Hadamard matrix, and the remaining $(n-4)\times(n-4)$ matrix is called C. In this case $CC^T=(k-a_{ii})I_{u_1,\ldots,u_8}+a_{ij}J_{u_1,\ldots,u_8}$, where $k=n-4$, $a_{ii}=-4$, $a_{12}=-2$, $a_{13}=-2$, $a_{14}=0$, $a_{15}=-2$, $a_{16}=0$, $a_{17}=0$, $a_{18}=2$, $a_{23}=0$, $a_{24}=-2$, $a_{25}=0$, $a_{26}=-2$, $a_{27}=2$, $a_{28}=0$, $a_{34}=-2$, $a_{35}=0$, $a_{36}=2$, $a_{37}=-2$, $a_{38}=0$, $a_{45}=2$, $a_{46}=0$, $a_{47}=0$, $a_{48}=-2$, $a_{56}=-2$, $a_{57}=-2$, $a_{58}=0$, $a_{67}=0$, $a_{68}=-2$ and $a_{78}=-2$.

Then, with $k=4t-4$, *det* $CC^T=n^{n-12}\det D$, where D is the following matrix:

$$\begin{bmatrix}
4t-4u_1 & -2u_2 & -2u_3 & 0 & -2u_5 & 0 & 0 & 2u_8\\
-2u_1 & 4t-4u_2 & 0 & -2u_4 & 0 & -2u_6 & 2u_7 & 0\\
-2u_1 & 0 & 4t-4u_3 & -2u_4 & 0 & 2u_6 & -2u_7 & 0\\
0 & -2u_2 & -2u_3 & 4t-4u_4 & 2u_5 & 0 & 0 & -2u_8\\
-2u_1 & 0 & 0 & 2u_4 & 4t-4u_5 & -2u_6 & -2u_7 & 0\\
0 & -2u_2 & 2u_3 & 0 & -2u_5 & 4t-4u_6 & 0 & -2u_8\\
0 & 2u_2 & -2u_3 & 0 & -2u_5 & 0 & 4t-4u_7 & -2u_8\\
2u_1 & 0 & 0 & -2u_4 & 0 & -2u_6 & -2u_7 & 4t-4u_8
\end{bmatrix}.$$

In this particular case, as we have worked by hand, we have used the properties of determinants to further simplify our determinants before turning to the computer. However a computer program can work from this point straightforwardly. Divide the elements of each column by 2 so we have, *det* $CC^T=2^8n^{n-12}\det E$, where

$$E=\begin{bmatrix}
2t-2u_1 & -u_2 & -u_3 & 0 & -u_5 & 0 & 0 & u_8\\
-u_1 & 2t-2u_2 & 0 & -u_4 & 0 & -u_6 & u_7 & 0\\
-u_1 & 0 & 2t-2u_3 & -u_4 & 0 & u_6 & -u_7 & 0\\
0 & -u_2 & -u_3 & 2t-2u_4 & u_5 & 0 & 0 & -u_8\\
-u_1 & 0 & 0 & u_4 & 2t-2u_5 & -u_6 & -u_7 & 0\\
0 & -u_2 & u_3 & 0 & -u_5 & 2t-2u_6 & 0 & -u_8\\
0 & u_2 & -u_3 & 0 & -u_5 & 0 & 2t-2u_7 & -u_8\\
u_1 & 0 & 0 & -u_4 & 0 & -u_6 & -u_7 & 2t-2u_8
\end{bmatrix}.$$

Now, add twice the 8th row to the first; add twice the 7th row to the second; add the 8th row to the second, third and 5th rows; and add the 7th row to the first, fourth, and 6th rows. Take row four from rows one, two, three and half row four from row seven. Divide rows 1,2,3,4,5,6, by 2 and put 2^6 into the determinant multiplier. Take row 6 from rows 1,2,5 and 7 and divide rows 1, 2, 3 and 5 by t and put t^4 into the determinant multiplier. Now we have $\det CC^T = 2^6 n^{n-8} \det F$, where

$$F = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 & -1 & 2 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -u_3 & t-u_4 & 0 & 0 & t-u_7 & -u_8 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -u_5 & t-u_6 & t-u_7 & -u_8 \\ 0 & u_2 & 0 & -t+u_4 & 0 & -t+u_6 & 0 & u_8 \\ u_1 & 0 & 0 & -u_4 & 0 & -u_6 & -u_7 & 2t-2u_8 \end{bmatrix}.$$

Add u_5 times row 5 to row 6, take u_1 times row 1 from row 8, take u_2 times row 2 from row 7 and add u_3 times row 3 to row 4. Now expand the determinant to obtain $2^6 n^{n-8} det\ G$ where G is given by

$$G = \begin{bmatrix} t-u_3-u_4 & 0 & t-u_3-u_7 & u_3-u_8 \\ 0 & t-u_5-u_6 & t-u_5-u_7 & u_5-u_8 \\ -t+u_2+u_4 & -t+u_2+u_6 & 0 & u_8-u_2 \\ u_1-u_4 & u_1-u_6 & u_1-u_7 & 2t-2u_1-2u_8 \end{bmatrix}.$$

Hence $\det C = 2^3 n^{\frac{1}{2}n-4} \sqrt{\det G}$. We now use

$$u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8 = 4t-4;$$

$$0 \le t-u_1-u_2 \le 4; \quad u_1+u_2+a_2 = t; \quad 0 \le t-u_3-u_4 \le 4; \quad u_3+u_4+a_4 = t;$$

$$0 \le t-u_5-u_6 \le 4; \quad u_5+u_6+a_6 = t; \quad 0 \le t-u_7-u_8 \le 4; \quad u_7+u_8+a_8 = t,$$

where $0 \le a_2,\ a_4,\ a_6,\ a_8 \le 4$ are the slack variables and $a_2+a_4+a_6+a_8 = 4$.
So the determinant becomes

$$2^6(4t)^{n-8} \begin{bmatrix} a_4 & 0 & a_8 & t-u_4-u_8 \\ 0 & a_6 & a_8 & t-u_6-u_8 \\ -a_2 & -a_2 & a_2+a_8 & u_2-u_8 \\ t-u_2-u_4 & t-u_2-u_6 & u_2-u_8 & 2t-u_2-u_4-u_6-u_8 \end{bmatrix}.$$

We have evaluated the determinant of the theorem for all different values of a_i, by computer, and obtained the results in Table 1.

a_2	a_4	a_6	a_8	Value of determinant
				$X = 2t - u_2 - u_4 - u_6 - u_8$
0	4	0	0	0
0	0	4	0	0
0	0	0	4	0
0	3	1	0	$-3(u_2 - u_8)^2$
0	3	0	1	$-3(t - u_2 - u_6)^2$
0	2	2	0	$-4(u_2 - u_8)^2$
0	2	1	1	$2X - 2(u_2 - u_8)^2 - 2(t - u_2 - u_6)^2 - (t - u_2 - u_4)^2$
0	2	0	2	$-4(t - u_2 - u_6)^2$
0	1	3	0	$-3(u_2 - u_8)^2$
0	1	2	1	$2X - 2(u_2 - u_8)^2 - (t - u_2 - u_6)^2 - 2(t - u_2 - u_4)^2$
0	1	1	2	$2X - (u_2 - u_8)^2 - 2(t - u_2 - u_6)^2 - 2(t - u_2 - u_4)^2$
0	1	0	3	$-3(t - u_2 - u_6)^2$
0	0	3	1	$-3(t - u_2 - u_4)^2$
0	0	2	2	$-4(t - u_2 - u_4)^2$
0	0	1	3	$-3(t - u_2 - u_4)^2$
4	0	0	0	0
3	0	1	0	$-3(t - u_4 - u_8)^2$
3	0	0	1	$-3(u_4 - u_6)^2$
2	0	2	0	$-4(t - u_4 - u_8)^2$
2	0	1	1	$2X - 2(u_4 - u_6)^2 - 2(t - u_4 - u_8)^2 - (t - u_2 - u_4)^2$
2	0	0	2	$-4(u_4 - u_6)^2$
1	0	3	0	$-3(t - u_4 - u_8)^2$
1	0	2	1	$2X - (u_4 - u_6)^2 - 2(t - u_4 - u_8)^2 - 2(t - u_2 - u_4)^2$
1	0	1	2	$2X - 2(u_4 - u_6)^2 - (t - u_4 - u_8)^2 - 2(t - u_2 - u_4)^2$
1	0	0	3	$-3(u_4 - u_6)^2$
3	1	0	0	$-3(t - u_6 - u_8)^2$
2	2	0	0	$-4(t - u_6 - u_8)^2$
2	1	0	1	$2X - 2(u_4 - u_6)^2 - (t - u_2 - u_6)^2 - 2(t - u_6 - u_8)^2$
1	3	0	0	$-3(t - u_6 - u_8)^2$
1	2	0	1	$2X - (u_4 - u_6)^2 - 2(t - u_6 - u_8)^2 - 2(t - u_2 - u_6)^2$
1	1	0	2	$2X - 2(u_4 - u_6)^2 - (t - u_6 - u_8)^2 - 2(t - u_2 - u_6)^2$
2	1	1	0	$2X - (u_8 - u_2)^2 - 2(t - u_6 - u_8)^2 - 2(t - u_4 - u_8)^2$
1	2	1	0	$2X - 2(u_2 - u_8)^2 - 2(t - u_6 - u_8)^2 - (t - u_4 - u_8)^2$
1	1	2	0	$2X - 2(u_2 - u_8)^2 - (t - u_6 - u_8)^2 - 2(t - u_4 - u_8)^2$
1	1	1	1	$4X - (t - u_2 - u_4)^2 - (t - u_2 - u_6)^2 - (t - u_4 - u_8)^2$
				$-(t - u_6 - u_8)^2 - (u_2 - u_8)^2 - (u_4 - u_6)^2$

Table 1

We have chosen to work with the slack variable a_i rather than the u_i's as this leads to simpler expressions for calculating by hand.

For the case $a_2 = 1, a_4 = 1, a_6 = 1, a_8 = 1$ we take the constraints: $u_i + u_{i+1} = t - 1$ for i =1, 3, 5, 7 and find the expression $4X - (t - u_2 - u_4)^2 - (t - u_2 - u_6)^2 - (t - u_4 - u_8)^2 - (t - u_6 - u_8)^2 - (u_2 - u_8)^2 - (u_4 - u_6)^2$ for the determinant. We now evaluate this at the vertices of the convex hull (u_2, u_4, u_6, u_8) where exactly two of the variables (u_7, u_8) or (u_4, u_6) are $t - 1$ and two are zero to get the value 4. All the other vertices, except 12 which give determinant 1, give non-positive values for the determinant. (We found the 12 cases by exhaustive enumeration of the cases). However the determinant of CC^T must be a square and so the negative values are impossible. Hence the maximum determinant is 4.

For the convex hull where exactly one of the u_i is $t - 2$, exactly one is $t - 1$ and two are zero and two of the corresponding a_j are 1, one is 2 and one is zero. We get the maximum determinant is 1. Without loss of generality, as the expressions in Table 1 are symmetric in the variables in the sense that if $u_j = t-2$ and $u_i = 0$, then u_i occurs in all four terms of the expression and u_j occurs in the terms $(u_i - u_j)^2$ or $(t - u_i - u_j)^2$ which have coefficient 1.

Now we have two cases corresponding to different expressions. For the first case the expression for the determinant is

$$4t - 2u_i - 2u_j - 2u_k - 2u_\ell - (u_i - u_j)^2 - 2(t - u_i - u_k)^2 - 2(t - u_i - u_\ell)^2. \tag{6}$$

Then the determinant is 1 if $a_i = 0,\ a_j = 2,\ a_k = 1,\ a_\ell = 1, u_i = t-1,\ u_j = t-2,\ u_k = 0,\ u_\ell = 0$, so $u_{i-1} = 1,\ u_{j-1} = 0,\ u_{k-1} = t-1,\ u_{\ell-1} = t-1$.

In the other case the expression for the determinant is

$$4t - 2u_i - 2u_j - 2u_k - 2u_\ell - 2(u_i - u_k)^2 - (t - u_i - u_j)^2 - 2(t - u_i - u_\ell)^2. \tag{7}$$

The determinant is 1 if $a_i = 0,\ a_j = 2,\ a_k = 1,\ a_\ell = 1, u_i = t-1,\ u_j = 0,\ u_k = t-2,\ u_\ell = 0$, so $u_{i-1} = 1,\ u_{j-1} = t-2,\ u_{k-1} = 1,\ u_{\ell-1} = t-1$.

We now form a $4 \times (n-4)$ matrix by choosing the appropriate columns given at the beginning of this proof using the values of $u_1,\ u_2,\ u_3, \cdots, u_8$ found in the two cases. If we choose the first row and first column as all ones, which we can always do, there is at most one way, up to permutation of columns 2, 3, and 4 to complete the $4 \times (n-4)$ matrix to form an Hadamard matrix.

We now have to systematically look through all the possible $4 \times (n-4)$ submatrices which can be formed satisfying equations (6) and (7). Some of them could have no completion to an Hadamard matrix and some could have only one completion. In practice we found a single completion in each case.

We first consider an example of a solution to equation (6). For

$$4t - 2u_2 - 2u_4 - 2u_6 - 2u_8 - (u_2 - u_8)^2 - 2(t - u_2 - u_4)^2 - 2(t - u_2 - u_6)^2$$

then the determinant is 1 if $a_2 = 0,\ a_4 = 1,\ a_6 = 1,\ a_8 = 2, u_2 = t-1,\ u_4 = 0,\ u_6 = 0,\ u_8 = t-2$, so $u_1 = 1,\ u_3 = t-1,\ u_5 = t-1,\ u_7 = 0$.

This gives the $4 \times (n-4)$ submatrix:

$$\begin{array}{ccccc} & \overbrace{}^{t-1} & \overbrace{}^{t-1} & \overbrace{}^{t-1} & \overbrace{}^{t-2} \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & - \\ 1 & 1 & - & 1 & - \\ 1 & - & 1 & 1 & - \end{array}$$

By considering the inner products of these rows (the first with the other 3 is 2 and the other pairs are orthogonal) we find that this can be extended to an Hadamard matrix by using the 4×4 submatrix (or a column permutation thereof)

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ - & - & - & 1 \\ 1 & - & - & - \\ - & 1 & - & - \end{array}$$

as the upper lefthand corner (or lower righthand corner) of the Hadamard matrix. We note this 4×4 matrix has determinant 8. We now consider an example of a solution to equation (7). For the determinant expression

$$4t - 2u_2 - 2u_4 - 2u_6 - 2u_8 - 2(u_4 - u_6)^2 - (t - u_2 - u_6)^2 - 2(t - u_6 - u_8)^2.$$

The determinant is 1 if $a_2 = 2,\ a_4 = 1,\ a_6 = 0,\ a_8 = 1,\ u_2 = 0,\ u_4 = t - 2,\ u_6 = t - 1,\ u_8 = 0$ so $u_1 = t - 2,\ u_3 = 1,\ u_5 = 1,\ u_7 = t - 1$.

We find, similarly, this has a unique solution up to permutation of the same 4×4 submatrix.

The calculations for all other submatrices are similar. In each case the 4×4 submatrix had determinant 8.

Hence we have the $(n-4) \times (n-4)$ minors of an Hadamard matrix of order n are zero, $8n^{\frac{n}{2}-4}$ or $16n^{\frac{n}{2}-4}$. □

6 $(n-j) \times (n-j)$ minors of Hadamard matrices

We now outline the method to evaluate the $(n-5), (n-6), \ldots, (n-j)$ minors. For the $(n-j) \times (n-j)$ minors we evaluate

$$detCC^T = n^{n-2^{j-1}-j} detD \tag{8}$$

where D is the $2^{j-1} \times 2^{j-1}$ matrix given in Section 2.

The Algorithm for $(n-j) \times (n-j)$ minors of an $n \times n$ Hadamard matrix, H.

Step1: Generate ± 1 matrices M, of order j with first row and column all $+1$.

Step2: Form the general matrix, $N = [M \quad U_j]$, of size $j \times n$ for the first j rows of an $n \times n$ Hadamard matrix H, where U_j is given in (3).

Step3: For each M consider all $\binom{j}{3}$ subsets of three rows of N and use the Distribution Lemma with $\sum_{i=1}^{2^{j-1}} u_i = n - j$ to form 4 equations in the variables $u_1, \ldots, u_{2^{j-1}}$ for each subset. A total number of $4\binom{j}{3}$ equations from which $\binom{j}{2} + 1$ are the different ones.

Step 4: For each M search for all feasible solutions to the different equations generated at Step 3.

Step 5 For each M and for each feasible solution found in Step 4 use the matrix D to find all possible values of the $(n-j) \times (n-j)$ minors.

Implementation of the algorithm

How to efficiently generate the set of equations

We note that any triple of rows of N $\underline{a}, \underline{b}, \underline{c}$ and the equation $\sum_{i=1}^{2^{j-1}} u_i = n - j$ allows us to see that, writing the inner product of rows $\underline{a}, \underline{b}, \underline{c}$ in N allows us to obtain 4 equations

$$
\begin{aligned}
u_1 + u_2 + \ldots + u_{2^{j-1}} &= n - j \\
\underline{m}_a \cdot \underline{m}_b + L_1(u_1, \ldots, u_{2^{j-1}}) &= 0 \\
\underline{m}_a \cdot \underline{m}_c + L_2(u_1, \ldots, u_{2^{j-1}}) &= 0 \\
\underline{m}_b \cdot \underline{m}_c + L_3(u_1, \ldots, u_{2^{j-1}}) &= 0
\end{aligned}
\tag{9}
$$

where L_1, L_2, L_3 are linear combinations of $u_1, \ldots, u_{2^{j-1}}$. Solving the equations (9) will allow bounds on sums of subsets of at most 2^{j-3} u_k in terms of $\underline{m}_a \cdot \underline{m}_b, \underline{m}_a \cdot \underline{m}_c, \underline{m}_b \cdot \underline{m}_c$ and $\frac{n}{4}$.

Example 3 Let $n = 16$ and $j = 5$. We write the first five rows of H as

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	−	−	−	−	−	−	−	−
M	1	1	1	1	−	−	−	−	1	1	1	1	−	−	−	−
	1	1	−	−	1	1	−	−	1	1	−	−	1	1	−	−
	1	−	1	−	1	−	1	−	1	−	1	−	1	−	1	−

recalling that $u_1, u_2, \ldots, u_{16}$ are the number of columns of the kind indicated. M is the 5×5 submatrix that will be removed in forming the $(n-5) \times (n-5)$ minors. From the order we have

$$\sum_{i=1}^{16} u_i = 11 \tag{10}$$

We first see by the Distribution Lemma that

$$0 \le u_i \le 4 \tag{11}$$

Also we have the constraints:

$$4 - u_{1+4j} - u_{2+4j} - u_{3+4j} - u_{4+4j} \le 5, \quad j = 0, 1, 2, 3$$

By careful counting, we see that if the first row and column of M are $+1$, there are 50 matrices M with distinct inner product vectors.

The inner product of rows i and j of N plus (10) are

$$
\begin{aligned}
u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9+u_{10}+u_{11}+u_{12}+u_{13}+u_{14}+u_{15}+u_{16} &= 11 \\
(\underline{m}_1 \cdot \underline{m}_2)+u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8-u_9-u_{10}-u_{11}-u_{12}-u_{13}-u_{14}-u_{15}-u_{16} &= 0 \\
(\underline{m}_1 \cdot \underline{m}_3)+u_1+u_2+u_3+u_4-u_5-u_6-u_7-u_8+u_9+u_{10}+u_{11}+u_{12}-u_{13}-u_{14}-u_{15}-u_{16} &= 0 \\
(\underline{m}_2 \cdot \underline{m}_3)+u_1+u_2+u_3+u_4-u_5-u_6-u_7-u_8-u_9-u_{10}-u_{11}-u_{12}+u_{13}+u_{14}+u_{15}+u_{16} &= 0 \\
(\underline{m}_1 \cdot \underline{m}_4)+u_1+u_2-u_3-u_4+u_5+u_6-u_7-u_8+u_9+u_{10}-u_{11}-u_{12}+u_{13}+u_{14}-u_{15}-u_{16} &= 0 \\
(\underline{m}_2 \cdot \underline{m}_4)+u_1+u_2-u_3-u_4+u_5+u_6-u_7-u_8-u_9-u_{10}+u_{11}+u_{12}-u_{13}-u_{14}+u_{15}+u_{16} &= 0 \\
(\underline{m}_3 \cdot \underline{m}_4)+u_1+u_2-u_3-u_4-u_5-u_6+u_7+u_8+u_9+u_{10}-u_{11}-u_{12}+u_{13}+u_{14}-u_{15}-u_{16} &= 0 \\
(\underline{m}_1 \cdot \underline{m}_5)+u_1-u_2+u_3-u_4+u_5-u_6+u_7-u_8+u_9-u_{10}+u_{11}-u_{12}+u_{13}-u_{14}+u_{15}-u_{16} &= 0 \\
(\underline{m}_2 \cdot \underline{m}_5)+u_1-u_2+u_3-u_4+u_5-u_6+u_7-u_8-u_9+u_{10}-u_{11}+u_{12}-u_{13}+u_{14}-u_{15}+u_{16} &= 0 \\
(\underline{m}_3 \cdot \underline{m}_5)+u_1-u_2+u_3-u_4-u_5+u_6-u_7+u_8+u_9-u_{10}+u_{11}-u_{12}-u_{13}+u_{14}-u_{15}+u_{16} &= 0 \\
(\underline{m}_4 \cdot \underline{m}_5)+u_1-u_2-u_3+u_4+u_5-u_6-u_7+u_8+u_9-u_{10}-u_{11}+u_{12}+u_{13}-u_{14}-u_{15}+u_{16} &= 0
\end{aligned}
$$

This can of course be simplified analytically but as we propose using a computer package this is not necessary. All possible solutions of these inequalities should be used in the determinant of the corresponding matrix D to obtain numerical results. In this case D is, writing $m = n - 5$,

$$\begin{bmatrix} m & -3 & -3 & -1 & -3 & -1 & -1 & 1 & -3 & -1 & -1 & 1 & -1 & 1 & 1 & 3 \\ & m & -1 & -3 & -1 & -3 & 1 & -1 & -1 & -3 & 1 & -1 & 1 & -1 & 3 & 1 \\ & & m & -3 & -1 & 1 & -3 & -1 & -1 & 1 & -3 & -1 & 1 & 3 & -1 & 1 \\ & & & m & 1 & -1 & -1 & -3 & 1 & -1 & -1 & -3 & 3 & 1 & 1 & -1 \\ & & & & m & -3 & -3 & -1 & -1 & 1 & 1 & 3 & -3 & -1 & -1 & 1 \\ & & & & & m & -1 & -3 & 1 & -1 & 3 & 1 & -1 & -3 & 1 & -1 \\ & & & & & & m & -3 & 1 & 3 & -1 & 1 & -1 & 1 & -3 & -1 \\ & & & & & & & m & 3 & 1 & 1 & -1 & 1 & -1 & -1 & -3 \\ & & & & & & & & m & -3 & -3 & -1 & -3 & -1 & -1 & 1 \\ & & & & & & & & & m & -1 & -3 & -1 & -3 & 1 & -1 \\ & & & & & & & & & & m & -3 & -1 & 1 & -3 & -1 \\ & & & & & & & & & & & m & 1 & -1 & -1 & -3 \\ & & & & & & & & & & & & m & -3 & -3 & -1 \\ & & & & & & & & & & & & & m & -1 & -3 \\ & & & & & & & & & & & & & & m & -3 \\ & & & & & & & & & & & & & & & m \end{bmatrix}$$

where every constant in column i of D is multiplied by u_i, the diagonals becoming $m = n - 5u_i$, and D is, in fact, symmetric.

□

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7 Appendix: Determinant Simplification Theorems

Example 4

$$CC^T = (k - \lambda)I_v + \lambda J_v \tag{12}$$

is the $v \times v$ matrix

$$CC^T = \begin{bmatrix} k & \lambda & \cdots & \lambda \\ \lambda & k & \cdots & \lambda \\ \lambda & \lambda & \cdots & \lambda \\ \vdots & \vdots & & \vdots \\ \lambda & \lambda & \cdots & k \end{bmatrix}$$

□

Example 5

$$CC^T = (k - a_{ii})I_{u,v} + a_{ij}J_{u,v} \tag{13}$$

where $(a_{ij}) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is the $(u+v) \times (u+v)$ matrix

$$CC^T = \begin{bmatrix} \overbrace{\begin{matrix} k & a & \cdots & a \\ a & k & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & k \end{matrix}}^{u} & \overbrace{\begin{matrix} b & b & \cdots & b \\ b & b & \cdots & b \\ \vdots & \vdots & & \vdots \\ b & b & \cdots & b \end{matrix}}^{v} \\ \begin{matrix} b & b & \cdots & b \\ b & b & \cdots & b \\ \vdots & \vdots & & \vdots \\ b & b & \cdots & b \end{matrix} & \begin{matrix} k & a & \cdots & a \\ a & k & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & k \end{matrix} \end{bmatrix}$$

□

Lemma 3 *Suppose C is a matrix of order $v \times v$ satisfying $CC^T = (k-\lambda)I_v + \lambda J_v$ given in (12). Then, det $CC^T = (k + (v-1)\lambda)(k-\lambda)^{v-1}$.*

Proof. Use the determinant simplification theorem. □

Lemma 4 *Suppose C is a matrix of order $(u+v) \times (u+v)$, where $n = u+v$, satisfying $CC^T = (k - a_{ii})I_{u,v} + a_{ij}J_{u,v}$ given in (13). Then, det $CC^T = (k-a)^{n-2}$det D where*

$$D = \begin{bmatrix} k + (u-1)a & ub \\ vb & k + (v-1)a \end{bmatrix}.$$

Then, det $CC^T = (k-a)^{n-2}\left((k-a)^2 + an(k-a) + uv(a^2 - b^2)\right)$.

Proof. Use the determinant simplification theorem and expand D. □

Example 6

$$CC^T = (k - a_{ii})I_{u,v,w,x} + a_{ij}J_{u,v,w,x} \tag{14}$$

where $(a_{ij}) = \begin{bmatrix} a & b & c & d \\ b & a & e & f \\ c & e & a & g \\ d & f & g & a \end{bmatrix}$ is the $(u+v+w+x) \times (u+v+w+x)$ matrix

$$CC^T = \left[\begin{array}{cccc} \overbrace{\begin{matrix} k & a & \cdots & a \end{matrix}}^{u} & \overbrace{\begin{matrix} b & b & \cdots & b \end{matrix}}^{v} & \overbrace{\begin{matrix} c & c & \cdots & c \end{matrix}}^{w} & \overbrace{\begin{matrix} d & d & \cdots & d \end{matrix}}^{x} \\ \begin{matrix} a & k & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & k \end{matrix} & \begin{matrix} b & b & \cdots & b \\ \vdots & \vdots & & \vdots \\ b & b & \cdots & b \end{matrix} & \begin{matrix} c & c & \cdots & c \\ \vdots & \vdots & & \vdots \\ c & c & \cdots & c \end{matrix} & \begin{matrix} d & d & \cdots & d \\ \vdots & \vdots & & \vdots \\ d & d & \cdots & d \end{matrix} \\ \\ \begin{matrix} b & b & \cdots & b \\ b & b & \cdots & b \\ \vdots & \vdots & & \vdots \\ b & b & \cdots & b \end{matrix} & \begin{matrix} k & a & \cdots & a \\ a & k & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & k \end{matrix} & \begin{matrix} e & e & \cdots & e \\ e & e & \cdots & e \\ \vdots & \vdots & & \vdots \\ e & e & \cdots & e \end{matrix} & \begin{matrix} f & f & \cdots & f \\ f & f & \cdots & f \\ \vdots & \vdots & & \vdots \\ f & f & \cdots & f \end{matrix} \\ \\ \begin{matrix} c & c & \cdots & c \\ c & c & \cdots & c \\ \vdots & \vdots & & \vdots \\ c & c & \cdots & c \end{matrix} & \begin{matrix} e & e & \cdots & e \\ e & e & \cdots & e \\ \vdots & \vdots & & \vdots \\ e & e & \cdots & e \end{matrix} & \begin{matrix} k & a & \cdots & a \\ a & k & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & k \end{matrix} & \begin{matrix} g & g & \cdots & g \\ g & g & \cdots & g \\ \vdots & \vdots & & \vdots \\ g & g & \cdots & g \end{matrix} \\ \\ \begin{matrix} d & d & \cdots & d \\ d & d & \cdots & d \\ \vdots & \vdots & & \vdots \\ d & d & \cdots & d \end{matrix} & \begin{matrix} f & f & \cdots & f \\ f & f & \cdots & f \\ \vdots & \vdots & & \vdots \\ f & f & \cdots & f \end{matrix} & \begin{matrix} g & g & \cdots & g \\ g & g & \cdots & g \\ \vdots & \vdots & & \vdots \\ g & g & \cdots & g \end{matrix} & \begin{matrix} k & a & \cdots & a \\ a & k & \cdots & a \\ \vdots & \vdots & & \vdots \\ a & a & \cdots & k \end{matrix} \end{array}\right].$$

□

Lemma 5 *Suppose C is a matrix of order $(u+v+w+x) \times (u+v+w+x)$, where $n = u+v+w+x$, satisfying $CC^T = (k - a_{ii})I_{u,v,w,x} + a_{ij}J_{u,v,w,x}$ given by (14). Then*

$$det\ CC^T = (k-a)^{n-4} det\ D$$

where

$$D = \begin{bmatrix} k+(u-1)a & vb & wc & xd \\ ub & k+(v-1)a & we & xf \\ uc & ve & k+(w-1)a & xg \\ ud & vf & wg & k+(x-1)a \end{bmatrix}. \tag{15}$$

Proof. Use the determinant simplification theorem. □

Theorem 3 (Determinant Simplification Theorem) *Let*

$$CC^T = (k - a_{ii})I_{b_1,b_2,\cdots,b_z} + a_{ij}J_{b_1,b_2,\cdots,b_z}$$

then

$$det\ CC^T = \Pi_{i=1}^{z}(k - a_{ii})^{b_i - 1} det\ D \tag{16}$$

where

$$D = \begin{bmatrix} k + (b_1 - 1)a_{11} & b_2 a_{12} & b_3 a_{13} & \cdots & b_z a_{1z} \\ b_1 a_{21} & k + (b_2 - 1)a_{22} & b_3 a_{23} & \cdots & b_z a_{2z} \\ \vdots & \vdots & \vdots & & \vdots \\ b_1 a_{z1} & b_2 a_{z2} & b_3 a_{z2} & \cdots & k + (b_z - 1)a_{zz} \end{bmatrix}$$

Proof. We note the matrix CC^T has k down the diagonal and elsewhere the elements are defined by the block of elements a_{ij}.

We start with the first row and subtract it from the 2nd to the b_1th row. Then take the first row of the 2nd block (the $b_1 + 1$st row) and subtract it from the $b_1 + 2$th to $b_1 + b_2$th rows. We continue this way with each new block.

Now the first column has: $a_{11} - k\ b_1 - 1$ times, then a_{21} followed by $b_2 - 1$ rows of zero, then a_{31} followed by $b_3 - 1$ rows of zero, and so on until we have a_{z1} followed by $b_z - 1$ rows of zero. We now add columns 2 to b_1 to the first column: each of the columns $b_1 + 2$nd to $b_1 + \ldots + b_2$th to the $b_1 + 1$st column; and so on until finally add each of the columns $b_1 + \cdots + b_{z-1} + 2$nd to $b_1 + \cdots + b_{z-1}$th to the $b_1 + \cdots + b_{z-1} + 1$st column.

The rows which contain zero in the first column will have $k - a_{ii}$ on the diagonal and all other elements zero. They can then be used to zero every element in their respective columns.

We now expand the determinant, taking into the coefficient those rows and columns which contain $k - a_{ii}$ as required. The remaining matrix to be evaluated is D given in the enunciation.
□