

On amicable sequences and orthogonal designs

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Abstract

In this paper we give a general theorem which can be used to multiply the length of amicable sequences keeping the amicability property and the type of the sequences. As a consequence we have that if there exist two, four or eight amicable sequences of length m and type (a_1, a_2) , (a_1, a_2, a_3, a_4) or (a_1, a_2, \dots, a_8) then there exist amicable sequences of length $\ell \equiv 0 \pmod{m}$ and of the same type. We also present a theorem that produces a set of $2v$ amicable sequences from a set of v (not necessary amicable) sequences and a construction method for amicable sequences of type $(a_1, a_1, a_2, a_2, \dots, a_v, a_v)$ from v pairs of disjoint $(0, \pm 1)$ amicable sequences.

Using these results we can obtain many infinite classes of orthogonal designs. Actually, if there exists an orthogonal design of order n and of type (a_1, a_2, \dots, a_v) , which is constructed from sequences, then there exists an infinite family of orthogonal designs of the same type which is constructed from appropriate sequences.

Key words and phrases: Construction, sequences, orthogonal designs, amicable sets.

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1 Introduction

An *orthogonal design* of order n and type (s_1, s_2, \dots, s_u) denoted $OD(n; s_1, s_2, \dots, s_u)$ in the variables x_1, x_2, \dots, x_u , is a matrix A of order n with entries in the set $\{0, \pm x_1, \pm x_2, \dots, \pm x_u\}$ satisfying

$$AA^T = \sum_{i=1}^u (s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n . Let A_1, A_2 be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2\}$ satisfying $A_1 A_1^T + A_2 A_2^T = (s_1 x_1^2 + s_2 x_2^2) I_n$. Then

$$D = \begin{pmatrix} A_1 & A_2 \\ -A_2^T & A_1^T \end{pmatrix} \quad \text{or} \quad D = \begin{pmatrix} A_1 & A_2 R \\ -A_2 R & A_1 \end{pmatrix}. \quad (1)$$

is an $OD(2n; s_1, s_2)$.

Let $B_i, i = 1, 2, 3, 4$ be circulant matrices of order n with entries in $\{0, \pm x_1, \pm x_2, \dots, \pm x_u\}$ satisfying

$$\sum_{i=1}^4 B_i B_i^T = \sum_{i=1}^u (s_i x_i^2) I_n.$$

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Then the Goethals-Seidel array

$$G = \begin{pmatrix} B_1 & B_2R & B_3R & B_4R \\ -B_2R & B_1 & B_4^T R & -B_3^T R \\ -B_3R & -B_4^T R & B_1 & B_2^T R \\ -B_4R & B_3^T R & -B_2^T R & B_1 \end{pmatrix} \quad (2)$$

where R is the back-diagonal identity matrix, is an $OD(4n; s_1, s_2, \dots, s_u)$. See page 107 of [1] for details.

A pair of matrices A, B is said to be amicable (anti-amicable) if $AB^T - BA^T = 0$ ($AB^T + BA^T = 0$). Following [5] a set $\{A_1, A_2, \dots, A_{2n}\}$ of square real matrices is said to be *amicable* if

$$\sum_{i=1}^n \left(A_{\sigma(2i-1)} A_{\sigma(2i)}^T - A_{\sigma(2i)} A_{\sigma(2i-1)}^T \right) = 0 \quad (3)$$

for some permutation σ of the set $\{1, 2, \dots, 2n\}$. For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. So

$$\sum_{i=1}^n \left(A_{2i-1} A_{2i}^T - A_{2i} A_{2i-1}^T \right) = 0. \quad (4)$$

Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. Throughout the paper R_k denotes the back diagonal identity matrix of order k .

A set of matrices $\{B_1, B_2, \dots, B_n\}$ of order m with entries in $\{0, \pm x_1, \pm x_2, \dots, \pm x_u\}$ is said to satisfy an additive property of type (s_1, s_2, \dots, s_u) if

$$\sum_{i=1}^n B_i B_i^T = \sum_{i=1}^u (s_i x_i^2) I_m. \quad (5)$$

Let $\{A_i\}_{i=1}^8$ be an amicable set of circulant matrices (or group developed of type 1) of type (s_1, s_2, \dots, s_u) of order t . Then the Kharaghani array from [5]

$$H = \begin{pmatrix} A_1 & A_2 & A_4 R_n & A_3 R_n & A_6 R_n & A_5 R_n & A_8 R_n & A_7 R_n \\ -A_2 & A_1 & A_3 R_n & -A_4 R_n & A_5 R_n & -A_6 R_n & A_7 R_n & -A_8 R_n \\ -A_4 R_n & -A_3 R_n & A_1 & A_2 & -A_8^T R_n & A_7^T R_n & A_6^T R_n & -A_5^T R_n \\ -A_3 R_n & A_4 R_n & -A_2 & A_1 & A_7^T R_n & A_8^T R_n & -A_5^T R_n & -A_6^T R_n \\ -A_6 R_n & -A_5 R_n & A_8^T R_n & -A_7^T R_n & A_1 & A_2 & -A_4^T R_n & A_3^T R_n \\ -A_5 R_n & A_6 R_n & -A_7^T R_n & -A_8^T R_n & -A_2 & A_1 & A_3^T R_n & A_4^T R_n \\ -A_8 R_n & -A_7 R_n & -A_6^T R_n & A_5^T R_n & A_4^T R_n & -A_3^T R_n & A_1 & A_2 \\ -A_7 R_n & A_8 R_n & A_5^T R_n & A_6^T R_n & -A_3^T R_n & -A_4^T R_n & -A_2 & A_1 \end{pmatrix} \quad (6)$$

is an $OD(8m; s_1, s_2, \dots, s_u)$.

The Kharaghani array has been used in a number of papers [2, 3, 5, 6, 7] to obtain infinitely many families of orthogonal designs.

A set $\{A_i\}_{i=1}^4$ is said to be a *short amicable set* of length m and type (u_1, u_2, u_3, u_4) if (4) and (5) are satisfied for $n = 4$ and $u \leq 4$. Short amicable sets can be used in either the Goethals-Seidel array or the *short Kharaghani array*

$$\begin{bmatrix} A & B & CR & DR \\ -B & A & DR & -CR \\ -CR & -DR & A & B \\ -DR & CR & -B & A \end{bmatrix} \quad (7)$$

to form an $OD(4m; u_1, u_2, u_3, u_4)$.

Short amicable sets were introduced in [4] where they are used to find infinite families of orthogonal designs.

A set of sequences $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m-1}\}$, $k = 1, 2, \dots, 2v$ where $a_{k,j} \in \{0, \pm x_1, \pm x_2, \dots, \pm x_p\}$, $j = 0, 1, \dots, m-1$ and $k = 1, 2, \dots, 2v$ is said to be a set of $2v$ *amicable sequences* of length m and type (u_1, u_2, \dots, u_p) if the circulant matrices which are constructed from these satisfy the equations (4) and (5).

Given the sequence $A = \{a_1, a_2, \dots, a_n\}$ of length n the *non-periodic autocorrelation function (NPAF)* $N_A(s)$ is defined as

$$N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \quad s = 0, 1, \dots, n-1, \quad (8)$$

Given A as above of length n the *periodic autocorrelation function (PAF)* $P_A(s)$ is defined, reducing $i+s$ modulo n , as

$$P_A(s) = \sum_{i=1}^n a_i a_{i+s}, \quad s = 0, 1, \dots, n-1. \quad (9)$$

We define the *NPAF (PAF)* of a set of sequences the sum of the corresponding *NPAF (PAF)* of the individual sequences.

Suppose $C = \text{circ}(c_0, c_1, \dots, c_{n-1})$ is a circulant matrix of order n .

Let

$$T_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

of order n , be the shift matrix. Then we can write $C = c_0 I + c_1 T_n + \dots + c_{n-1} T_n^{n-1}$. Note that $T_n^n = I$ the identity matrix of order n . We say the Hall polynomial of C is $\sum_{i=0}^{n-1} c_i x^i$. The Hall polynomial of C^T is $\sum_{i=0}^{n-1} c_i x^{n-i}$.

2 Multiplication of the length of amicable sets of sequences

Theorem 1 Let $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m-1}\}$, $k = 1, 2, \dots, 2v$ where $a_{k,j} \in \{0, \pm x_1, \pm x_2, \dots, \pm x_p\}$, $j = 0, 1, \dots, m-1$ and $k = 1, 2, \dots, 2v$ be a set of $2v$ amicable sequences of length m and type (u_1, u_2, \dots, u_p) . Then there exist a set of $2v$ amicable sequences of length $\ell \equiv (0 \pmod m) = mi$ for all $i = 1, 2, \dots$ and type (u_1, u_2, \dots, u_p) .

Proof. Let i be a constant integer. We use the map T_m^k to define sequences A_k and the map $S_\ell^k = T_m^k$ to define sequences B_k

$$B_k = \sum_{j=0}^{m-1} a_{k,j} S_\ell^j, \quad k = 1, 2, \dots, 2v$$

Now

$$\sum_{k=1}^{2v} A_k A_k^T = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{2v} \left(a_{k,j} a_{k,x} T_m^{j-x} \right) = \left(\sum_{k=1}^p u_k x_k^2 \right) I_m.$$

Thus we have that

- (i) If m is odd then the coefficients of T_m^σ , $\sigma = -(m-1), \dots, -1, 1, \dots, m-1$ is zero, and the coefficient of T_m^0 is $\sum_{k=1}^p u_k x_k^2$. That means

$$\sum_{\substack{j,x=0 \\ j-x=\sigma}}^{m-1} \sum_{k=1}^{2v} a_{k,j} a_{k,x} = 0 \quad \text{and} \quad \sum_{j=0}^{m-1} \sum_{k=1}^{2v} a_{k,j}^2 = \sum_{k=1}^p u_k x_k^2 \quad (10)$$

- (ii) If m is even, $m = 2n$ then we have that $T_m^n = T_m^{-n}$ and so the coefficients of T_m^σ , $\sigma = -(2n-1), \dots, -(n+1), -(n-1), \dots, -1, 1, \dots, n-1, n+1, \dots, 2n-1$ are zero, the coefficient of T_m^n plus the coefficient of T_m^{-n} is zero and the coefficient of T_m^0 is $\sum_{k=1}^p u_k x_k^2$. That means

$$\sum_{\substack{j,x=0 \\ j-x=\sigma \\ \sigma \neq \pm n}}^{m-1} \sum_{k=1}^{2v} a_{k,j} a_{k,x} = 0, \quad \sum_{\substack{j,x=0 \\ j-x=\pm n}}^{m-1} \sum_{k=1}^{2v} a_{k,j} a_{k,x} = 0 \quad \text{and} \quad \sum_{j=0}^{m-1} \sum_{k=1}^{2v} a_{k,j}^2 = \sum_{k=1}^p u_k x_k^2 \quad (11)$$

Now

$$\sum_{k=1}^{2v} B_k B_k^T = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{2v} \left(a_{k,j} a_{k,x} S_\ell^{j-x} \right)$$

We have that the coefficients of S_ℓ^σ are equal to the coefficients of T_m^σ for all $\sigma = -(m-1), \dots, m-1$, and so using equations (10) or (11) we obtain

$$\sum_{k=1}^{2v} B_k B_k^T = \left(\sum_{k=1}^p u_k x_k^2 \right) I_{2mi} = \left(\sum_{k=1}^p u_k x_k^2 \right) I_\ell \quad (12)$$

Moreover

$$\sum_{k=1}^v \left(A_{2k-1} A_{2k}^T - A_{2k} A_{2k-1}^T \right) = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^v \left((a_{2k-1,j} a_{2k,x} - a_{2k,j} a_{2k-1,x}) T_m^{j-x} \right) = 0$$

and from these we have that

(i) if m odd, then the coefficients of T_m^σ , $\sigma = -(m-1), \dots, m-1$ are zero. That means

$$\sum_{\substack{j,x=0 \\ j-x=\sigma}}^{m-1} \sum_{k=1}^v (a_{2k-1,j} a_{2k,x} - a_{2k,j} a_{2k-1,x}) = 0 \quad (13)$$

(ii) if m is even, $m = 2n$ then the coefficients of T_m^σ , $\sigma = -(2n-1), \dots, -(n+1), -(n-1), \dots, n-1, n+1, \dots, 2n-1$ are zero and the coefficient of T_m^n plus the coefficient of T_m^{-n} is zero. That means

$$\sum_{\substack{j,x=0 \\ j-x=\sigma \\ \sigma \neq \pm n}}^{m-1} \sum_{k=1}^v (a_{2k-1,j} a_{2k,x} - a_{2k,j} a_{2k-1,x}) = 0 \text{ and } \sum_{\substack{j,x=0 \\ j-x=\pm n}}^{m-1} \sum_{k=1}^v (a_{2k-1,j} a_{2k,x} - a_{2k,j} a_{2k-1,x}) = 0 \quad (14)$$

Now

$$\sum_{k=1}^v (B_{2k-1} B_{2k}^T - B_{2k} B_{2k-1}^T) = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^v ((a_{2k-1,j} a_{2k,x} - a_{2k,j} a_{2k-1,x}) S_\ell^{j-x})$$

We have that the coefficients of S_ℓ^σ are equal to the coefficients of T_m^σ for all $\sigma = -(m-1), \dots, m-1$ and so using equations (13) or equations (14) we obtain

$$\sum_{k=1}^v (B_{2k-1} B_{2k}^T - B_{2k} B_{2k-1}^T) = 0 \quad (15)$$

Equations (12) and (15) show that $\{B_k\}_{k=1}^{2v}$ is an amicable set of sequences of length $\ell \equiv 0 \pmod{m}$, $\ell = mi$, $i = 1, 2, \dots$ and type (u_1, u_2, \dots, u_p) . \square

Corollary 1 Let $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m-1}\}$, $k = 1, 2$ where $a_{k,j} \in \{0, \pm x_1, \pm x_2\}$, $j = 0, 1, \dots, m-1$ and $k = 1, 2$ be a set of two amicable sequences of length m and type (u_1, u_2) . Then there exist a set of two amicable sequences of length $\ell \equiv (0 \pmod{m}) = mi$ and type (u_1, u_2) .

Proof. Use Theorem 1 with $2v = 2$ and $p = 2$.

Example 1 We have that $A_1 = 0T_4^0 + aT_4^1 + bT_4^2 - aT_4^3$ and $A_2 = 0T_4^0 + aT_4^1 + 0T_4^2 + aT_4^3$ is a set of two amicable sequences of length $m = 4$ and type $(1, 4)$. Corollary 1 gives a set of two amicable sequences of length $m = 4i$ and type $(1, 4)$ for all $i = 1, 2, \dots$.

Corollary 2 Let $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m-1}\}$, $k = 1, 2, 3, 4$ where $a_{k,j} \in \{0, \pm x_1, \pm x_2, \pm x_3, \pm x_4\}$, $j = 0, 1, \dots, m-1$ and $k = 1, 2, 3, 4$ be a set of four amicable sequences of length m and type (u_1, u_2, u_3, u_4) . Then there exist a set of four amicable sequences of length $\ell \equiv (0 \pmod{m}) = mi$ and type (u_1, u_2, u_3, u_4) .

Proof. Use Theorem 1 with $2v = 4$ and $p = 4$.

Example 2 We have that $A_1 = aT_3^0 - bT_3^1 + aT_3^2$, $A_2 = bT_3^0 + aT_3^1 + bT_3^2$ and $A_3 = aT_3^0 + aT_3^1 - aT_3^2$, $A_4 = bT_3^0 + bT_3^1 + bT_3^2$ is a set of four amicable sequences of length $m = 3$ and type $(6, 6)$. Corollary 2 gives a set of four amicable sequences of length $m = 3i$ and type $(6, 6)$ for all $i = 1, 2, \dots$.

Corollary 3 Let $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m-1}\}$, $k = 1, 2, \dots, 8$ where $a_{k,j} \in \{0, \pm x_1, \pm x_2, \dots, \pm x_8\}$, $j = 0, 1, \dots, m-1$ and $k = 1, 2, \dots, 8$ be a set of eight amicable sequences of length m and type (u_1, u_2, \dots, u_8) . Then there exist a set of eight amicable sequences of length $\ell \equiv 0 \pmod{m} = mi$ and type (u_1, u_2, \dots, u_8) .

Proof. Use Theorem 1 with $2v = 8$ and $p = 8$.

Example 3 We have that $A_1 = -aT_7^0 + aT_7^1 + aT_7^2 + gT_7^3 + aT_7^4 + eT_7^5 + cT_7^6$, $A_2 = -fT_7^0 + fT_7^1 + fT_7^2 - hT_7^3 + fT_7^4 + bT_7^5 - dT_7^6$, $A_3 = -gT_7^0 + gT_7^1 + gT_7^2 - aT_7^3 + gT_7^4 + cT_7^5 - eT_7^6$, $A_4 = -hT_7^0 + hT_7^1 + hT_7^2 + fT_7^3 + hT_7^4 + dT_7^5 + bT_7^6$, $A_5 = -eT_7^0 + eT_7^1 + eT_7^2 - cT_7^3 + eT_7^4 - aT_7^5 + gT_7^6$, $A_6 = -dT_7^0 + dT_7^1 + dT_7^2 - bT_7^3 + dT_7^4 - hT_7^5 + fT_7^6$, $A_7 = -bT_7^0 + bT_7^1 + bT_7^2 + dT_7^3 + bT_7^4 - fT_7^5 - hT_7^6$ and $A_8 = -cT_7^0 + cT_7^1 + cT_7^2 + eT_7^3 + cT_7^4 - gT_7^5 - aT_7^6$ is a set of eight amicable sequences of length $m = 7$ and type $(7, 7, 7, 7, 7, 7, 7, 7)$. Corollary 3 gives a set of eight amicable sequences of length $m = 7i$ and type $(7, 7, 7, 7, 7, 7, 7, 7)$ for all $i = 1, 2, \dots$.

Remark 1 Using Corollaries 1, 2 and 3 as indicated by the examples and using array (1), (2) or (7) and (6) respectively we obtain many infinite classes of orthogonal designs.

3 Construction of amicable sets of sequences from non amicable sets of sequences

Lemma 1 Let $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m-1}\}$, $k = 1, 2, \dots, v_1$, where $a_{k,j} \in \{0, \pm x_1, \pm x_2, \dots, \pm x_p\}$, $j = 0, 1, \dots, m-1$ and $k = 1, 2, \dots, v_1$ be a set of v_1 amicable sequences of length m and type (u_1, u_2, \dots, u_p) and $B_r = \{b_{r,0}, b_{r,1}, \dots, b_{r,m-1}\}$, $r = 1, 2, \dots, v_2$, where $b_{r,s} \in \{0, \pm y_1, \pm y_2, \dots, \pm y_q\}$, $s = 0, 1, \dots, m-1$ and $r = 1, 2, \dots, v_2$ be a set of v_2 amicable sequences of length m and type (t_1, t_2, \dots, t_q) .

Then there exist a set of $v_1 + v_2$ amicable sequences of length m and type $(u_1, u_2, \dots, u_p, t_1, t_2, \dots, t_q)$.

Proof. These are the sequences A_k , $k = 1, 2, \dots, v_1$ and B_k , $k = 1, 2, \dots, v_2$ together. Λ

Corollary 4 Let $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m_1-1}\}$, $k = 1, 2, \dots, v_1$, where $a_{k,j} \in \{0, \pm x_1, \pm x_2, \dots, \pm x_p\}$, $j = 0, 1, \dots, m_1-1$ and $k = 1, 2, \dots, v_1$ be a set of v_1 amicable sequences of length m_1 and type (u_1, u_2, \dots, u_p) and $B_r = \{b_{r,0}, b_{r,1}, \dots, b_{r,m_2-1}\}$, $r = 1, 2, \dots, v_2$, where $b_{r,s} \in \{0, \pm y_1, \pm y_2, \dots, \pm y_q\}$, $s = 0, 1, \dots, m_2-1$ and $r = 1, 2, \dots, v_2$ be a set of v_2 amicable sequences of length m_2 and type (t_1, t_2, \dots, t_q) .

Then there exist a set of $v_1 + v_2$ amicable sequences of length $\ell \cdot i$ where $\ell = [m_1, m_2]$ is the least common multiple (l.c.m.) of m_1 and m_2 and type $(u_1, u_2, \dots, u_p, t_1, t_2, \dots, t_q)$.

Proof. Since ℓ is the least common multiple of m_1 and m_2 then $\ell = m_1 \cdot i_1 = m_2 \cdot i_2$. Using theorem 1 we can construct a set of v_1 amicable sequences of length ℓ and type (u_1, u_2, \dots, u_p) and a set of v_2 amicable sequences of length ℓ and type (t_1, t_2, \dots, t_q) . Now using Lemma 1 we obtain a set of $v_1 + v_2$ amicable sequences of length ℓ and type $(u_1, u_2, \dots, u_p, t_1, t_2, \dots, t_q)$.

Using theorem 1 again in the derived sequences we have the result. Λ

Example 4 We have that $A_1 = \{e, f\}$, $A_2 = \{e, -f\}$, $A_3 = \{e, 0\}$, $A_4 = \{f, 0\}$ is a short amicable sets of length 2 and type $(3, 3)$. We also have that $A_1 = \{a, a, b, -b\}$, $A_2 =$

$\{c, c, d, -d\}$, $A_3 = \{d, d, -c, c\}$, $A_4 = \{b, b, -a, a\}$ is a short amicable set of length 4 and type $(4, 4, 4, 4)$. Now $\ell = [4, 2] = 4$ and thus from corollary 4 we obtain eight amicable sequences of length $\ell \cdot i$ and type $(3, 3, 4, 4, 4, 4)$ for all $i = 1, 2, \dots$.

Theorem 2 (Doubling the number of sequences) *Let $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m-1}\}$, $k = 1, 2, \dots, v$ where $a_{k,j} \in \{0, \pm x_1, \pm x_2, \dots, \pm x_p\}$, $j = 0, 1, \dots, m-1$ and $k = 1, 2, \dots, v$ be v sequences with $PAF=0$ (or $NPAF=0$) of length m and type (u_1, u_2, \dots, u_p) . Then there exist a set of $2v$ amicable sequences of length m and type $(2u_1, 2u_2, \dots, 2u_p)$ with $PAF=0$ (or $NPAF=0$).*

Proof. Set $B_{2k-1} = B_{2k} = \text{circ}(A_k)$, $k = 1, 2, \dots, v$. Then

$$\sum_{k=1}^{2v} B_k B_k^T = 2 \cdot \sum_{k=1}^v A_k A_k^T = \left(\sum_{i=1}^p 2u_i x_i^2 \right) I_m$$

and

$$B_{2k-1} B_{2k}^T - B_{2k} B_{2k-1}^T = A_k A_k^T - A_k A_k^T = 0, \quad k = 1, 2, \dots, v.$$

Thus $\{B_k\}_{k=1}^{2v}$ is a set of $2v$ amicable sequences of length m and type $(2u_1, 2u_2, \dots, 2u_p)$. Λ

4 More Constructions

Theorem 3 *Let (X_k, Y_k) , $k = 1, 2, \dots, v$ be v pairs of sequences of lengths m_k with the properties*

$$Z_k Z_k^T + W_k W_k^T = p_k I_{m_k} \tag{16}$$

$$Z_k W_k^T - W_k Z_k^T = 0 \tag{17}$$

$$Z_k * W_k = 0 \tag{18}$$

for all $k = 1, 2, \dots, v$, where $Z_k = \text{circ}(X_k)$ and $W_k = \text{circ}(Y_k)$. Then there exist a set of $2v$ amicable sequences of length $\ell \equiv 0 \pmod{[m_1, m_2, \dots, m_v]}$, where $[m_1, m_2, \dots, m_v]$ is the least common multiple (l.c.m.) of m_1, m_2, \dots, m_v and of type $(p_1, p_1, p_2, p_2, \dots, p_v, p_v)$ on the set $\{a_1, a_2, \dots, a_{2v}\}$ of commuting variables.

Proof. Set

$$B_k = a_{2k} X_k + a_{2k-1} Y_k, \quad \text{and} \quad C_k = -a_{2k-1} X_k + a_{2k} Y_k, \quad k = 1, 2, \dots, v$$

Condition (18) gives that B_k , $k = 1, 2, \dots, v$ and C_k , $k = 1, 2, \dots, v$ are sequences of lengths m_k , $k = 1, 2, \dots, v$ and type $(p_1, p_1, p_2, p_2, \dots, p_v, p_v)$.

For any k and by simple calculations using conditions (16) and (17) we have that

$$B_k B_k^T + C_k C_k^T = (p_k a_{2k-1}^2 + p_k a_{2k}^2) I_{m_k} \quad \text{and} \quad B_k C_k^T - C_k B_k^T = 0$$

Now from theorem 1, for all $k=1, 2, \dots, v$, there are sequences D_k and E_k of length $\ell \equiv 0 \pmod{[m_1, m_2, \dots, m_v]}$ and $k = 1, 2, \dots, v$ with the desirable properties. By lemma 1 we have the result. Λ

Example 5 Set $Z_1 = \{1\}$, $W_1 = \{0\}$, $Z_2 = \{1, 0\}$, $W_2 = \{0, 1\}$, $Z_3 = \{1, 1, 1, -1\}$, $W_3 = \{0, 0, 0, 0\}$, $Z_4 = \{0, 1, 0, -1, 0, 1\}$ and $W_4 = \{0, 0, 1, 0, 1, 0\}$. These are four pair of sequences of lengths 1, 2, 4 and 6 satisfying conditions (16), (17) and (18) with $p_1 = 1$, $p_2 = 2$, $p_3 = 4$ and $p_4 = 5$. We have that $[1, 2, 4, 6] = 12$ and from theorem 3 we obtain eight sequences of length $\ell \equiv 0 \pmod{12}$ and of type $(1, 1, 2, 2, 4, 4, 5, 5)$ on the set $\{a_1, a_2, \dots, a_8\}$ of commuting variables and can be used in the Kharaghani array (6) to obtain an infinite class of orthogonal designs $OD(8\ell; 1, 1, 2, 2, 4, 4, 5, 5)$.

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