Values of Minors of $(1,-1)$ Incidence Matrices of SBIBDs and Their Application to the Growth Problem

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Abstract

We obtain explicit formulae for the values of the $v-j$ minors, $j=0,1,2$ of $(1,-1)$ incidence matrices of $SBIBD(v,k,\lambda)$. This allows us to obtain explicit information on the growth problem for families of matrices with moderate growth. An open problem remains to establish whether the $(1,-1)$ CP incidence matrices of SBIBD(v,k,λ), can have growth greater than v for families other than Hadamard families.

Key Words and Phrases: Incidence matrices, SBIBD, minors, Gaussian elimination, growth, complete pivoting.
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1 Introduction

We evaluate the $v-j,\quad j=0,1,2$ minors for $(1,-1)$ incidence matrices of certain SBIBDs. For the purpose of this paper we will define a SBIBD(v,k,λ) to be a $v\times v$ matrix, B, with entries 0 or 1, which has exactly k entries $+1$ and $v-k$ entries 0 in each row and column and for which the inner product of any distinct pairs of rows and columns is λ. The $(1,-1)$ incidence matrix of B is obtained by letting $A=2B-J$, where J is the $v\times v$ matrix with entries all $+1$. We write I for the identity matrix of order v.

Then we have

$$BB^T=(k-\lambda)I+\lambda J \tag{1}$$

and

$$AA^T=4(k-\lambda)I+(v-4(k-\lambda))J \tag{2}$$

The determinant simplification theorem (see the Appendix) shows that

$$detB=(k-\lambda)^{\frac{v-1}{2}}\sqrt{k+(v-1)\lambda}=k(k-\lambda)^{\frac{1}{2}(v-1)}$$

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and since $\lambda(v-1) = k^2 - k$,

$$\det A = 2^{v-1}(k-\lambda)^{\frac{v-1}{2}}|v-2k| \tag{3}$$

or with $x = v - 4k + 4\lambda$,

$$\det A = (v-x)^{\frac{1}{2}(v-1)}|v-2k|.$$

We see the determinant is greatest for values of x close to zero. Indeed $x = 0$ for Hadamard matrices.

In this paper we study the application of the computed values of the minors to the growth problem. Let $A = [a_{ij}] \in \mathcal{R}^{n\times n}$. We reduce A to upper triangular form by using Gaussian elimination with complete pivoting (GECP) [9]. Let $A^{(k)} = [a_{ij}^{(k)}]$ denote the matrix obtained after the first k pivoting operations, so $A^{(n-1)}$ is the final upper triangular matrix. A diagonal entry of that final matrix will be called a pivot. Matrices with the property that no exchanges are actually needed during GECP are called completely pivoted (CP). Let $g(n, A) = \max_{i,j,k} |a_{ij}^{(k)}|/|a_{11}^{(0)}|$ denote the growth associated with GECP on A and $g(n) = \sup\{\, g(n, A)/A \in \mathcal{R}^{n\times n}\,\}$. The problem of determining $g(n)$ for various values of n is called the growth problem.

The determination of $g(n)$ remains a mystery. Wilkinson [9] conjectured that $g(n, A) \leq n \forall\ A \in \mathcal{R}^{n\times n}$. This conjecture is now known to be false [4]. One of the curious frustrations of the growth problem is that it is quite difficult to construct any examples of $n \times n$ matrices A other than Hadamard for which $g(n, A)$ is even close to n. In order to obtain matrices with large growth sophisticated numerical optimization techniques must be applied [4]. By using such methods, matrices with growth larger than $n = 13, 14, 15, 16, 18, 20, 25$ were specified. For a special category of orthogonal matrices $H \in \mathcal{R}^{n\times n}$ with elements ± 1 and $HH^T = nI$, it has been observed that $g(n, H) = n$. This equality has been proved for a certain class of $n \times n$ Hadamard matrices [3]. It has also been observed that weighing matrices of order n can give $g(n, H) = n - 1$ [5].

Explicit information for the pivot structure for families of matrices achieving moderate growth is derived and an open problem concerning the possibility of finding $(1, -1)$ incidence matrices of SBIBD(v, k, λ) having growth greater than v is posed. We conjecture

Conjecture (The growth conjecture for $(1, -1)$ incidence matrices of SBIBD(v, k, λ))

Let A be an $v \times v$ CP $(1, -1)$ incidence matrices of SBIBD(v, k, λ). Let $x = v - 4(k - \lambda)$ and v, k, λ not trivial. Reduce A by GE. Then we conjecture

(i) $g(v, A) = \frac{(v-x)(v-2k)(v-1)}{2(v-2k-1)(v-k)}$, or $\frac{(v-x)(v-2k)(v-1)}{2k(v-2k+1)}$;

(ii) The last pivot is equal to $\frac{(v-x)(v-2k)(v-1)}{2(v-2k-1)(v-k)}$, or $\frac{(v-x)(v-2k)(v-1)}{2k(v-2k+1)}$;

(iii) The second last pivot is equal to $\frac{(v-x)}{2}$ or $\frac{v-x}{2} \cdot \frac{k(v-2k+1)}{(v-2k-1)(v-k)}$;

(iv) Every pivot before the last has magnitude at most $\frac{v-x}{2}$.

Notation 1. Write A for a matrix of order n whose initial pivots are derived from matrices with CP structure. Write $A(j)$ for the absolute value of the determinant of the $j \times j$ principal submatrix in the upper lefthand corner of the matrix A and $A[j]$ for the absolute value of the determinant of the $(n-j)\times(n-j)$ principal submatrix in the bottom righthand corner of the matrix A. Throughout this paper when we have used i pivots we then find all possible values of the $A(n-i)$ minors. Hence, if any minor is CP it must have one of these values. The magnitude of the pivots appearing after the application of GE operations on a CP matrix W is given by

$$p_j = W(j)/W(j-1), \quad j = 1,2,\ldots,n, \quad W(0)=1. \tag{4}$$

In particular for a CP SBIBD(v,k,λ), A,

$$p_v = A(v)/A(v-1), \quad p_{v-1} = A(v-1)/A(v-2.) \tag{5}$$

We use the notation M_j to denote the $j\times j$ minor of A.

In all determinants studied with an underlying SBIBD(v,k,λ) the minors of size $v-j$ where $j>\lambda$ may need special analysis. When $\lambda=j$ the determinant of the $n-j$ minor obtained by using the two columns with λ ones in their first λ places gives a special case. This has been calculated for $\lambda=1$ but resulted in the same determinants as Table 2. We have not considered in this paper the $v-2\times v-2$ minors of SBIBD$(v,k,2)$.

For completeness we give the determinant simplification theorem in Appendix as we use it extensively in this paper.

2 The $(v-1)\times(v-1)$ minors

Clearly the determinant of the (±1) incidence matrix of an $SBIBD(v,k,\lambda)$ and the (±1) incidence matrix of its complementary $SBIBD(v,v-k,v+2k-\lambda)$ are negatives of each other. Hence their determinants have the same absolute value.

We show the $(v-1)\times(v-1)$ subdeterminants of the (±1) incidence matrix of an $SBIBD(v,k,\lambda)$ take two values. We show one value arises if the (1,1) and (1,2) elements have the same sign (+) and the other occurs if the (1,1) and (1,2) elements have different signs. These also correspond to forming the $(v-1)\times(v-1)$ submatrix by removing the first row and column of the $SBIBD(v,k,\lambda)$ and its complementary $SBIBD(v,v-k,v-2k+\lambda)$.

Theorem 1 *The $(v-1)\times(v-1)$ minors of the $(1,-1)$ incidence matrix of an $SBIBD(v,k,\lambda)$, A, have value*

$$(v-x)^{\frac{1}{2}(v-3)}\sqrt{(v-2k\pm1)^2x+(x-1)(-vx-v+2x)}. \tag{6}$$

$$= \begin{cases} (v-x)^{\frac{1}{2}(v-3)}2(v-k)\frac{(v-2k-1)}{(v-1)} \\ (v-x)^{\frac{1}{2}(v-3)}2k\frac{(v-2k+1)}{(v-1)} \end{cases}$$

where $x=v-4(k-\lambda)$ and $\lambda=k\frac{(k-1)}{(v-1)}$. For $x=1$ the minor is

$$(v-1)^{\frac{1}{2}(v-3)}(v-2k\pm1).$$

Proof. The inner product of any two rows of the $(1,-1)$ incidence matrix of any $SBIBD(v,k,\lambda)$ is $x=v-4(k-\lambda)$. We have two cases:

	1	$1\ 1\ \cdots\ -1\ -1$
	1	
$k-1$	$\vdots$	C_1
	1	
	$-$	
$v-k$	$\vdots$	C_2
	$-$	

Figure 1: The Generic Form for Case$(v-1,1)$

	$-$	$1\ 1\ \cdots\ -1\ -1$
	1	
k	$\vdots$	C_1
	1	
	$-$	
$v-k-1$	$\vdots$	C_2
	$-$	

Figure 2: The Generic Form For Case$(v-1,2)$

Case$(v-1,1)$ We rearrange the rows and columns of A until we have the matrix structure in Figure 1, where C_1 is rows $2,\ldots,k$ and columns $2,\ldots,v$, while C_2 is rows $k+1,\ldots,v$ and columns $2,\ldots,v$. The inner product of any pair of rows of C_1 is $x-1$; the inner product of any pair of rows of C_2 is $x-1$; the inner product of any pair of rows where one row is in C_1 and the other row is in C_2 is $x+1$. Thus we have, with $C^T = \left[C_1^T\ C_2^T\right]$, the determinant simplification theorem with $q=v-k$, $p=k-1$ gives

$$\begin{aligned}\det\,(CC^T) &= (v-x)^{v-3}\begin{vmatrix} v-1+(x-1)(p-1) & (x+1)q \\ (x+1)p & v-1+(x-1)(q-1)\end{vmatrix} \\ &= (v-x)^{v-3}[(v-2k+1)^2x+(x-1)(-vx-v+2x)]. \qquad (7) \\ &= (v-x)^{(v-3)}4(v-k)^2\frac{(v-2k-1)^2}{(v-1)^2}\end{aligned}$$

Case$(v-1,2)$ We rearrange the rows and columns of A until we have the matrix arranged as in Figure 2. C_1 is rows $2,\ldots,k+1$ and columns $2,\ldots,v$, while C_2 is rows $k+2,\ldots,v$ and columns $2,\ldots,v$. The inner product of any pair of rows of C_1 is $x-1$; the inner product of any pair of rows of C_2 is $x-1$; the inner product of any pair of rows where one row is in C_1 and the other row is in C_2 is $x+1$.

Thus we have, using the determinant simplification theorem with $p=k$ and $q=v-k-1$,

$$\det\,(CC^T) = (v-x)^{v-3}[(v-2k-1)^2x+(x-1)(-vx-v+2x)], \qquad (8)$$

$$
\begin{array}{c|cc|cc}
 & 1 & b & 1 & \dots \\
 & c & d & 1 & \dots \\
\hline
 & 1 & 1 & & \\
p & \vdots & \vdots & C_1 & \\
 & 1 & 1 & & \\
 & 1 & - & & \\
q & \vdots & \vdots & C_2 & \\
 & 1 & - & & \\
 & - & 1 & & \\
r & \vdots & \vdots & C_3 & \\
 & - & 1 & & \\
 & - & - & & \\
s & \vdots & \vdots & C_4 & \\
 & - & - & &
\end{array}
$$

Figure 3: The Generic Matrix for the Seven Cases for Minors $A(v-2)$

$$= (v-x)^{(v-3)} 4k^2 \frac{(v-2k+1)^2}{(v-1)^2}$$

as required. The result for $x = 1$ is obtained by substitution and rearrangement. □

Each $SBIBD(v, k, \lambda)$ has a complementary $SBIBD(v, v-k, v-2k+\lambda)$. The second result (8) can also be obtained from the first (7) above by applying the first result (7) to the complementary $SBIBD(v, k, \lambda)$.

3 The $(v-2) \times (v-2)$ minors

We now consider the $(v-2) \times (v-2)$ minors. We first observe that while $\lambda = 1$ causes no change to the value of the $(v-1) \times (v-1)$ determinant, there is potential for the $(v-2) \times (v-2)$ to differ. As before we consider all cases.

We choose the (1,1) element to be 1 (otherwise we negate the whole matrix, obtaining a matrix whose determinant has the same absolute value, and proceed as before). We choose the matrix to be CP in the first two steps and thereafter find all possible determinants of $A(v-2)$ so if the matrix is CP it must have one of these determinants.

We consider seven cases after rearranging the rows and columns of A: called Case $(v-2, i)$, $i = 1, 2, 3, 4, 5, 6, 7$. Only cases i=1,2,3 can possibly be CP. We have the matrix arranged as in Figure 3. Here p, q, r and u are the number of columns beginning with the corresponding two first elements,$(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$ respectively as descibed in Table 1, and $p + q + r + u = v - 2$.

Write $x = v - 4(k - \lambda)$. Then a simple variant of the determinant simplification theorem gives the required determinant of CC^T, in this case, as

	1	b	p	q	r	u
	c	d				
Case$(v-2,1)$	1	1	$\lambda-1$	$k-\lambda-1$	$k-\lambda$	$v-2k+\lambda$
	1	–				
Case$(v-2,2)$	1	1	$\lambda-1$	$k-\lambda$	$k-\lambda-1$	$v-2k+\lambda$
	–	1				
Case$(v-2,3)$	1	–	λ	$k-\lambda-1$	$k-\lambda$	$v-2k+\lambda-1$
	–	–				
Case$(v-2,4)$	1	1	$\lambda-2$	$k-\lambda$	$k-\lambda$	$v-2k+\lambda$
	1	1				
Case$(v-2,5)$	1	1	$\lambda-1$	$k-\lambda$	$k-\lambda$	$v-2k+\lambda-1$
	–	–				
Case$(v-2,6)$	1	–	λ	$k-\lambda-1$	$k-\lambda-1$	$v-2k+\lambda$
	–	1				
Case$(v-2,7)$	1	–	λ	$k-\lambda-2$	$k-\lambda$	$v-2k+\lambda$
	1	–				

Table 1: Parameter Values for Cases$(v-2,i)$, $i=1,\ 2,\cdots,\ 7$

$$(v-x)^{v-6}\begin{vmatrix} v-x+p(x-2) & qx & rx & u(x+2) \\ px & v-x+q(x-2) & r(x+2) & ux \\ px & q(x+2) & v-x+r(x-2) & ux \\ p(x+2) & qx & rx & v-x+u(x-2) \end{vmatrix}. \tag{9}$$

Now using $P=v-x-4p$, $Q=v-x-4q$, $R=v-x-4r$, and $U=v-x-4u$ we obtain

$$\det\,(CC^T) = (v-x)^{v-6}[4(x+1)(pqRU+prQU+quPR+ruPQ) \tag{10}$$
$$+(x+2)(pQRU+qPRU+rPQU+uPQR)+PQRU].$$

We now consider the seven cases summarized in Table 1 and find by lengthy but straightforward calculation, the minors given in Table 2.

Lemma 1 *The value of the* $(v-2)\times(v-2)$ *minors obtained from Case*$(v-1,1)$ *and Case*$(v-1,2)$ *are equal.*

Proof. This can be seen from Table 2 or by noting the 2×2 principal submatrices are permutations of each other. □

Hence we have

Theorem 2 *The* $(v-2)\times(v-2)$ *minors of the* $(1,-1)$ *incidence matrix of an SBIBD*$(v,\ k,\ \lambda)$, *A, have values as summarized in Table 2.*

	1	b	P	Q	R	U	det C$/(v-x)^{\frac{v-5}{2}}$
	c	d					
Case$(v-2,1)$	1	1	$2(v-x-2k+2)$	4	0	$-2(v+x-2k)$	$4(v-2k-1)\frac{(v-k)}{(v-1)}$
	1	$-$					
Case$(v-2,2)$	1	1	$2(v-x-2k+2)$	0	4	$-2(v+x-2k)$	$4(v-2k-1)\frac{(v-k)}{(v-1)}$
	$-$	1					
Case$(v-2,3)$	1	$-$	$2(v-x-2k)$	4	0	$-2(v+x-2k+2)$	$4k\frac{(v-2k+1)}{(v-1)}$
	$-$	$-$					
Case$(v-2,4)$	1	1	$2(v-x-2k+4)$	0	0	$-2(v+x-2k)$	0
	1	1					
Case$(v-2,5)$	1	1	$2(v-x-2k+2)$	0	0	$-2(v+x-2k+2)$	0
	$-$	$-$					
Case$(v-2,6)$	1	$-$	$2(v-x-2k)$	4	4	$-2(v+x-2k)$	$4\frac{((v-2k)^2-k)}{(v-1)}$
	$-$	1					
Case$(v-2,7)$	1	$-$	$2(v-x-2k)$	8	0	$-2(v+x-2k)$	0
	1	$-$					

Table 2: Determinant Values for Cases$(v-2,i)$, $i=1,\ 2,\cdots,\ 7$

For $x=1$ these give the minor as

$$2(v-1)^{\frac{1}{2}(v-5)}(v-2k\pm 1) \quad \text{or} \quad 0 \quad \text{or} \quad 2(v-1)^{\frac{1}{2}(v-5)}\sqrt{2[(v-2k)^2+2\pm 1-2v]}.$$

Proof. Use $p=\lambda-1$, $q=k-\lambda-1$, $r=k-\lambda$, $s=v-2k-\lambda$, $P=2(v-x-2k+2)$, $Q=4$, $R=0$ and $U=-2(v+x-2k)$ in (10) to get the the result for Case$(v-2,1)$ and Case$(v-2,2)$. The result for Case$(v-2,3)$ arises by replacing k by $v-k$.

The result for Case$(v-2,3)$ also comes from using $p=\lambda$, $q=k-\lambda$, $r=k-\lambda-1$, $s=v-2k+\lambda-1$, $P=2(v-x-2k)$, $Q=0$, $R=4$ and $U=-2(v+x-2k+2)$ in (10). Only these three cases could be part of a CP matrix.

The determinant of the $(v-2)\times(v-2)$ minors of the $1,-1$ incidence matrix of an $SBIBD$ obtained by deleting the minors with elements

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ - & - \end{bmatrix} \quad \begin{bmatrix} - & - \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} - & - \\ - & - \end{bmatrix}$$

is found, by using $Q=R=0$ in equation (10), to be zero. Thus the determinant in Case$(v-2,4)$ and Case$(v-2,5)$ is zero.

Case$(v-2,6)$ and Case$(v-2,7)$ are obtained by using the appropriate values for P, Q, R and U from Table 2 □

$\frac{\text{Case}(v)}{\text{Case}(v-1,1)}$	$\frac{(v-x)(v-2k)(v-1)}{2(v-2k-1)(v-k)}$
$\frac{\text{Case}(v)}{\text{Case}(v-1,2)}$	$\frac{(v-x)(v-1)(v-2k)}{2k(v-2k+1)}$

Table 3: Pivots p_v for SBIBD

$\frac{\text{Case}(v)}{\text{Case}(v-1,1)}$	$\frac{(v-1)(v-2k)}{v-2k+1}$
$\frac{\text{Case}(v)}{\text{Case}(v-1,2)}$	$\frac{(v-1)(v-2k)}{v-2k-1}$

Table 4: Pivots p_v for SBIBD for $x = 1$

Remark 1 We see, by observing the upper lefthand 2×2 matrix, that Case$(v-2,3)$ is the complement of Case$(v-2,1)$. □

4 Pivot structure for SBIBD(v, k, λ)

We observe that M_{v-1} has two values that may appear in CP matrices

$$Case(v-1,1) = (v-x)^{\frac{1}{2}(v-3)}2(v-k)\frac{(v-2k-1)}{(v-1)}, \quad Case(v-1,2) = (v-x)^{\frac{1}{2}(v-3)}2k\frac{(v-2k+1)}{(v-1)}$$

We observe that M_{v-2} has four values of which the following two may appear in CP matrices

$$Case(v-2, 1 \text{ or } 2) = 4(v-x)^{\frac{1}{2}(v-5)}(v-2k-1)\frac{(v-k)}{(v-1)}, \quad Case(v-2,3) = 4k(v-x)^{\frac{1}{2}(v-5)}\frac{(v-2k+1)}{(v-1)}.$$

We use (5). Then in Case$(v-1,1)$ and Case$(v-1,2)$ the pivots are given in Table 3. For $x = 1$ the pivots are given in Table 4.

Lemma 2 *The pivot* $p_v = \frac{(v-x)(v-1)(v-2k)}{2k(v-2k+1)} > v$ *for an SBIBD*(v, k, λ).
Proof. We note for a SBIBD(v, k, λ), $\lambda = k\frac{(k-1)}{(v-1)}$ so $x = 4 - 4k + 4k\frac{(k-1)}{(v-1)}$. We used Matlab to evaluate the pivot more exactly obtaining

$$p_v = \frac{(v-2k)((v-2k)^2 + (v-4)(v-1))}{2k(v-2k+1)} > v.$$

Now $v > k$ for all non trivial SBIBD so $p_v > v$ for $v \neq 2k$. □

We note those cases associated with Cases$(v-2, \ell)$, $\ell = 4, 5, 6$ or 7 cannot occur in CP matrices. For the remainder of this paper we will not consider the Cases $(v-2,4)$, $(v-2,5)$, $(v-2,6)$, $(v-2,7)$ further. We use (5). Then in Case$(v-1,1)$ and Case$(v-1,2)$ the pivots are given in Table 5. For $x = 1$ the pivots are given in Table 6.

$\frac{\text{Case}(v-1,1)}{\text{Case}(v-2,1)}$	$\frac{v-x}{2}$
$\frac{\text{Case}(v-1,1)}{\text{Case}(v-2,3)}$	$\frac{v-x}{2}\frac{(v-k)(v-2k-1)}{2k(v-2k+1)}$
$\frac{\text{Case}(v-1,2)}{\text{Case}(v-2,2)}$	$\frac{v-x}{2}\frac{k(v-2k+1)}{(v-2k-1)(v-k)}$
$\frac{\text{Case}(v-1,2)}{\text{Case}(v-2,3)}$	$\frac{v-x}{2}$

Table 5: Pivots p_{v-1} for SBIBD

$\frac{\text{Case}(v-1,1)}{\text{Case}(v-2,1)}$	$\frac{v-1}{2}$
$\frac{\text{Case}(v-1,1)}{\text{Case}(v-2,3)}$	$\frac{(v-1)(v-2k-1)}{2(v-2k+1)}$
$\frac{\text{Case}(v-1,2)}{\text{Case}(v-2,1)}$	$\frac{(v-1)(v-2k+1)}{2(v-2k-1)}$
$\frac{\text{Case}(v-1,2)}{\text{Case}(v-2,3)}$	$\frac{v-1}{2}$

Table 6: Pivots p_{v-1} for SBIBD where $x = 1$

Remark 2 We found for all $(1, -1)$ SBIBD(v, k, λ), except those related to Hadamard matrices, there is the theoretical possibility that the growth is greater than the order v. In practice we were unable to find any CP $(1, -1)$ SBIBD with growth $> v$ leaving this possibility as an intriguing open question. □

5 Pivot structure for SBIBD$(2s^2 + 2s + 1, s^2, \frac{1}{2}s(s-1))$

In this section we are especially interested in the growth problem for the Brouwer family SBIBD$(2s^2 + 2s + 1, s^2, \frac{1}{2}s(s-1))$, $(x = 1)$. The Brouwer family for s an odd prime power can be found in [2]. For $s = 2$ the $SBIBD(13, 4, 1)$ is well known as the projective plane of order 3. The result for $s = 4$ was given by Bridges, Hall, and Hayden [1].

Let d_v denote the maximum determinant of all $v \times v$ matrices with elements ± 1. It follows from Hadamard's inequality that $d_v \leq v^{v/2}$ and it is easily shown that equality can only hold if $v = 1$ or 2 or if $v \equiv 0 \pmod 4$. If $v \equiv 1 \pmod 4$, $v \neq 1$, Payne [6], showed that

$$d_v \leq (v-1)^{(v-1)/2}\sqrt{2v-1}$$

and equality can hold only if $v = 2s^2 + 2s + 1$, $s = 1, 2, 3, \ldots$. In this case we can write d_v as

$$d_v \leq 2^{s^2+s} s^{s^2+s} (s+1)^{s^2+s} (2s+1).$$

Raghavarao [7] constructed these $v \times v$ matrices with elements ± 1 for $v = 5, 13, 25$ with maximum determinant. Brouwer [2] constructed the $v \times v$ matrices with maximum determinant for $v = 2s^2 + 2s + 1$, when s is an odd prime power, i.e., for $v = 25, 61, 113, 181, 265, \ldots$.

Conjecture (The growth conjecture for Brouwer's SBIBD$(2s^2 + 2s + 1, s^2, \frac{1}{2}s(s-1))$)

Let A be an $v \times v$ CP SBIBD$(2s^2+2s+1, s^2, \frac{1}{2}s(s-1))$ of the Brouwer type. Reduce A by GE. Then we conjecture

(i) $g(v, A) = s(2s+1), \quad \text{or} \quad (s+1)(2s+1) \quad v > 13$;

(ii) The last pivot is equal to $s(2s+1)$ or $(s+1)(2s+1)$;

(iii) The second last pivot is equal to $s(s+1) = \frac{v-1}{2}$ or s^2 or $(s+1)^2$;

(iv) Every pivot before the last has magnitude at most $\frac{v-1}{2}$;

(v) The first four pivots are equal to $1, 2, 2, 4$;

(vi) The fifth pivot may be 2 or 3.

We prove (ii), (iii), (v) and (vi) in this paper.

Theorem 3 *Let A be the $v \times v$ SBIBD$(2s^2+2s+1, s^2, \frac{1}{2}s(s-1))$ design of Brouwer type. Reduce A by GECP, then the last two pivots are $s(2s+1)$, and $s(s+1) = \frac{v-1}{2}$ or s^2, respectively, for Case$(v-1,1)$ and $(2s+1)(s+1)$ and $s(s+1)$* or *$(s+1)^2$, respectively, for Case$(v-1,2)$.*

Proof. Since

$$\begin{aligned} M_v &= (2s^2+2s)^{s^2+s}(2s+1) \\ M_{v-1} &= 2(s+1)(2s^2+2s)^{s^2+s-1} \quad \text{or} \quad 2s(2s^2+2s)^{s^2+s-1} \\ M_{v-2} &= 4(s+1)(2s^2+2s)^{s^2+s-2} \quad \text{or} \quad 4s(2s^2+2s)^{s^2+s-2} \end{aligned}$$

using (5) we obtain the required result. □

We give some values for the family $SBIBD(2s^2+2s+1, s^2, \frac{1}{2}s(s-1))$.

v	s	$Case(v-1,1)$			$Case(v-1,2)$		
		p_v	p_{v-1}		p_v	p_{v-1}	
		$s(2s+1)$	$s(s+1)$ or	s^2	$(s+1)(2s+1)$	$s(s+1)$ or	$(s+1)^2$
5	1	3	2				
25	3	21	12	9	28	12	16
41	4	36	20	16	45	20	25
61	5	55	30	25	66	30	36

As the SBIBD for $v = 5$ is unique, we show, by computation, there is no entry for $v = 5$ in Case $(v-1,2)$.

By detecting the pivot structure of Brouwer's $SBIBD(2s^2+2s+1, s^2, \frac{1}{2}s(s-1))$, Table 7 was computed.

Remark 3 We experimented with the SBIBD(25,9,3). After testing 40000 equivalent matrices we observed that always the five first pivots were $1, 2, 2, 4, 2$ *or* 3 whereas the three last pivots in backward order were $\frac{24}{2}$ *or* $\frac{24}{5/2}$, $\frac{24}{2}$ *or* $\frac{24}{3/2}$, 21. □

s	v	**growth**	**Pivot Pattern**
3	25	21	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},4,4,\frac{9}{2},5.1825,5.4857,4.75,5.4737,$ $5.1923,\frac{16}{3},\frac{24}{10/3},\frac{16}{3},\frac{24}{16/5},\frac{24}{10/3},\frac{24}{3},\frac{24}{16/5},\frac{24}{5/2},\frac{24}{2},21)$
4	41	36	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},4,4,\frac{13}{3},4.9231,\ldots,$ $\frac{40}{4},\frac{40}{4},\frac{40}{2},\frac{40}{4},\frac{40}{2},\frac{40}{2},36)$
5	61	55	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},4,4,5,5.4,\ldots,\frac{60}{4},\frac{60}{2},\frac{60}{2},55)$

Table 7: Growth Factors and Pivots Patterns for small CP Brouwer SBIBD

The next result is easy to prove using a counting argument and noting the inner product of every pair of rows is $+1$ to see that the design always contains a 4×4 Hadamard matrix.

Proposition 1 *Let A be the $v \times v$ $(1,-1)$ incidence matrix of an SBIBD of the Brouwer type. Reduce A by GECP then the magnitudes of the first four pivots are $1,2,2$ and 4; the magnitude of $|a_{55}^{(4)}|$ is 2 or 3.*

Proof: Since the design always contains a 4×4 Hadamard matrix, this can be moved to be the 4×4 principal minor without changing the CP property.Thus the first four pivots will be $1,2,2$ and 4 [3]. Because every entry in $A^{(3)}$ is of magnitude $0,2$ or 4, pivoting on $a_{44}^{(3)}$ will only involve adding ± 1 or $\pm 1/2$ times the fourth row of $A^{(3)}$ to the rows below, and this will create only integer entries in $A^{(4)}$. Thus $|a_{55}^{(4)}|$ must be an integer satisfying the relation

$$A(1\,2\,3\,4\,5) = 16|a_{55}^{(4)}| \leq 4^{4/2}\sqrt{10-1} \Rightarrow |a_{55}^{(4)}| \leq 3.$$

where $A(1\,2\,3\,4\,5)$ denotes the determinant of the 5×5 principal submatrix of A. Thus $|a_{55}^{(4)}|$ must be $1,2$ or 3. To see that it cannot be 1 is to show that one could not have

$$A^{(4)} = \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 4 & \\ & & & & B \end{bmatrix}$$

where every entry of B is zero or ± 1; for, if that were true, then B would be a normalized $(v-4) \times (v-4)$ matrix, and so

$$|detB| \leq (v-4)^{\frac{v-4}{2}}.$$

But $|detB| = \frac{(v-1)^{\frac{v-1}{2}}\sqrt{2v-1}}{16}$ and it is easily checked that these cannot both hold when $v > 4$. $\square$

Proposition 2 *Let A be the 5×5 $(1,-1)$ incidence matrix of an SBIBD of the Brouwer type. Reduce A by GECP then the pivot structure is unique and is equal to $(1,2,2,4,3)$.*

Proof: Because of the above proposition we have that the first four pivots are $1,2,2,4$. It follows that $A(5) = (5-1)^{(5-1)/2}\sqrt{2 \cdot 5 - 1} = 48$ since $5 = 2s^2 + 2s + 1$, $s = 1$. Thus the fifth pivot will be defined by the relation $p_5 = \frac{A(5)}{A(4)} = \frac{48}{16} = 3$. $\square$

	Minor of $(2s^2+2s+1, s^2, \lambda)$ $\lambda = \frac{1}{2}(s^2-s))$	Minor of $(4s^2, 2s^2+s, s^2+s)$	Minor of $(4t-1, 2t-1, t-1)$
Case$(v-2,1)$ & Case$(v-2,2)$	$4(s+1)(2s^2+2s)^{s^2+s-2}$	$2(4s^2)^{2s^2-2}$	0
Case$(v-2,\ 3)$	$4s(2s^2+2s)^{s^2+s-2}$	$2(4s^2)^{2s^2-2}$	$4(4t)^{2t-3}$
Case$(v-2,4)$ & Case$(v-2,5)$	0	0	0
Case$(v-2,6)$	$4(2s^2+2s)^{s^2+s-2}$	0	$4(4t)^{2t-3}$
Case$(v-2,7)$	0	0	0

Table 8: Values of Large Minors of Some SBIBD

6 Other Families of SBIBDs

We now use (5) to obtain results for other families of SBIBD.

6.1 The Growth of the Finite Projective Planes SBIBD$(s^2+s+1, s+1, 1)$

Theorem 4 *Let A be the $v \times v$ $(1,-1)$ incidence matrix of the finite projective plane SBIBD$(s^2+s+1, s+1, 1)$, $s > 3$. Reduce A by GECP, then the last two pivots are*

$$p_v = \frac{2(s^2-s-1)}{s-2} \quad \text{or} \quad \frac{2s(s^2-s-1)}{s-1}$$

and

$$p_{v-1} = 2s \quad \text{or} \quad \frac{2(s-1)}{s-2}.$$

Proof. Since

$$M_v = 2^{s^2+s} s^{\frac{s^2+s}{2}}(s^2-s-1).$$
$$M_{v-1} = 2^{s^2+s-1} s^{\frac{s^2+s}{2}}(s-2) \quad \text{or} \quad 2^{s^2+s-1} s^{\frac{s^2+s-2}{2}}(s-1).$$
$$M_{v-2} = 2^{s^2+s-2} s^{\frac{s^2+s-2}{2}}(s-2) \quad \text{or} \quad 2^{s^2+s-2} s^{\frac{s^2+s-4}{2}}(s-1).$$

The values of the two last pivots are specified from the above formulae. □

Remark 4 We experimented with the SBIBD(31,6,1). Its $A(n-1)$ minor is $2^{14} \cdot 3 \cdot 10^{15}$ or $2^{17} \cdot 10^{14}$ depending on the case considered in Theorem 1. Our calculations always found the CP matrix had $M_{30} = 2^{14} \cdot 3 \cdot 10^{15}$ although our theory also allows for $M_{30} = 2^{17} \cdot 10^{14}$. Thus, in our examples, the last pivot was always equal to $\frac{38}{3}$ and not $\frac{95}{2}$. □

6.2 The Growth of the Menon-Hadamard Family SBIBD$(4s^2, 2s^2 \pm s, s^2 \pm s)$

Theorem 5 *Let A be the $v \times v$ regular Hadamard matrix from the SBIBD$(4s^2, 2s^2 \pm s, s^2 \pm s)$. Reduce A by GECP, then the last two pivots are v, and $\frac{v}{2}$.*

Proof. Since

$$M_v = (4s^2)^{2s^2}, \quad M_{v-1} = (4s^2)^{2s^2-1}, \quad M_{v-2} = 2(4s^2)^{2s^2-2}$$

the values of the two last pivots are $4s^2$, and $2s^2$, respectively. □

6.3 The Growth of the Hadamard Family SBIBD$(4t-1, 2t-1, t-1)$

Theorem 6 *Let A be the $v \times v$ $(1, -1)$ incidence matrix of the Hadamard family SBIBD$(4t - 1, 2t - 1, t - 1)$. Reduce A by GECP, then the last two pivots are $2t$, and $2t$.*

Proof. The last two pivots are computed straightforwardly using formula (5) since

$$M_v = (4t)^{2t-1}, \quad M_{v-1} = 2(4t)^{2t-2}, \quad M_{v-2} = 4(4t)^{2t-3}.$$

Thus, the values of the two last pivots are $2t$, and $2t$. □

We give the results for some families of $SBIBD(v, k, \lambda)$ in Table 8.

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7 Appendix: The Determinant Simplification Theorem

We use the notation

$$CC^T = (k - a_{ii})I_{b_1,b_2,\cdots,b_z} + a_{ij}J_{b_1,b_2,\cdots,b_z}$$

for a matrix of blocks with integer multiples. For example the matrix

$$CC^T = (k - a_{ii})I_{u,v,w,x} + a_{ij}J_{u,v,w,x} \tag{11}$$

where $(a_{ij}) = \begin{bmatrix} a & b & c & d \\ b & a & e & f \\ c & e & a & g \\ d & f & g & a \end{bmatrix}$ is the $(u+v+w+x) \times (u+v+w+x)$ matrix

$$CC^T = \begin{bmatrix} \overbrace{\begin{matrix} k\,a & \cdots & a \\ a\,k & \cdots & a \\ \vdots\,\vdots & & \vdots \\ a\,a & \cdots & k \end{matrix}}^{u} & \overbrace{\begin{matrix} b\,b & \cdots & b \\ b\,b & \cdots & b \\ \vdots\,\vdots & & \vdots \\ b\,b & \cdots & b \end{matrix}}^{v} & \overbrace{\begin{matrix} c\,c & \cdots & c \\ c\,c & \cdots & c \\ \vdots\,\vdots & & \vdots \\ c\,c & \cdots & c \end{matrix}}^{w} & \overbrace{\begin{matrix} d\,d & \cdots & d \\ d\,d & \cdots & d \\ \vdots\,\vdots & & \vdots \\ d\,d & \cdots & d \end{matrix}}^{x} \\ \begin{matrix} b\,b & \cdots & b \\ b\,b & \cdots & b \\ \vdots\,\vdots & & \vdots \\ b\,b & \cdots & b \end{matrix} & \begin{matrix} k\,a & \cdots & a \\ a\,k & \cdots & a \\ \vdots\,\vdots & & \vdots \\ a\,a & \cdots & k \end{matrix} & \begin{matrix} e\,e & \cdots & e \\ e\,e & \cdots & e \\ \vdots\,\vdots & & \vdots \\ e\,e & \cdots & e \end{matrix} & \begin{matrix} f\,f & \cdots & f \\ f\,f & \cdots & f \\ \vdots\,\vdots & & \vdots \\ f\,f & \cdots & f \end{matrix} \\ \begin{matrix} c\,c & \cdots & c \\ c\,c & \cdots & c \\ \vdots\,\vdots & & \vdots \\ c\,c & \cdots & c \end{matrix} & \begin{matrix} e\,e & \cdots & e \\ e\,e & \cdots & e \\ \vdots\,\vdots & & \vdots \\ e\,e & \cdots & e \end{matrix} & \begin{matrix} k\,a & \cdots & a \\ a\,k & \cdots & a \\ \vdots\,\vdots & & \vdots \\ a\,a & \cdots & k \end{matrix} & \begin{matrix} g\,g & \cdots & g \\ g\,g & \cdots & g \\ \vdots\,\vdots & & \vdots \\ g\,g & \cdots & g \end{matrix} \\ \begin{matrix} d\,d & \cdots & d \\ d\,d & \cdots & d \\ \vdots\,\vdots & & \vdots \\ d\,d & \cdots & d \end{matrix} & \begin{matrix} f\,f & \cdots & f \\ f\,f & \cdots & f \\ \vdots\,\vdots & & \vdots \\ f\,f & \cdots & f \end{matrix} & \begin{matrix} g\,g & \cdots & g \\ g\,g & \cdots & g \\ \vdots\,\vdots & & \vdots \\ g\,g & \cdots & g \end{matrix} & \begin{matrix} k\,a & \cdots & a \\ a\,k & \cdots & a \\ \vdots\,\vdots & & \vdots \\ a\,a & \cdots & k \end{matrix} \end{bmatrix}.$$

We now give a theorem proved similarly to the proof for finding the determinant of an SBIBD in [8, Theorem3,p32].

Theorem 7 (Determinant Simplification Theorem) *Let*

$$CC^T = (k - a_{ii})I_{b_1,b_2,\cdots,b_z} + a_{ij}J_{b_1,b_2,\cdots,b_z}$$

then

$$det\ CC^T = \Pi_{i=1}^{z}(k - a_{ii})^{b_i - 1} det\ D \tag{12}$$

where

$$D = \begin{bmatrix} k + (b_1 - 1)a_{11} & b_2 a_{12} & b_3 a_{13} & \cdots & b_z a_{1z} \\ b_1 a_{21} & k + (b_2 - 1)a_{22} & b_3 a_{23} & \cdots & b_z a_{2z} \\ \vdots & \vdots & \vdots & & \vdots \\ b_1 a_{z1} & b_2 a_{z2} & b_3 a_{z2} & \cdots & k + (b_z - 1)a_{zz} \end{bmatrix}$$

Corollary 1 *Suppose* C *is the matrix of order* $(u+v+w+x)\times(u+v+w+x)$, *where* $n=u+v+w+x$, *for which* CC^T *is given above, satisfying* $CC^T=(k-a_{ii})I_{u,v,w,x}+a_{ij}J_{u,v,w,x}$. *Then*

$$det\ CC^T=(k-a)^{n-4}det\ D$$

where

$$D=\begin{bmatrix} k+(u-1)a & vb & wc & xd \\ ub & k+(v-1)a & we & xf \\ uc & ve & k+(w-1)a & xg \\ ud & vf & wg & k+(x-1)a \end{bmatrix}. \tag{13}$$