

# New orthogonal designs and sequences with two and three variables in order 28

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## Abstract

We give new sets of sequences with entries from  $\{0, \pm a, \pm b, \pm c\}$  on the commuting variables  $a, b, c$  and zero autocorrelation function. Then we use these sequences to construct some new orthogonal designs.

We show the necessary conditions for the existence of an  $OD(28; s_1, s_2, s_3)$  plus the condition that  $(s_1, s_2, s_3) \neq (1, 5, 20)$  are sufficient conditions for the existence of an  $OD(28; s_1, s_2, s_3)$ . We also show the necessary conditions for the existence of an  $OD(28; s_1, s_2, s_3)$  constructed using four circulant matrices are sufficient conditions for the existence of  $4 - NPAF(s_1, s_2, s_3)$  sequences of length  $n$  for all lengths  $n \geq 7$ .

We establish asymptotic existence results for  $OD(4N; s_1, s_2)$  for  $2 \leq s_1 + s_2 \leq 28$ . We show the necessary conditions for the existence of an  $OD(28; s_1, s_2)$  with  $25 \leq s_1 + s_2 \leq 28$ , constructed using four circulant matrices, plus the condition that  $(s_1, s_2) \neq (1, 26), (2, 25), (7, 19), (8, 19)$  or  $(13, 14)$ , are sufficient conditions for the existence of  $4 - NPAF(s_1, s_2)$  sequences of length  $n$  for all lengths  $n \geq 7$ .

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## 1 Introduction

Throughout this paper we will use the definition and notation of Koukouvinos, Mitrouli, Seberry and Karabelas [4].

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**Theorem 1 (Geramita and Seberry and Lemma 3 [1, 2])** Write  $(s_i, s_j)_p$  for the Hilbert norm residue symbol. The following conditions are necessary for the existence of an  $OD(4n; s_1, s_2, \dots, s_\ell)$  in orders with  $n$  odd:

- i) for  $\ell = 2$ ;  $(-1, s_1 s_2)_p (s_1, s_2)_p = 1$  for all primes  $p$ ;
- ii) for  $\ell = 3$ ;  $(s_1, s_2)_p (s_1, s_3)_p (s_2, s_3)_p (-1, s_1 s_2 s_3)_p = 1$  for all primes  $p$ ;  
if  $s_1 + s_2 + s_3 = n - 1$  then  $s_1 s_2 s_3$  is a square and  
 $(s_1, s_2)_p (s_1, s_3)_p (s_2, s_3)_p = 1$  for all primes  $p$ ;

The Goethals-Seidel construction can only be used if

- iii) there exists an integer matrix  $P$ , with all entries of modulus  $\leq n$  which satisfies  $PP^T = \text{diag}(s_1, s_2, s_3)$ .

In this paper there are no 2- or 3-tuples which satisfy (i) and (ii) which do not also satisfy (iii). However in others orders, such as 20, this does happen.

## 2 New orthogonal designs

**Theorem 2** There exist orthogonal designs  $OD(4n; s_1, s_2, s_3)$  where  $(s_1, s_2, s_3)$  is one of the 3-tuples

$$\begin{array}{ccccc} (1, 1, 17) & (1, 3, 14) & (1, 6, 11) & (1, 8, 11) & (1, 8, 16) \\ (1, 9, 16) & (2, 5, 7) & (2, 7, 10) & (2, 7, 13) & (2, 8, 18) \\ (3, 4, 14) & (3, 6, 8) & (3, 7, 11) & (3, 8, 10) & (3, 9, 14) \\ (4, 4, 13) & (4, 5, 14) & (5, 5, 10) & (7, 8, 13) & \end{array}$$

for all  $n \geq 7$ , constructed using four circulant matrices in the Goethals-Seidel array.

**Proof.** We use the 4-*NPAF* sequences given in the Appendices A and B, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain the required orthogonal designs. Since these sequences have zero non-periodic autocorrelation function, the sequences are first padded with sufficient zeros added to the end to make their length  $n \geq 7$ .  $\square$

**Theorem 3** There exist orthogonal designs  $OD(4n; s_1, s_2, s_3)$ , constructed using four circulant matrices in the Goethals-Seidel array, where  $(s_1, s_2, s_3)$  is one of the 3-tuples

(1, 1, 17)	(1, 3, 14)	(1, 6, 11)	(1, 8, 11)	(1, 8, 16)
(1, 8, 17)	(1, 9, 16)	(2, 4, 22)	(2, 5, 7)	(2, 7, 10)
(2, 7, 13)	(2, 8, 18)	(2, 9, 11)	(2, 9, 17)	(3, 4, 14)
(3, 6, 8)	(3, 6, 16)	(3, 6, 17)	(3, 7, 11)	(3, 7, 15)
(3, 8, 10)	(3, 8, 15)	(3, 9, 14)	(3, 10, 15)	(4, 4, 13)
(4, 5, 14)	(4, 10, 11)	(5, 5, 10)	(5, 5, 18)	(5, 9, 14)
(5, 10, 10)	(6, 7, 8)	(6, 9, 11)	(7, 8, 10)	(7, 8, 13)
(8, 8, 9)	(8, 9, 11)	(9, 9, 10).		

**Proof.** We use the sequences given in Appendices B and C, which have zero periodic and non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain the required orthogonal designs.

The designs for (3, 6, 16), (3, 8, 15), (4, 6, 11), (4, 9, 13), (8, 8, 9), and (8, 9, 9) are from [3].  $\square$

**Theorem 4** *If  $OD(28; 1, 5, 20)$  and  $OD(20; 3, 7, 8)$  exist they cannot be constructed using four circulant matrices in the Goethals-Seidel array.*

**Proof.** By an exhaustive search.  $\square$

**Theorem 5** *There are no orthogonal designs  $OD(4n; s_1, s_2, s_3)$  where  $(s_1, s_2, s_3)$  is one of the 3-tuples*

(1, 3, 22)	(1, 5, 19)	(2, 5, 15)	(2, 6, 11)	(2, 6, 17)
(2, 11, 11)	(2, 11, 13)	(2, 11, 15)	(3, 7, 10)	(3, 11, 14)
(4, 5, 19)	(5, 6, 14)	(5, 6, 15)	(5, 7, 10)	(5, 7, 14)
(6, 8, 11)	(7, 10, 11),			

*constructed using four circulant matrices in the Goethals-Seidel array.*

**Proof.** There is no an integer sum-fill matrix  $P$  as described in [4, Lemma 3].  $\square$

**Theorem 6** *There are no  $4-NPAF(s_1, s_2, s_3)$  and  $4-NPAF(s_1, s_2, s_3, s_4)$  of length 5 for the following 3- and 4-tuples*

(1, 3, 14)	(1, 4, 13)	(1, 5, 20)	(3, 7, 8)	(1, 3, 6, 8)	(1, 4, 4, 9)
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**Proof.** An exhaustive search showed the 3-tuples were not given  $4-NPAF$  sequences.

The the existence of  $4-NPAF(1, 3, 6, 8)$  of length 5, after equating the last two variables would prove the existence of  $4-NPAF(1, 3, 14)$  of length

5: hence the nonexistence of the  $4 - NPAF(1, 3, 14)$  of length 5 gives the nonexistence of  $4 - NPAF(1, 3, 6, 8)$ .

Also, the existence of  $4 - NPAF(1, 4, 4, 9)$  of length 5, after equating the last two variables would prove the existence of  $4 - NPAF(1, 4, 13)$  of length 5: the nonexistence of the  $4 - NPAF(1, 4, 13)$  of length 5 gives the nonexistence of  $4 - NPAF(1, 4, 4, 9)$ .  $\square$

**Theorem 7** *The necessary conditions plus the condition that  $(s_1, s_2, s_3) \neq (1, 5, 20)$  are sufficient conditions for the existence of an  $OD(28; s_1, s_2, s_3)$ .*

**Proof.** See Table 1.

## 2.1 Two variable designs

**Lemma 1** *The necessary conditions for the existence of an  $OD(28; s_1, s_2)$  plus the condition that  $(s_1, s_2) \neq (1, 26), (2, 25), (7, 19), (8, 19)$ , or  $(13, 14)$  are sufficient for the existence of  $4 - NPAF(s_1, s_2)$  sequences.*

**Proof.** The theoretical necessary conditions are used to prove the following 2-tuples  $(s_1, s_2)$  do not give orthogonal designs  $OD(28; s_1, s_2)$ :

$$\begin{array}{cccccc} (1, 7) & (3, 5) & (4, 7) & (5, 12) & (7, 9) & (9, 15) & (12, 13) \\ (1, 15) & (3, 13) & (4, 15) & (5, 19) & (7, 16) & (10, 17) & (12, 15) \\ (1, 23) & (3, 20) & (4, 23) & (5, 22) & (7, 17) & (11, 13) & \\ (2, 14) & (3, 21) & (5, 11) & (6, 10) & (8, 14) & (11, 16) & \end{array}$$

The specific cases mentioned in the enunciation have been eliminated by exhaustive search.  $\square$

**Theorem 8** *There are no  $4 - NPAF(s_1, s_2, s_3)$  of length 7 for the following 3-tuples*

$$\begin{array}{cccccc} (1, 1, 25) & (1, 1, 26) & (1, 2, 25) & (1, 8, 17) & (1, 8, 18) & (1, 8, 19) \\ (1, 9, 13) & (1, 10, 14) & (1, 13, 13) & (1, 13, 14) & (2, 4, 22) & (2, 7, 19) \\ (2, 9, 17) & (2, 13, 13) & (3, 6, 17) & (3, 7, 15) & (3, 10, 15) & (4, 4, 18) \\ (4, 9, 13) & (4, 10, 11) & (5, 5, 18) & (5, 9, 9) & (5, 9, 14) & (6, 7, 8) \\ (6, 9, 11) & (7, 8, 10) & (8, 9, 11) & (9, 9, 9) & (9, 9, 10) & \end{array}$$

**Proof.** By exhaustive search there are no  $4 - NPAF(s_1, s_2)$  for the following 2-tuples

$$(1, 26) \quad (2, 25) \quad (7, 19) \quad (8, 19) \quad (13, 14).$$

Hence, there can be no  $4 - NPAF(s_1, s_2, s_3)$  of length 7 for the following 3-tuples

(1, 1, 25)    (1, 1, 26)    (1, 2, 25)    (1, 8, 18)    (1, 8, 19)    (1, 13, 13)  
 (1, 13, 14)    (2, 7, 19).

There are no 4- $NPAF(s_1, s_2, s_3)$  of length 7 for the following 3-tuples

(1, 8, 17)    (1, 9, 13)    (1, 10, 14)    (2, 4, 22)    (2, 9, 17)    (2, 13, 13)  
 (3, 6, 17)    (3, 7, 15)    (3, 10, 15)    (4, 4, 18)    (4, 9, 13)    (4, 10, 11)  
 (5, 5, 18)    (5, 9, 9)    (5, 9, 14)    (6, 7, 8)    (6, 9, 11)    (7, 8, 10)  
 (8, 9, 11)    (9, 9, 9)    (9, 9, 10)

by exhaustive search. □

**Remark 1** The 236 3-tuples given in Table 1 are possible types of orthogonal designs in order 28. There are no cases unresolved. We use

- $n$  if they are made from 4-sequences with zero  $NPAF$  and length  $n$ ;
- $P$  if they are made from 4-sequences with zero  $PAF$ ;
- $F$  if they are made from the  $OD(28; 4, 4, 9, 9)$ ;
- $X$  if they are not known because the integer sum-fill matrix does not exist;
- $Y$  if it does not exist by exhaustive search for length 7.

(1, 1, 1)	1	(1, 5, 16)	7	(2, 4, 8)	5	(3, 4, 8)	5	(4, 8, 12)	7
(1, 1, 2)	1	(1, 5, 19)	<i>X</i>	(2, 4, 9)	5	(3, 4, 14)*	<i>P</i>	(4, 8, 16)	7
(1, 1, 4)	2	(1, 5, 20)	<i>Y</i>	(2, 4, 11)	5	(3, 4, 18)	7	(4, 9, 9)	6
(1, 1, 5)	3	(1, 6, 8)	5	(2, 4, 12)	7	(3, 6, 6)	5	(4, 9, 10)	7
(1, 1, 8)	3	(1, 6, 11)	5	(2, 4, 16)	7	(3, 6, 8)	5	(4, 9, 13)†	7
(1, 1, 9)	7	(1, 6, 12)	7	(2, 4, 17)	7	(3, 6, 9)	5	(4, 10, 10)	7
(1, 1, 10)	3	(1, 6, 14)	7	(2, 4, 18)	7	(3, 6, 11)	5	(4, 10, 11)	<i>P</i>
(1, 1, 13)	5	(1, 6, 18)	7	(2, 4, 19)	7	(3, 6, 12)	7	(4, 10, 14)	7
(1, 1, 16)	7	(1, 6, 21)	7	(2, 4, 22)	<i>P</i>	(3, 6, 16)†	7	(5, 5, 5)	7
(1, 1, 17)*	<i>P</i>	(1, 8, 8)	7	(2, 5, 5)	3	(3, 6, 17)	<i>P</i>	(5, 5, 8)	7
(1, 1, 18)	6	(1, 8, 9)	5	(2, 5, 7)	5	(3, 6, 18)	7	(5, 5, 9)	5
(1, 1, 20)	6	(1, 8, 11)	5	(2, 5, 8)	5	(3, 6, 19)	7	(5, 5, 10)	5
(1, 1, 25)	<i>P</i>	(1, 8, 12)	7	(2, 5, 13)	6	(3, 7, 8)	6	(5, 5, 13)	<i>P</i>
(1, 1, 26)	<i>P</i>	(1, 8, 16)	7	(2, 5, 15)	<i>X</i>	(3, 7, 10)	<i>X</i>	(5, 5, 16)	7
(1, 2, 2)	2	(1, 8, 17)	<i>P</i>	(2, 5, 18)	7	(3, 7, 11)	7	(5, 5, 18)	<i>P</i>
(1, 2, 3)	2	(1, 8, 18)	<i>P</i>	(2, 6, 7)	5	(3, 7, 15)	<i>P</i>	(5, 6, 9)	7
(1, 2, 4)	2	(1, 8, 19)	<i>P</i>	(2, 6, 9)	5	(3, 7, 18)	7	(5, 6, 14)	<i>X</i>
(1, 2, 6)	3	(1, 9, 9)	7	(2, 6, 11)	<i>X</i>	(3, 8, 9)	7	(5, 6, 15)	<i>X</i>
(1, 2, 8)	3	(1, 9, 10)	5	(2, 6, 12)	6	(3, 8, 10)*	<i>P</i>	(5, 7, 8)	7
(1, 2, 9)	3	(1, 9, 13)*	<i>P</i>	(2, 6, 13)	7	(3, 8, 15)†	7	(5, 7, 10)	<i>X</i>
(1, 2, 11)	5	(1, 9, 16)	7	(2, 6, 16)	7	(3, 9, 14)	7	(5, 7, 14)	<i>X</i>
(1, 2, 12)	5	(1, 9, 18)	7	(2, 6, 17)	<i>X</i>	(3, 10, 15)	<i>P</i>	(5, 8, 8)	7
(1, 2, 16)	7	(1, 10, 10)	7	(2, 7, 10)	7	(3, 11, 14)	<i>X</i>	(5, 8, 13)	7
(1, 2, 17)	5	(1, 10, 11)	7	(2, 7, 12)	7	(4, 4, 4)	3	(5, 9, 9)*	<i>P</i>
(1, 2, 18)	6	(1, 10, 14)	<i>P</i>	(2, 7, 13)	7	(4, 4, 5)	5	(5, 9, 10)*	<i>P</i>
(1, 2, 19)	6	(1, 13, 13)	<i>P</i>	(2, 7, 19)	<i>P</i>	(4, 4, 8)	7	(5, 9, 14)	<i>P</i>
(1, 2, 22)	7	(1, 13, 14)	<i>P</i>	(2, 8, 8)	5	(4, 4, 9)	5	(5, 10, 10)	7
(1, 2, 25)	<i>P</i>	(2, 2, 2)	2	(2, 8, 9)	5	(4, 4, 10)	5	(6, 6, 6)	7
(1, 3, 6)	3	(2, 2, 4)	2	(2, 8, 10)	5	(4, 4, 13)	7	(6, 6, 12)	7
(1, 3, 8)	3	(2, 2, 5)	3	(2, 8, 13)	7	(4, 4, 16)	7	(6, 7, 8)	<i>P</i>
(1, 3, 14)	6	(2, 2, 8)	3	(2, 8, 16)	7	(4, 4, 17)	7	(6, 8, 9)	7
(1, 3, 18)	6	(2, 2, 9)	5	(2, 8, 18)	7	(4, 4, 18)	<i>P</i>	(6, 8, 11)	<i>X</i>
(1, 3, 22)	<i>X</i>	(2, 2, 10)	5	(2, 9, 9)	5	(4, 4, 20)	7	(6, 8, 12)	<i>P</i>
(1, 3, 24)	7	(2, 2, 13)	5	(2, 9, 11)	6	(4, 5, 5)	5	(6, 9, 11)	<i>P</i>
(1, 4, 4)	5	(2, 2, 16)	7	(2, 9, 12)	7	(4, 5, 6)	5	(7, 7, 7)	7
(1, 4, 5)	5	(2, 2, 17)	7	(2, 9, 17)	<i>P</i>	(4, 5, 9)	5	(7, 7, 14)	7
(1, 4, 8)	5	(2, 2, 18)	6	(2, 10, 10)	6	(4, 5, 14)*	<i>P</i>	(7, 8, 10)	<i>P</i>
(1, 4, 9)	5	(2, 2, 20)	7	(2, 10, 12)	6	(4, 5, 16)	7	(7, 8, 13)	7
(1, 4, 10)	5	(2, 3, 4)	3	(2, 11, 11)	<i>X</i>	(4, 5, 19)	<i>X</i>	(7, 10, 11)	<i>X</i>
(1, 4, 13)	7	(2, 3, 6)	3	(2, 11, 13)	<i>X</i>	(4, 6, 8)	5	(8, 8, 8)	7
(1, 4, 16)	7	(2, 3, 7)	3	(2, 11, 15)	<i>X</i>	(4, 6, 11)†	7	(8, 8, 9)†	7
(1, 4, 17)	7	(2, 3, 9)	5	(2, 13, 13)	<i>P</i>	(4, 6, 12)	7	(8, 8, 10)	7
(1, 4, 18)	7	(2, 3, 10)	7	(3, 3, 3)	3	(4, 6, 14)	7	(8, 9, 9)†	7
(1, 4, 20)	7	(2, 3, 15)	7	(3, 3, 6)	3	(4, 6, 18)	7	(8, 9, 11)	<i>P</i>
(1, 5, 5)	3	(2, 3, 16)	7	(3, 3, 12)	7	(4, 8, 8)	7	(8, 10, 10)	7
(1, 5, 6)	3	(2, 4, 4)	3	(3, 3, 15)	7	(4, 8, 9)	7	(9, 9, 9)	<i>P</i>
(1, 5, 9)	5	(2, 4, 6)	3	(3, 4, 6)	5	(4, 8, 11)	7	(9, 9, 10)	<i>P</i>
(1, 5, 14)	5								

Table 1: Known 3-variable designs in order 28: for the 3-tuples marked by

\* the corresponding orthogonal design is known for  $n \geq 6$  and orders  $\geq 24$ .

† the corresponding orthogonal design is known for  $n \geq 7$  from [3].

### 3 Asymptotic Results

Tables 2, 3 and 4 indicate the smallest known length,  $\ell$ , such that  $4 - NPAF(4n; s_1, s_2)$  sequences, with  $\sigma = s_1 + s_2 \leq 28$ , exist for every length  $\geq \ell$ .

**Theorem 9** *The known asymptotic results for  $4 - NPAF(s_1, s_2)$  with  $\sigma = s_1 + s_2 \leq 28$  are summarized in Tables 2, 3, 4.*

**Proof.** All the results quoted except those improved here, may be found in Geramita and Seberry [2, p168] and [4, 5].  $\square$

**Theorem 10** *Suppose  $(s_1, s_2)$  satisfies the necessary conditions to be the type of orthogonal design. Then an  $OD(4N; s_1, s_2)$  exists for*

- (i)  $N \geq 2$  for  $2 \leq s_1 + s_2 \leq 8$ ;
- (ii)  $N \geq 4$  for  $9 \leq s_1 + s_2 \leq 16$ ;
- (iii)  $N \geq 5$  for  $17 \leq s_1 + s_2 \leq 20$  except possibly for the 2-tuples  $(3, 16)$ ,  $(6, 13)$ ,  $(7, 11)$ ,  $(7, 12)$  which are the types of orthogonal designs for orders  $4N$ ,  $N \geq 6$ ;
- (iv)  $N \geq 6$  for  $21 \leq s_1 + s_2 \leq 24$ ;
- (v)  $N \geq 7$  for  $25 \leq s_1 + s_2 \leq 28$ .

**Proof.** Every 2-tuple with  $2 \leq s_1 + s_2 \leq 8$  is the type of an  $OD(8; s_1, s_2)$  from [2, p.368]. This combined with Table 2 gives the result.

Every 2-tuple with  $9 \leq s_1 + s_2 \leq 16$  is the type of an  $OD(16; s_1, s_2)$  from [2, p.389]. This combined with Table 3 gives the result. Geramita and Seberry give the existence of  $OD(24; s_1, s_2)$  for  $(s_1, s_2) = (3, 16)$ ,  $(6, 13)$ ,  $(7, 11)$  and  $(7, 12)$ . Hence, using Table 2 these are the types of orthogonal designs in all orders  $4N$ ,  $N \geq 6$ . The remainder of the 2-tuples for which  $17 \leq s_1 + s_2 \leq 20$  are given in Table 3.

All 2-tuples  $2 \leq s_1 + s_2 \leq 24$  are the types of an  $OD(24; s_1, s_2)$  from [2, p.391]. This combined with Tables 3 and 4 gives the result for (iv).

Table 3 gives the result for the first part of (v) for all 2-tuples except  $(1, 26)$ ,  $(2, 25)$ ,  $(7, 19)$   $(8, 19)$  and  $(13, 14)$ . Table 1 shows an  $OD(28; s_1, s_2)$  exists for each of these 2-tuples. [2, p.394 and p.395] shows each of these 5 2-tuples is the type of an orthogonal design in orders 32 and 40. Now [5] gives  $OD(4m; s_1, s_2)$ , for these  $(s_1, s_2)$  with  $m \geq 9$  and so we have that these designs exist for all  $N \geq 7$ .  $\square$

$s_1, s_2$	$\sigma$	$\ell$	$s_1, s_2$	$\sigma$	$\ell$	$s_1, s_2$	$\sigma$	$\ell$	$s_1, s_2$	$\sigma$	$\ell$	$s_1, s_2$	$\sigma$	$\ell$
(1, 1)	2	1	(1, 2)	3	1	(1, 4)	5	2	(1, 6)	7	3	(1, 8)	9	3
			(1, 3)	4	1	(1, 5)	6	2	(1, 7)	<i>no</i>		(1, 9)	10	3
			(2, 2)	4	1	(2, 3)	5	2	(2, 5)	7	3	(2, 7)	9	3
						(2, 4)	6	2	(2, 6)	8	2	(2, 8)	10	3
						(3, 3)	6	2	(3, 4)	7	3	(3, 6)	9	3
									(3, 5)	<i>no</i>		(3, 7)	10	3
									(4, 4)	8	2	(4, 5)	9	3
												(4, 6)	10	3
												(5, 5)	10	3

Table 2: The indicated  $4 - NPAF(s_1, s_2)$  sequences with  $1 \leq s_1 + s_2 \leq 10$  exist for every length  $N \geq \ell$ .

$s_1, s_2$	$\sigma$	$\ell$	$s_1, s_2$	$\sigma$	$\ell$	$s_1, s_2$	$\sigma$	$\ell$	$s_1, s_2$	$\sigma$	$\ell$	$s_1, s_2$	$\sigma$	$\ell$
(1, 10)	11	3	(1, 12)	13	4	(1, 14)	15	5	(1, 16)	17	5	(1, 18)	19	5
(1, 11)	12	3	(1, 13)	14	5	(2, 14)	<i>no</i>		(1, 17)	18	5	(1, 19)	20	5
(2, 9)	11	5	(2, 11)	13	5	(2, 13)	15	5	(2, 15)	17	5	(2, 17)	19	5
(2, 10)	12	3	(2, 12)	14	5	(2, 14)	<i>no</i>		(2, 16)	18	5	(2, 18)	20	5
(3, 8)	11	3	(3, 10)	13	5	(3, 12)	15	5	(3, 14)	17	5	(3, 16)	19	7
(3, 9)	12	3	(3, 11)	14	5	(3, 13)	<i>no</i>		(3, 15)	18	5	(3, 17)	20	5
(4, 7)	<i>no</i>		(4, 9)	13	5	(4, 11)	15	5	(4, 13)	17	5	(4, 15)	<i>no</i>	
(4, 8)	12	3	(4, 10)	14	5	(4, 12)	16	5	(4, 14)	18	5	(4, 16)	20	5
(5, 6)	11	3	(5, 8)	13	5	(5, 10)	15	5	(5, 12)	<i>no</i>		(5, 14)	19	5
(5, 7)	12	3	(5, 9)	14	5	(5, 11)	<i>no</i>		(5, 13)	18	5	(5, 15)	20	5
(6, 6)	12	3	(6, 7)	13	5	(6, 9)	15	5	(6, 11)	17	5	(6, 13)	19	7
			(6, 8)	14	5	(6, 10)	<i>no</i>		(6, 12)	18	5	(6, 14)	20	5
			(7, 7)	14	4	(7, 8)	15	5	(7, 10)	17	5	(7, 12)	19	7
						(7, 9)	<i>no</i>		(7, 11)	18	7	(7, 13)	20	5
						(8, 8)	16	5	(8, 9)	17	5	(8, 11)	19	5
									(8, 10)	18	5	(8, 12)	20	5
									(9, 9)	18	5	(9, 10)	19	5
												(9, 11)	20	5
												(10, 10)	20	5

Table 3: The indicated  $4 - NPAF(s_1, s_2)$  sequences with  $11 \leq s_1 + s_2 \leq 20$  exist for every length  $N \geq \ell$ .



$s_1, s_2$	$\sigma$	$\ell$	$s_1, s_2$	$\sigma$	$\ell$	$s_1, s_2$	$\sigma$	$\ell$	$s_1, s_2$	$\sigma$	$\ell$
(1,20)	21	6	(1,22)	23	7	(1,24)	25	7	(1,26)	27	9
(1,21)	22	6	(1,23)	no		(1,25)	26	7	(1,27)	28	7
(2,19)	21	6	(2,21)	23	7	(2,23)	25	7	(2,25)	27	9
(2,20)	22	6	(2,22)	24	6	(2,24)	26	7	(2,26)	28	7
(3,18)	21	6	(3,20)	no		(3,22)	25	7	(3,24)	27	7
(3,19)	22	7	(3,21)	no		(3,23)	26	7	(3,25)	28	7
(4,17)	21	7	(4,19)	23	7	(4,21)	25	7	(4,23)	no	
(4,18)	22	6	(4,20)	24	6	(4,22)	26	7	(4,24)	28	7
(5,16)	21	7	(5,18)	23	7	(5,20)	25	7	(5,22)	no	
(5,17)	22	7	(5,19)	no		(5,21)	26	7	(5,23)	28	7
(6,15)	21	7	(6,17)	23	7	(6,19)	25	7	(6,21)	27	7
(6,16)	22	6	(6,18)	24	6	(6,20)	26	7	(6,22)	28	7
(7,14)	21	7	(7,16)	no		(7,18)	25	7	(7,20)	no	
(7,15)	22	7	(7,17)	no		(7,19)	26	9	(7,21)	28	7
(8,13)	21	7	(8,15)	23	7	(8,17)	25	7	(8,19)	27	9
(8,14)	no		(8,16)	24	6	(8,18)	26	7	(8,20)	28	7
(9,12)	21	7	(9,14)	23	7	(9,16)	25	7	(9,18)	27	7
(9,13)	22	6	(9,15)	no		(9,17)	26	7	(9,19)	28	7
(10,11)	21	6	(10,13)	23	7	(10,15)	25	7	(10,17)	no	
(10,12)	22	6	(10,14)	24	6	(10,16)	26	7	(10,18)	28	7
(11,11)	22	6	(11,12)	23	7	(11,14)	25	7	(11,16)	no	
			(11,13)	no		(11,15)	26	7	(11,17)	28	7
			(12,12)	24	6	(12,13)	no		(12,15)	no	
						(12,14)	26	7	(12,16)	28	7
						(13,13)	26	7	(13,14)	27	9
									(13,15)	28	7
									(14,14)	28	7

Table 4: The indicated  $4 - NPAF(s_1, s_2)$  sequences with  $21 \leq s_1 + s_2 \leq 28$  exist for every length  $N \geq \ell$ .

## References

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**Appendix A:** Order 20 (Sequences with zero nonperiodic autocorrelation function)

Design	$A_1$	$A_2$	$A_3$	$A_4$
(1, 3, 14)	do not exist with length 5			
(1, 4, 13)	do not exist with length 5			
(1, 6, 11)	$-c \quad b \quad a \quad -b \quad c$	$b \quad c \quad c \quad 0 \quad c$	$b \quad c \quad -c \quad 0 \quad -c$	$-c \quad b \quad c \quad b \quad -c$
(1, 8, 11)	$c \quad b \quad a \quad -b \quad -c$	$c \quad b \quad b \quad -c \quad c$	$b \quad -c \quad b \quad c \quad -c$	$c \quad c \quad c \quad -b \quad b$
(2, 5, 7)	$b \quad a \quad -b \quad 0 \quad c$	$-c \quad 0 \quad -c \quad a \quad c$	$c \quad 0 \quad b \quad 0 \quad c$	$-c \quad 0 \quad b \quad 0 \quad b$
(3, 6, 8)	$b \quad -c \quad b \quad c \quad a$	$-b \quad c \quad 0 \quad c \quad a$	$0 \quad -c \quad a \quad -c \quad -b$	$-b \quad -c \quad b \quad c \quad 0$
(7, 10)	$-b \quad b \quad b \quad 0 \quad a$	$a \quad 0 \quad -b \quad b \quad b$	$a \quad b \quad a \quad b \quad -a$	$a \quad -b \quad 0 \quad -b \quad a$

**Appendix B:** Order 28 (Sequences with zero periodic autocorrelation function)

Design	$A_1$	$A_2$	$A_3$	$A_4$
(1, 8, 17)	$a \quad -c \quad -c \quad c \quad -c \quad c \quad c$	$b \quad b \quad 0 \quad -c \quad c \quad c \quad c$	$b \quad b \quad -c \quad -c \quad c \quad -c \quad 0$	$-b \quad b \quad -b \quad c \quad b \quad c \quad c$
(2, 4, 22)	$-c \quad c \quad c \quad -c \quad -c \quad -b \quad a$	$-c \quad c \quad -c \quad c \quad a \quad b \quad c$	$c \quad -c \quad c \quad c \quad c \quad -b \quad c$	$-c \quad c \quad c \quad c \quad c \quad b \quad -c$
(2, 9, 11)	$a \quad b \quad 0 \quad -c \quad -b \quad 0 \quad 0$	$a \quad -b \quad c \quad c \quad b \quad -c \quad 0$	$b \quad b \quad -c \quad c \quad b \quad 0 \quad 0$	$c \quad -c \quad c \quad c \quad c \quad b \quad -b$
(2, 9, 17)	$a \quad -b \quad c \quad -b \quad b \quad c \quad b$	$a \quad c \quad -c \quad c \quad -c \quad -c \quad -c$	$b \quad b \quad -c \quad c \quad b \quad -c \quad c$	$-b \quad -c \quad b \quad c \quad c \quad c \quad c$
(3, 6, 17)	$a \quad b \quad -c \quad c \quad c \quad 0 \quad c$	$a \quad -c \quad b \quad c \quad -c \quad 0 \quad -c$	$a \quad -b \quad -b \quad -c \quad -c \quad c \quad c$	$-b \quad b \quad c \quad -c \quad c \quad c \quad c$
(3, 7, 11)	$a \quad b \quad 0 \quad c \quad -c \quad c \quad 0$	$a \quad b \quad 0 \quad c \quad 0 \quad -c \quad 0$	$c \quad c \quad b \quad -a \quad b \quad -c \quad 0$	$c \quad b \quad -b \quad -b \quad c \quad c \quad 0$
(3, 7, 15)	$a \quad -b \quad b \quad b \quad c \quad c \quad 0$	$a \quad b \quad -c \quad -c \quad c \quad c \quad -c$	$a \quad c \quad -b \quad -c \quad -b \quad -c \quad 0$	$b \quad -c \quad c \quad -c \quad 0 \quad -c \quad -c$
(3, 10, 15)	$a \quad -b \quad b \quad b \quad c \quad b \quad c$	$a \quad b \quad -b \quad -b \quad c \quad -c \quad -c$	$a \quad -c \quad -b \quad -c \quad -c \quad c \quad c$	$b \quad -c \quad b \quad -c \quad -c \quad -c \quad -c$
(4, 9, 13)	$a \quad a \quad -b \quad -c \quad c \quad b \quad 0$	$b \quad b \quad -c \quad c \quad b \quad -c \quad c$	$c \quad c \quad c \quad c \quad -c \quad b \quad -b$	$a \quad -a \quad -b \quad -c \quad -c \quad b \quad 0$
(4, 10, 11)	$a \quad a \quad -b \quad b \quad 0 \quad -c \quad c$	$-a \quad a \quad -c \quad b \quad c \quad c \quad -b$	$b \quad b \quad b \quad c \quad -c \quad 0 \quad -c$	$-b \quad b \quad c \quad c \quad 0 \quad b \quad c$
(5, 5, 18)	$a \quad b \quad -c \quad b \quad c \quad -c \quad c$	$a \quad -c \quad a \quad c \quad b \quad -b \quad -b$	$a \quad -c \quad -a \quad -c \quad -c \quad c \quad -c$	$-c \quad -c \quad c \quad c \quad c \quad c \quad c$
(5, 9, 14)	$a \quad -b \quad b \quad -c \quad c \quad c \quad c$	$a \quad b \quad -c \quad a \quad c \quad -b \quad -c$	$b \quad -b \quad -b \quad -b \quad -c \quad -b \quad c$	$-a \quad c \quad c \quad a \quad c \quad -c \quad c$
(5, 10, 10)	$-a \quad a \quad a \quad b \quad c \quad b \quad c$	$a \quad 0 \quad a \quad -b \quad -c \quad -c \quad c$	$b \quad 0 \quad -b \quad -b \quad c \quad c \quad -c$	$-b \quad b \quad b \quad -c \quad 0 \quad -c \quad b$
(6, 7, 8)	$-c \quad c \quad c \quad 0 \quad c \quad b \quad a$	$c \quad -c \quad -c \quad 0 \quad -c \quad b \quad a$	$b \quad a \quad 0 \quad -a \quad b \quad 0 \quad 0$	$-b \quad a \quad -b \quad a \quad b \quad 0 \quad 0$
(6, 9, 11)	$a \quad -c \quad c \quad c \quad c \quad 0 \quad c$	$a \quad -b \quad b \quad c \quad -c \quad 0 \quad -c$	$a \quad b \quad -c \quad a \quad -b \quad -b \quad b$	$-a \quad b \quad b \quad a \quad c \quad b \quad -c$
(7, 8, 10)	$-a \quad a \quad a \quad b \quad b \quad -c \quad 0$	$-a \quad b \quad b \quad c \quad -c \quad c \quad 0$	$a \quad -b \quad -c \quad -c \quad b \quad -b \quad b$	$a \quad -c \quad a \quad c \quad c \quad 0 \quad c$
(8, 9, 11)	$-c \quad b \quad -b \quad a \quad c \quad -c \quad a$	$b \quad a \quad a \quad c \quad a \quad -a \quad -b$	$c \quad -a \quad b \quad b \quad b \quad -c \quad a$	$-c \quad c \quad c \quad c \quad b \quad c \quad -b$
(9, 9, 10)	$a \quad a \quad a \quad -b \quad b \quad -c \quad c$	$-b \quad b \quad c \quad b \quad b \quad b \quad -c$	$a \quad b \quad -a \quad -b \quad c \quad -c \quad -c$	$-a \quad a \quad -a \quad c \quad a \quad c \quad c$

**Appendix C: Order 28 (Sequences with zero nonperiodic autocorrelation function)**

Design	$A_1$							$A_2$							$A_3$							$A_4$						
(1, 1, 17)	-c	a	c	0	0	0	0	-c	b	c	0	0	0	0	c	c	c	-c	-c	c	-c	c	c	c	0	c	-c	c
(1, 3, 14)	-c	0	-c	a	c	0	c	c	0	0	0	b	-c	c	c	c	-c	0	0	c	b	-c	-c	0	0	-c	0	b
(1, 8, 16)	-c	-b	c	a	-c	b	c	b	b	0	c	-c	c	-c	b	-c	-c	0	b	c	c	-c	-b	b	-c	0	-c	-c
(1, 9, 16)	-c	-b	c	a	-c	b	c	c	b	-c	b	-c	b	c	-c	0	-c	b	-c	-b	-c	c	0	c	b	-c	-b	-c
(2, 7, 10)	-c	0	a	0	c	b	-c	b	a	-b	0	-b	c	0	-c	0	0	b	-c	0	0	-c	b	c	0	c	b	c
(2, 7, 13)	a	b	-c	0	-c	0	0	a	-b	c	0	c	0	0	c	-c	-b	b	b	c	c	c	c	b	-c	b	c	-c
(2, 8, 18)	b	c	c	a	-c	-c	b	-b	-c	c	a	-c	c	-b	b	c	-c	c	c	c	-b	b	c	c	c	-c	c	-b
(3, 4, 14)	c	c	a	-c	c	-c	0	c	b	0	a	0	-b	c	-c	c	a	-c	-c	-c	0	c	b	0	0	0	b	-c
(3, 6, 16)	c	b	c	a	-c	0	-c	c	b	-c	0	c	a	-c	c	b	-c	b	-c	-a	c	c	0	c	-b	c	b	c
(3, 7, 11)	c	0	a	b	0	c	-c	c	0	-c	0	a	b	0	-c	-c	-b	a	-b	c	0	-c	0	-c	-b	b	b	-c
(3, 8, 10)	b	c	c	0	a	-c	b	-b	-c	c	0	a	c	-b	b	-c	0	a	0	-c	-b	b	c	0	0	0	c	-b
(3, 8, 15)	b	c	c	0	a	-c	b	b	c	-c	0	-a	-c	b	b	-c	-c	a	c	-c	-b	b	c	c	c	-c	c	-b
(3, 9, 14)	c	-b	0	b	c	c	a	-c	b	-c	a	-c	-b	c	-c	b	0	-b	-c	c	a	-c	b	c	b	c	b	-c
(4, 4, 13)	a	-c	-c	0	c	c	a	b	-c	c	0	-c	c	b	a	c	0	0	0	c	-a	b	c	0	c	0	c	-b
(4, 5, 14)	a	0	-b	0	b	0	a	c	0	-c	b	c	0	c	a	-c	b	c	b	-c	-a	c	c	c	c	-c	c	-c
(4, 6, 11)	b	0	a	0	a	0	-b	c	c	c	b	-c	c	0	c	-c	c	b	-c	-c	0	b	0	-a	-c	a	0	b
(5, 5, 13)	b	c	c	a	-c	-c	b	a	-c	c	-b	-c	c	a	a	c	0	0	0	c	-a	b	c	0	c	0	c	-b
(5, 10, 10)	b	c	c	a	c	-c	b	-b	0	-b	a	-c	a	b	c	-c	-c	0	-b	b	b	b	0	-c	-a	b	a	-c
(7, 8, 13)	-c	b	-a	a	a	b	-c	-a	c	b	-c	c	b	c	a	-c	-b	c	c	b	c	-c	b	a	c	a	-b	c
(8, 8, 9)	b	-c	a	c	b	0	a	b	-c	b	c	-a	0	-a	b	-c	-a	-c	-b	-c	a	b	-c	-b	0	a	c	-a
(8, 9, 9)	a	c	c	0	c	-c	a	a	b	-c	0	-c	-b	a	a	b	-b	b	b	b	-a	a	c	b	-c	-b	c	-a
(4, 22)	b	0	a	a	-b	-b	b	-b	0	-a	a	b	b	b	b	b	-b	b	-b	b	b	-b	b	b	b	-b	b	b
(5, 23)	b	-b	-a	a	a	b	b	-b	-b	a	-b	a	b	b	b	b	-b	b	-b	b	b	-b	b	b	b	-b	b	b
(11, 14)	-a	a	a	-b	-b	b	b	b	b	a	0	b	-a	a	a	0	a	a	-b	b	-b	-b	b	a	b	b	0	-a
(11, 15)	a	b	-b	a	b	b	-a	a	b	b	0	-a	-b	a	a	b	-b	0	a	-b	a	a	b	b	b	-b	b	-a
(11, 17)	-a	a	a	-b	b	b	b	b	a	-b	-b	-a	a	-b	-b	b	a	a	b	a	-b	a	b	-b	b	-a	b	b