On circulant best matrices and their applications

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Abstract

Call four type 1 $(1,-1)$ matrices, X_1, X_2, X_3, X_4, of the same group of order m (odd) with the properties (i) $(X_i-I)^T = -(X_i-I)$, $i = 1,2,3$, (ii) $X_4^T = X_4$ and the diagonal elements are positive, (iii) $X_iX_j = X_jX_i$ and (iv) $X_1X_1^T + X_2X_2^T + X_3X_3^T + X_4X_4^T = 4mI_m$, *best matrices*. We use a computer to give, for the first time, all inequivalent best matrices of odd order $m \le 31$. Inequivalent best matrices of order m, m odd, can be used to find inequivalent skew-Hadamard matrices of order $4m$. We use best matrices of order $\frac{1}{4}(s^2+3)$ to construct new orthogonal designs, including new $OD(2s^2+6; 1,1,2,2,s^2,s^2)$.

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1 Introduction and basic definitions

A $(1,-1)$ matrix of order n is called a Hadamard matrix if $HH^T = H^TH = nI_n$, where H^T is the transpose of H and I_n is the identity matrix of order n. A $(1,-1)$ matrix A of order n is said to be of skew type if $A - I_n$ is skew-symmetric. If A is a skew type Hadamard matrix then A is said to be a skew-Hadamard matrix. Two $(1,-1)$ matrices A, B of order n are said to be amicable if $AB^T = BA^T$.

Let G be an additive abelian group of order n with elements $g_1, g_2, \ldots, g_n$ and X a subset of G. Define the type 1 $(1,-1)$ incidence matrix $M = (m_{ij})$ of order n of X to be

$$m_{ij} = \begin{cases} +1 & if\ g_j - g_i \in X \\ -1 & otherwise \end{cases}$$

and the type 2 $(1,-1)$ incidence matrix $N = (n_{ij})$ of order n of X to be

$$n_{ij} = \begin{cases} +1 & if\ g_j + g_i \in X \\ -1 & otherwise \end{cases}$$

In particular, if G is cyclic the matrices M and N are called circulant and back circulant respectively. In this case $m_{ij} = m_{1,j-i+1}$ *and* $n_{ij} = n_{1,i+j-1}$ respectively (indices should be reduced modulo n).

Definition 1 Let X_1, X_2, X_3, X_4 be four type 1 $(1,-1)$ matrices on the same group of order m (odd) with the properties

(i) $(X_i - I)^T = -(X_i - I)$, $i = 1,2,3$

(ii) $X_4^T = X_4$ and the diagonal elements are positive

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(iii) $X_iX_j = X_jX_i$

(iv) $X_1X_1^T + X_2X_2^T + X_3X_3^T + X_4X_4^T = 4mI_m$

Call such matrices *best matrices* of order m.

Pre and post multiplying equation (iv) by e and e^T, respectively, where e is the $1 \times m$ matrix of all ones gives that $4m - 3 = s^2$ where s is odd integer.

In this paper we consider circulant best matrices, so condition (iii) is trivially satisfied. Hence, multiplying on the left by e^T (the $1 \times m$ vector of one's) and on the right by e both sides of (iv) we conclude that circulant (or type 1) best matrices can only exist for orders m of which $4m = 1^2 + 1^2 + 1^2 + a^2$, where a is the sum of the elements of the first row of the symmetric matrix X_4 and a is an odd integer.

An *orthogonal design* of order n and type $(s_1, s_2, \ldots, s_u)$ $(s_i > 0)$, denoted $OD(n; s_1, s_2, \ldots, s_u)$, on the commuting variables $x_1, x_2, \ldots, x_u$, is an $n \times n$ matrix A with entries from $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ such that

$$AA^T = (\sum_{i=1}^{u} s_i x_i^2) I_n$$

Alternatively, the rows of A are formally orthogonal and each row has precisely s_i entries of the type $\pm x_i$. In [1], where this was first defined, it was mentioned that

$$A^T A = (\sum_{i=1}^{u} s_i x_i^2) I_n$$

and so our alternative description of A applies equally well to the columns of A. It was also shown in [1] that $u \le \rho(n)$, where $\rho(n)$ (Radon's function) is defined by $\rho(n) = 8c + 2^d$, when $n = 2^a b$, b odd, $a = 4c + d$, $0 \le d < 4$. For more details and constructions of orthogonal designs the reader can consult the book of Geramita and Seberry [2].

In section 2 we describe briefly the method of construction, in section 3 we give all inequivalent circulant best matrices of odd order $m \le 31$, and in section 4 we use best matrices to construct some new orthogonal designs and families of Hadamard matrices.

2 Method of construction

In order to describe our construction for best matrices, we need a few more definitions. Let n be a positive integer.

Definition 2 Four subsets S_0, S_1, S_2, S_3 of $\{1, 2, ..., n-1\}$ are called $4-(n; n_0, n_1, n_2, n_3; \lambda)$ *supplementary difference sets (sds)* modulo n if $|S_k| = n_k$ for $k = 0, 1, 2, 3$ and for each $m \in \{1, 2, ..., n-1\}$ we have $\lambda_0(m) + \ldots + \lambda_3(m) = \lambda$, where $\lambda_k(m)$ is the number of solutions (i, j) of the congruence $i - j \equiv m (mod\ n)$ with $i, j \in S_k$.

Suppose that S_k are $4-(n; n_0, n_1, n_2, n_3; \lambda)$ sds modulo n having the following additional properties:

$$n + \lambda = n_0 + n_1 + n_2 + n_3 \quad (1)$$

$$i \in S_k \Longleftrightarrow n - i \notin S_k, \ k = 0, 1, 2 \quad (2)$$

$$i \in S_t \Longleftrightarrow n - i \in S_t, \ t = 3 \quad (3)$$

where in (2) and (3) it is assumed that $i \in \{1, 2, \ldots, n-1\}$.

Let $a_k = (a_{k_0}, a_{k_1}, \ldots, a_{k_{n-1}})$, $k = 0, 1, 2, 3$, be the row vector defined by

$$a_{k_i} = \begin{cases} -1 & if \quad i \in S_k \\ 1 & otherwise \end{cases}$$

Furthermore let A_k, $k = 0, 1, 2, 3$ be the circulant matrices with first row a_k. Then it can be easily verified that A_0, A_1, A_2, A_3 are four matrices of order n as described in definition 1.

Let r be an integer relatively prime to n, and set

$$S_k' = \{r_i \ (mod\ n) : i \in S_k\} \subset \{1, 2, \ldots, n-1\}$$

for $k = 0, 1, 2, 3$. These sets are also $4-(n; n_0, n_1, n_2, n_3; \lambda)$ sds modulo n satisfying the conditions $(1), (2), (3)$. We shall say that such quadruples S_0, S_1, S_2, S_3 and S_0', S_1', S_2', S_3' are equivalent.

We now give a brief description of the method of computation used to find the necessary sds's. The numbers n_i are easy to determine (see [6]). We first generate a number of subsets of size n_i of $\{1, 2, ..., n\}$ having the required symmetry properties (2) or (3), and at the same time compute the corresponding set of differences. We store the multiplicities of these differences in a file, say f_i, saving only sets of differences with different multiplicities. After creating these files for each of the sizes $n_0, ..., n_3$, we try to match the items in the four files to produce an sds. This is done by examining items in two files only, say f_0 and f_1 and creating a new file in which we record the pairs which produce different total multiplicities of the differences. The procedure is repeated with the remaining two files f_2 and f_3. Finally the resulting two files are examined in order to find a perfect match.
The results that we found applying this algorithm are presented in the next section.

3 The inequivalent supplementary difference sets

In this section we give for the first time all inequivalent supplementary difference sets which satisfy the condition (1), (2), (3) for all odd $m \leq 31$.

Table 1.
All inequivalent circulant best matrices of odd order $m \leq 31$

m = 3; $4-(3; 1, 1, 1, 0; 0)$

$$S_0 = \{1\}, \quad S_1 = \{1\}, \quad S_2 = \{1\}, \quad S_3 = \emptyset$$

m = 7; $4-(7; 3, 3, 3, 6; 8)$

$$S_0 = \{1, 3, 5\}, \quad S_1 = \{1, 2, 3\}, \quad S_2 = \{1, 4, 5\}, \quad S_3 = \{1, 2, 3, 4, 5, 6\}$$

m = 13; $4-(13; 6, 6, 6, 10; 15)$

1. $S_0 = \{1, 3, 5, 6, 9, 11\}$, $S_1 = \{2, 6, 8, 9, 10, 12\}$,
$S_2 = \{1, 4, 5, 6, 10, 11\}$, $S_3 = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

2. $S_0 = \{2, 6, 89, 10, 12\}$, $S_1 = \{1, 4, 5, 6, 10, 11\}$,
$S_2 = \{1, 2, 3, 4, 6, 8\}$, $S_3 = \{1, 2, 3, 4, 6, 7, 9, 10, 11, 12\}$

Table 1. (continued)

m = 21; $4-(21; 10, 10, 10, 6; 15)$

1. $S_0 = \{1, 3, 6, 10, 12, 13, 14, 16, 17, 19\}$, $S_1 = \{1, 2, 3, 4, 7, 8, 10, 12, 15, 16\}$,
$S_2 = \{1, 2, 5, 11, 12, 13, 14, 15, 17, 18\}$, $S_3 = \{4, 6, 10, 11, 15, 17\}$,

2. $S_0 = \{1, 3, 4, 5, 6, 8, 10, 12, 14, 19\}$, $S_1 = \{1, 2, 3, 4, 7, 9, 10, 13, 15, 16\}$,
$S_2 = \{3, 4, 6, 7, 8, 9, 11, 16, 19, 20\}$, $S_3 = \{2, 8, 9, 12, 13, 19\}$,

3. $S_0 = \{1, 3, 4, 6, 10, 12, 13, 14, 16, 19\}$, $S_1 = \{2, 3, 4, 7, 8, 9, 10, 15, 16, 20\}$,
$S_2 = \{1, 2, 3, 4, 5, 7, 8, 11, 12, 15\}$, $S_3 = \{1, 3, 8, 13, 18, 20\}$,

4. $S_0 = \{1, 2, 4, 6, 7, 9, 10, 13, 16, 18\}$, $S_1 = \{4, 5, 10, 12, 13, 14, 15, 18, 19, 20\}$,
$S_2 = \{1, 2, 3, 4, 5, 6, 9, 11, 13, 14\}$, $S_3 = \{1, 5, 9, 12, 16, 20\}$,

5. $S_0 = \{1, 2, 4, 6, 7, 9, 10, 13, 16, 18\}$, $S_1 = \{1, 2, 3, 4, 8, 11, 12, 14, 15, 16\}$,
$S_2 = \{1, 6, 11, 12, 13, 14, 16, 17, 18, 19\}$, $S_3 = \{2, 6, 8, 13, 15, 19\}$,

6. $S_0 = \{1, 3, 7, 10, 12, 13, 15, 16, 17, 19\}$, $S_1 = \{1, 2, 3, 4, 6, 7, 8, 11, 12, 16\}$,
$S_2 = \{1, 3, 4, 5, 6, 11, 12, 13, 14, 19\}$, $S_3 = \{1, 5, 9, 12, 16, 20\}$,

7. $S_0 = \{1, 2, 4, 6, 7, 9, 10, 13, 16, 18\}$, $S_1 = \{1, 2, 3, 4, 6, 7, 8, 11, 12, 16\}$,
$S_2 = \{1, 7, 8, 9, 11, 15, 16, 17, 18, 19\}$, $S_3 = \{2, 4, 9, 12, 17, 19\}$,

m = 31; $4-(31; 15, 15, 15, 10; 24)$

1. $S_0 = \{1, 5, 6, 8, 10, 13, 14, 16, 19, 20, 22, 24, 27, 28, 29\}$,
$S_1 = \{2, 3, 4, 5, 6, 7, 10, 12, 13, 14, 16, 20, 22, 23, 30\}$,
$S_2 = \{1, 3, 4, 5, 12, 16, 17, 18, 20, 21, 22, 23, 24, 25, 29\}$,
$S_3 = \{3, 6, 8, 12, 13, 18, 19, 23, 25, 28\}$,

2. $S_0 = \{1, 4, 7, 10, 12, 14, 15, 18, 20, 22, 23, 25, 26, 28, 29\}$,
$S_1 = \{1, 2, 6, 12, 13, 14, 16, 20, 21, 22, 23, 24, 26, 27, 28\}$,
$S_2 = \{5, 6, 7, 9, 10, 11, 12, 14, 15, 18, 23, 27, 28, 29, 30\}$,
$S_3 = \{2, 3, 10, 12, 14, 17, 19, 21, 28, 29\}$,

4 Constructions using best matrices

Theorem 1 *Suppose there exist best matrices of order $t = \frac{1}{4}(s^2+3)$, then there exists an $OD(8t; 1, 1, 2, 2, s^2, s^2)$.*

Proof. Suppose $I + X$,$I + X_2$,$I + X_3$ and X_4 are the circulant best matrices of order t. Let a, b, c, d, e, f be commuting variables. Define

$$X = \frac{1}{2}(X_2 + X_3); Y = \frac{1}{2}(X_2 - X_3)$$

So

$$X^T = -X,\ Y^T = -Y,\ XX^T + YY^T = \frac{1}{2}(X_2X_2^T + X_3X_3^T).$$

Now define

$$
\begin{aligned}
A_1 &= aI + bX_1\\
A_2 &= dX_4\\
A_3 &= cI - dX_1\\
A_4 &= bX_4\\
A_5 &= eI + bX + dY\\
A_6 &= fI - dX + bY\\
A_7 &= eI - bX - dY\\
A_8 &= fI + dX - bY
\end{aligned}
$$

It is straightforward to check, using the properties of best matrices, that $A_1, A_2, \ldots, A_8$ satisfy the additive property and

$$\sum_{i=1}^{8} A_i A_i^2 = (a^2 + c^2 + 2e^2 + 2f^2 + (4t-3)b^2 + (4t-3)d^2)I_t.$$

We now check if the matrices form an amicable set. First we see

$$
\begin{aligned}
A_1A_2^T &= adX_4 + bdX_1X_4\\
A_2A_1^T &= adX_4 - bdX_1X_4\\
A_3A_4^T &= cbX_4 - bdX_1X_4\\
A_4A_3^T &= cbX_4 + bdX_1X_4
\end{aligned}
$$

So

$$A_1A_2^T - A_2A_1^T + A_3A_4^T - A_4A_3^T = 0.$$

Then we have

$$
\begin{aligned}
A_5A_6^T &= efI + bfX + dfY + edX + bdX^2 + d^2XY - ebY - b^2XY - dbY^2\\
A_6A_5^T &= efI - bfX - dfY - edX + bdX^2 + d^2XY + ebY - b^2XY - dbY^2\\
A_7A_8^T &= efI - bfX - dfY - edX + bdX^2 + d^2XY + ebY - b^2XY - dbY^2\\
A_8A_7^T &= efI + bfX + dfY + edX + bdX^2 + d^2XY - ebY - b^2XY - dbY^2
\end{aligned}
$$

So

$$A_5A_6^T - A_6A_5^T + A_7A_8^T - A_8A_7^T = 0.$$

Hence $A_1 \ldots A_8$ are amicable set of circulant matrices satisfying the additive property. Hence we may use them in Kharaghani array [3] to form $OD(8t; 1, 1, 2, 2, 4t-3, 4t-3)$. □

Remark 1 We note there is no construction known which gives $OD(8t; 1, 1, 2, 2, 4t-3, 4t-3)$.

Hence we have $OD(56; 1, 1, 2, 2, 25, 25)$, $OD(104; 1, 1, 2, 2, 49, 49)$, $OD(168; 1, 1, 2, 2, 81, 81)$ and $OD(248; 1, 1, 2, 2, 121, 121)$ for the first time.

Theorem 2 *Suppose there are best matrices of order m then there exists an $OD(4m; 1, 1, 1, 4m-3)$.*

Proof. Let x_1, x_2, x_3 and x_4 be four commuting variables. Write $I + B_1$, $I + B_2$, $I + B_3$ and B_4 for the best matrices of order m. Further write $A_1 = x_1 I + x_4 B_1$, $A_2 = x_2 I + x_4 B_2$, $A_3 = x_3 I + x_4 B_3$ and $A_4 = x_4 B_4$ for the four circulant (or type 1) matrices of order m satisfying

$$A_1A_1^T + A_2A_2^T + A_3A_3^T + A_4A_4^T = (x_1^2 + x_2^2 + x_3^2 + (4m-3)x_4^2)I_m.$$

Let $R = r_{ij}$, where $r_{ij} = 1$ for $i + j = m + 1$ and 0 otherwise. Then using the Goethals-Seidel array

$$\begin{bmatrix} A_1 & A_2R & A_3R & A_4R \\ -A_2R & A_1 & A_4^TR & -A_3^TR \\ -A_3R & -A_4^TR & A_1 & A_2^TR \\ -A_4R & A_3^TR & -A_2^TR & A_1 \end{bmatrix},$$

is the required $OD(4m; 1, 1, 1, 4m-3)$. □

Corollary 2 *Let m be the order of best matrices. Then an $OD(4m; 1, 1, 1, 4m-3)$ exists.*

Corollary 3 *Let $m \in \{3, 7, 13, 21, 31\}$. Then an $OD(4m; 1, 1, 1, 4m-3)$ exists.*

Corollary 4 *Let m be the order of best matrices. Then there exist up to 8 inequivalent skew-Hadamard, and Hadamard, matrices of order $4m$.*

Proof. Let X_1, X_2, X_3, X_4 be best matrices of order m. Then choosing $A_1 = X_1$, $A_2 = I \pm (X_2 - I)$, $A_3 = I \pm (X_3 - I)$ and $A_4 = \pm X_4$, in the Goethals-Seidel array gives the required result, (Note choosing $A_2 = \pm I + (X_2 - I)$, and $A_3 = \pm I + (X_3 - I)$ is an alternative choice.) □

We have constructed the Hadamard matrices of order 28 made, using as A_1, A_2, A_3 and A_4, the first rows given below in the Goethals-Seidel array

```
1 1 1-1 1-1-1;   1 1 1-1 1-1-1;   1 1 1-1 1-1-1;   1-1-1-1-1-1-1
1 1 1-1 1-1-1;   1 1 1-1 1-1-1;  -1 1 1-1 1-1-1;   1-1-1-1-1-1-1
1 1 1-1 1-1-1;  -1 1 1-1 1-1-1;   1 1 1-1 1-1-1;   1-1-1-1-1-1-1
1 1 1-1 1-1-1;  -1 1 1-1 1-1-1;  -1 1 1-1 1-1-1;   1-1-1-1-1-1-1
```

We believe that the four Hadamard matrices thus produced are H-inequivalent and inequivalent skew-Hadamard matrices.

Corollary 5 *Suppose there are best matrices of order m and an Hadamard matrix, H, of order $4m/3$, then there is an Hadamard matrix of order $4m(4m-3)/3$.*

Proof. Use the best matrices to make an $OD(4m; 1, 1, 1, 4m-3)$.

Write J for the $4m/3 - 1 \times 4m/3 - 1$ matrix of all ones. Normalize the Hadamard matrix, H, of order $4m/3$ so that its first row and column is all ones, then discard the first row and column to obtain the core of the Hadamard matrix, B, of order $4m/3 - 1$, which satisfies $BJ = -J$ and $BB^T = 4m/3I_{4m/3-1} - J_{4m/3-1}$. Then replacing the variables of the $OD(4m; 1, 1, 1, 4m-3)$ by J, J, J and B, which satisfy

$$3JJ^T + (4m-3)BB^T = (4m-3)J + 4m(4m-3)/3I - (4m-3)J = 4m(4m-3)/3I,$$

gives the required matrix. □

Example 1 We have found best matrices of orders $m = 3$ and 21. These give Hadamard matrices or orders 36 and 2268. These orders are not new, but, since Kimura [4, 5] has found some 487 inequivalent Hadamard matrices of order 28 which can be used in the corollary for $m = 21$ we may have constructed new, inequivalent, Hadamard matrices of order 2268. Since the variables can also be replaced by J, $\pm J$, $\pm J$ and $\pm B$ there is further potential for inequivalent Hadamard matrices. □

Corollary 6 *Suppose there are best matrices of order m and a symmetric Hadamard matrix of order h*

1. $h = 4(m+1)/3$;
2. $h = 4(m+2)/3$;
3. $h = 4(m+3)/3$,

then there is an Hadamard matrix of order $4m(h-1)$.

Proof. Use the best matrices to make an $OD(4m; 1, 1, 1, 4m-3)$.

Normalize the symmetric Hadamard matrix of order h so that its first row and column is all ones, then discard the first row and column to obtain the symmetric core of the symmetric Hadamard matrix, B, which satisfies $BJ = -J$ and $BB^T = hI_{h-1} - J_{h-1}$. Write $K = J - 2I$. Then

$$KJ^T = JK^T; \quad KB^T = BK^T; \quad JB^T = BJ^T.$$

Then replacing the variables of the $OD(4m; 1, 1, 1, 4m-3)$ by

1. J, J, K and B:
2. J, K, K and B:
3. K, K, K and B

which satisfy

$$2JJ^T + KK^T + (4m-3)BB^T = 2(h-1)J + (h-5)J + 4I + h(4m-3)I - (4m-3)J = 4m(h-1)I;$$

$$JJ^T + 2KK^T + (4m-3)BB^T = (h-1)J + 2(h-5)J + 8I + h(4m-3)I - (4m-3)J = 4m(h-1)I;$$

$$3KK^T + (4m-3)BB^T = 3(h-5)J + 12I + h(4m-3)I - (4m-3)J = 4m(h-1)I,$$

respectively giving the required matrices. □

Example 2 From above we have four sequences of lengths $m = 3,\ 7,\ 13,\ 21$ and 31 which are the first rows for best matrices. Then using Corollary 5 and the best matrices of orders 3 and 21 we obtain Hadamard matrices of order 36 and 2268. Using Corollary 6 we obtain Hadamard matrices of orders $84 = 4 \cdot 21$, $308 = 4 \cdot 77$, $988 = 4 \cdot 247 = 4 \cdot 13 \cdot 19$, $2604 = 4 \cdot 851 = 4 \cdot 21 \cdot 31$ and $5332 = 4 \cdot 31 \cdot 43$. None of these orders are new but there are possibly inequivalent Hadamard matrices. □

Corollary 7 *Suppose there are best matrices of order m, a back-circulant $SBIBD(v, k, \lambda)$ and an Hadamard matrix with circulant core, B, of order*

1. $v = 4(k-\lambda) + 4m/3 - 1$;

2. $v = (8k - 8\lambda + 4m)/3 - 1$;

3. $v = 4(k - \lambda + m)/3 - 1$;

then there is an Hadamard matrix of order $4mv$.

Proof. Form the $OD(4m; 1,1,1,4m-3)$ as before.

As before B satisfies $BJ = -J$ and $BB^T = (v+1)I_v - J_v$. Let A be the ± 1 incidence matrix of the $SBIBD(v,k,\lambda)$ then $AJ = (2k-v)J$ and $AA^T = 4(k-\lambda)I + (v - 4(k-\lambda))J$. We note $AB^T = BA^T$ as A is back-circulant and B is circulant. We now replace the variables of the $OD(4m; 1,1,1,4m-3)$ by (1) A, A, A and B, (2) A, A, J and B, and (3) A, J, J and B, respectively, which satisfy

$$3AA^T + (4m-3)BB^T = 12(k-\lambda)I + 3(v-4(k-\lambda))J + (4m-3)(v+1)I - (4m-3)J = 4mvI,$$

$$2AA^T + JJ^T + (4m-3)BB^T = 8(k-\lambda)I + 2(v-4(k-\lambda))J + vJ + (4m-3)(v+1)I - (4m-3)J = 4mvI,$$

$$AA^T + 2JJ^T + (4m-3)BB^T = 4(k-\lambda)I + (v-4(k-\lambda))J + 2vJ + (4m-3)(v+1)I - (4m-3)J = 4mvI,$$

gives the required matrices. □

References

[1] A.V. Geramita, J.M. Geramita and J. Seberry Wallis, Orthogonal designs, *Linear and Multilinear Algebra*, 3 (1976), 281-306.

[2] A.V. Geramita, and J. Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York-Basel, 1979.

[3] H. Kharaghani, Arrays for orthogonal designs, *J. Combin. Designs*, to appear.

[4] H. Kimura, Classification of Hadamard matrices of order 28 with Hall sets, *Discrete Math.,* 128 (1994), 257-268.

[5] H. Kimura, Classification of Hadamard matrices of order 28, *Discrete Math.,* 133 (1994), 171-180.

[6] J.Seberry Wallis, Hadamard matrices, Part IV,*Combinatorics: Room Squares, Sum free sets and Hadamard matrices*, Lecture Notes in Mathematics, Vol.292, eds. W.D.Wallis A.P.Street and J.Seberry Wallis, Springer-Verlag, Berlin-Heidelberg, New York, 1972.