On F-Squares and their Critical Sets

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Abstract

We define the notion of critical set of an F-square, following the definition of critical set in latin squares, and then give critical sets for certain classes of F-squares. We also generalise certain results obtained for critical sets of latin squares, and look at minimal such sets. We show that critical sets of F-squares need to be studied as well as critical sets for latin squares as the techniques used differ considerably. We obtain theorems for the sizes of critical sets of types $F(n; 1, n-1)$, $F(n; 1, 1, n-2)$, and $F(n; 2, 2, \ldots, 2)$.

1 Introduction

Let $n = \alpha_0 + \alpha_1 + \ldots + \alpha_{v-1}$, where α_i is a natural number for each i. A *frequency square* or *F-square* of *type* $F = F(n; \alpha_0, \alpha_1, \ldots, \alpha_{v-1})$ *and of order* n is an $n \times n$ array with entries chosen from the set $N = \{0, 1, \ldots, v-1\}$, such that each element i occurs α_i times in each row and in each column. An F-square F can also be thought of as the set of ordered triples $F = \{(i, j; k)\}$ where element k occurs in position (i, j). A subset of F will also be called a *subsquare* or *partial F-square*. A subsquare of an F-square is also said to be *embeded* in the F-square. Two subsquares of an F-square are called *isotopic*, if one can be transformed onto the other by rearranging rows, rearranging columns, and renaming elements. A non-empty subset S of $F = F(n; \alpha_0, \alpha_1, \ldots, \alpha_{v-1})$ is a *critical set (of F)* if,

1. F is the only F-square of order n which has element k in position (i, j) for each $(i, j; k) \in S$. (We then say that F is *uniquely completable from* S, and that S is *uniquely completable to* F),

2. (a) every proper subset of S is contained in at least two F-squares of type $F(n; \alpha_0, \alpha_1, \ldots, \alpha_{v-1})$
   or
   (b) for every $(i, j; k) \in S, l \in N, l \neq k \implies$ there does not exist any F-square of type $F(n; \alpha_0, \alpha_1, \ldots, \alpha_{v-1})$ which contains $(S/\{(i, j; k)\}) \cup \{(i, j; l)\}$

We note that a latin square is an F-square of type $F = F(n; 1, 1, \ldots, 1))$. A latin square $L = \{(i, j; k)\}$ of order n is called *back circulant* if $(i + j)\ mod\ n = k$ for every triple $(i, j; k) \in L$.

An F-square will usually have more than one critical set. We say that a critical set is *minimal* if it has minimum cardinality. The notation cr or $cr(F)$ will denote the size of a critical set. We denote by $scr(F(n; \alpha_0, \alpha_1, \ldots, \alpha_{v-1}))$ the size of the smallest critical set of

$(F(n;\alpha_0,\alpha_1,\ldots,\alpha_{v-1}))$. The size of the smallest critical set in any F-square of order n is denoted by $fscs(n)$, and that of the largest critical set in any F-square of order n is denoted by $flcs(n)$. In the case of latin squares only, these sizes are denoted by $scs(n)$ and $lcs(n)$ respectively. Curran and van Rees [5] evaluated $scs(n)$ for $n = 1,\ldots,5$, and recently Howse[9] extended this result to $n = 6$. The results they obtained are summarised as follows:

n	1	2	3	4	5	6
scs(n)	0	1	2	4	6	9

We conjecture that the size of the smallest critical set of type $(\mathrm{F}(n;1,\ldots,1,n-i))$ where there are i zeroes satisfies

$$scr(F(n;1,\ldots,n-i)) \leq scr(F(n;1,\ldots,1,n-i+1)) \leq scr(F(n;1,1,\ldots,1)) = scs(n).$$

Critical sets of latin squares were first studied by Smetaniuk [12], Curran and van Rees [5], and Cooper, Donovan and Seberry[1]. If n is even, let C_e be the set

$$C_e = \{(i,j;i+j) :\ i = 0,\ldots,\frac{n}{2}-1,\ \text{and } j = 0,\ldots,\ \frac{n}{2}-1-i\}$$

$$\cup\ \{(i,j;i+j) : i = \frac{n}{2}+1,\ldots,n-1 \text{ and } j = \frac{n}{2}-i,\ldots,n-1\}$$

where addition is taken modulo n. If n is odd, let C_o be given by

$$C_o = \{(i,j;i+j) :\ i = 0,\ldots,(n-3)/2 \text{ and } j = 0,\ \ldots,\ (n-3)/2-i\}$$

$$\cup\ \{(i,j;i+j) : i = (n-1)/2+1,\ \ldots,\ n-1 \text{ and } j = (n-1)/2-i,\ldots,n-1\}$$

where addition is reduced modulo n. Curran and van Rees showed that sets C_e and C_o both satisfy condition 1 of the definition of critical set, and that C_e was in fact critical. Cooper, Donovan and Seberry [1] later verified that both C_e and C_o are critical sets, and that in fact C_e is minimal. In summary they showed that:

Theorem 1 *C_e is a minimal critical set for a back circulant square of even order n, and $|C_e| = \frac{n^2}{4}$. C_o is a critical set for a back circulant square of odd order n, and $|C_o| = \frac{n^2-1}{4}$.*

Smetaniuk[12], using a different method, showed that the size of a minimal critical set of a latin square of order $2n$ is at most n^2. It is not yet established whether for n odd $|C_o| = \frac{n^2-1}{4}$ is minimal, but the evidence so far, is that this may be the case, see Howse[9].

Donovan and Cooper [7] later established another family of critical sets in back circulant latin squares, thus settling a conjecture by Nelder [11] that in the latin square representing the addition table of the integers modulo n, the upper triangle of entries bounded by, but not including the main right-to-left diagonal, is a critical set. The critical set in question is

$$A = \{(i,j;i+j) : i = 0,\ldots,n-2 \text{ and } j = 0,\ldots,n-2-i\}.$$

They further proved that:

Lemma 1 *(Donovan and Cooper) If L is any back circulant latin square of order n, and r is some integer such that $(n-3)/2 \leq r \leq n-2$, then the set*

$$B = \{(i,j;i+j) : i = 0,\ldots,r \text{ and } j = 0,\ldots,r-i\}$$

$$\cup\ \{(i,j;i+j) : i = r+2,\ldots,n-1 \text{ and } j = r+1-i,\ldots,n-1\}$$

is a critical set in L.

If P is a partial latin square then an element $p \in P$ is said to be *2-essential* if there exists a 2×2 latin subsquare S, such that $S \cap P = p$. P is called *2-critical* if it is uniquely completable to a latin square, and every element of P is 2-essential. Stinson and van Rees give a product construction on latin squares, and show that given a 2-critical set, a critical set of higher order can be constructed. Their construction could not be applied to certain types of critical sets in back circulant latin squares of odd order (among others).

Cooper, Donovan and Gower[3] determined a family of critical sets for latin squares that are the product of a latin square of order 2 with a back circulant latin square of odd order. In [16] Peddada, Seberry and Chaudhry showed that the uniquely completable sets of Cooper, Donovan and Gower are not minimal, and also gave a number of computer generated critical sets.

Let $P = \{(i, j; k) : i, j, k \in N\}$ be a partial F-square of order n. $|P|$ is said to be the *size* of the partial square and the set $\{(i, j) : (i, j; k) \in P \text{ for some } k \in N\}$ is called the *shape* of P.

We now look at general constructions of critical sets for some types of F-squares.

2 F-squares of type $F = F(n; t, n - t)$.

Theorem 2 *Let $t = 1$, so F is of the form $F(n; 1, n - 1)$. For every natural number $n \geq 2$, there exists a minimal critical set of size $fscs(n) = n - 1$.*

Proof. In such an F-square 0 occurs precisely once in every row and once in every column, while 1 occurs $n - 1$ times in every row and $n - 1$ times in every column. Consider the short diagonal $D' = \{(1, n - 1; 0), (2, n - 2; 0), \ldots, (n - 1, 1; 0)\}$. We claim that it is a critical set. Denote by D the shape of D'. Since 0 occurs once in rows 2 to $n - 1$, therefore 1 can fill every other cell in these rows. Also, 0 occurs once in columns 2 to $n - 1$, so 1 can fill every other cell (that is, those column cells that were not filled above). Thus the only cell not yet filled is $(0, 0)$, and this can be uniquely filled with the element 0. That is, D' is uniquely completable.
Consider the subset $D/\{(1, n-1; 0)$ of D. We can uniquely fill each cell in the rows $2, \ldots, n-1$ and in the columns $2, \ldots, n-2$, because the element 0 occurs once each in each of these rows and columns. Only cells $(0, 0), (0, n-1), (1, 0), (1, n-1)$ remain unfilled. Consider cell $(1, n-1)$. There are exactly two possible entries that can fill this cell, 0 or 1. If 0 then we obtain the previously filled F-square. If the entry is 1 then element 0 must be forced into cells $(0, n - 1)$ and $(1, 0)$, in which case 1 must again fill cell $(0, 0)$. But this then gives a complete F-square of the form $F(n; 1, n - 1)$. Thus $D/\{(1, n-1; 0)$ has at least two completions. A similar argument can be made for each of the other entries in D. Thus D' is a critical set.

Let E be any subset of F of size $n - 2$, and with every cell filled with element 0. Then each row i such that there exists $(i, j; 0) \in E$ can be filled uniquely, as can every column in E than contains a 0. There will be two rows that cannot be uniquely filled, and two columns, giving rise to exactly 4 cells that haven't been filled (the intersections of these rows and columns). Each of these unfilled rows/columns contains only $1's$. Thus filling any one of these cells will force the other cells to be filled uniquely. That is, each such E has at least two completions. Thus there cannot be any critical set of size $n - 2$ (or less), and D is of minimal size.

Clearly $fscs(n) = n - 1$. □

Remark. This is obviously the smallest critical set size taken over all critical sets of order n. That is, $fscs(n) = n - 1$. Further work, which will appear elsewhere, indicates that the largest critical set size for all critical sets of order n, is $flcs(n) \geq 7(\frac{n}{4})^2 - 2$ when n is even, and is $flcs(n) \geq 7(\frac{n-1}{4})^2 + 7(\frac{n-1}{4}) - 1$ when n is odd.

Remark. In the case of latin squares a uniquely completable partial latin square is not a critical set if and only if any subset of the partial latin square has at least two completions. In the case of F-squares, as can be seen in the proof of the theorems above and below, the situation differs in general. A uniquely completable partial F-square F may not be a critical set because every non-trivial subset of F does not lead to *any* legitimate F-square. Thus the idea of *latin interchanges* as discussed and used by several authors (see for example [1]) is not particularly useful for F-squares in general.

Theorem 3 *Every F-square of type $F(n; 2, n-2), n \geq 4$ has a critical set of size $2n-3$. For this type the element 0 occurs twice in each row and twice in each column, whereas element 1 occurs $n-2$ times in each row and in each column.*

Proof. Let $D' = \{(1, n-1; 0), (2, n-2; 0), \ldots, (n-1, 1; 0)\} \cup \{(2, n-1; 0), (3, n-2; 0), \ldots, (n-1, 2; 0)\}$ be the union of the two short diagonals indicated. Denote by D the shape of D'. Note that in D, rows $2, \ldots, n-1$ and columns $2, \ldots, n-1$ contain the element 0 twice each. Each cell in these rows/columns can be filled uniquely, with the element 1. Fill these. So far, only cells $(0,0), (0,1), (1,0)$ and $(1,1)$ haven't yet been filled. But row 0 has the element 1 occuring in each of its cells, except for cells $(0,0)$ and $(0,1)$. Thus these two cells can be filled uniquely with element 0 each. Column 0 now contains 1 in each of the cells $(2,0), \ldots, (n-1,0)$, and element 0 in cell $(0,0)$. Thus 0 can uniquely fill cell $(1,0)$. Element 1 is now forced uniquely into cell $(1,1)$. Thus D' is uniquely completable.

Consider the set $E = D/\{(2, n-1)\}$. Element 0 occurs twice in each of the rows/columns $3, \ldots, n-1$. Thus the rest of the cells in these rows/columns can be uniquely filled with the element 1. Fill these. Consider column 0. The cells common to this column and the union of the rows $3, \ldots, n-1$ are $(3,0), \ldots, (n-1,0)$, and these must necessarily now have been filled with the element 1. But there are $n-2$ occurances of 1 in this column, so the only two remaining cells, $(0,0)$ and $(1,0)$, must be filled with element 0.

So far, only cells $(0,1), (0,n-1), (1,1)$ and $(1,n-1)$ are unfilled. Each of these cells may be filled with either 0 or 1. If for example 0 goes into $(1,n-1)$ then element 1 goes into cell $(0,n-1)$, forcing 0 into $(0,1)$ and 1 into $(1,1)$. We obtain an F-square of type $F(n; 2, n-2)$. On the other hand if 1 goes into cell $(1,n-1)$ then 0 goes into cell $(0,n-1)$, 1 into cell $(0,1)$ and 0 into cell $(1,1)$, giving another F-square of type $F(n; 2, n-2)$. That is, E has two completions.

Similar arguments can be made for the partial F-square $D/(n-1,1)$. That is, $D/(n-1,1)$ has at least two completions.

Consider the set $D/\{(2, n-2)$. Each of rows $3, \ldots, n-1$ contain element 0 twice. Each of columns $3, \ldots, n-3$ and $n-1$ contain the element 0 twice. Thus the remaining cells in these rows/columns can be filled uniquely, with the element 1. Fill these. The as yet unfilled cells will be the intersections of the unfilled rows and columns: These are the cells $(0,0), (0,1), (0,n-2), (1,0), (1,1), (1,n-2), (2,0), (2,1), (2,n-2)$, nine in all. If cell $(2,n-2)$ is filled with 0 then we can uniquely complete to the above F-square. We will show that there is at least one other completion. Fill cell $(2,0)$ with element 1, and cell $(2,n-2)$ with element 1. Now column 1 contains 1 in all its cells, except cells $(0,0)$ and $(1,0)$. These can now be filled with element 0. Row 1 now contains 0 in cells $(1,0)$ and $(1,n-1)$. Thus any remaining cell in this row can be filled, with element 1. Fill these. Only cell $(0,n-2)$ in column $n-2$ hasn't yet been filled. Since 1 already occurs $n-2$ times in this column, therefore element 0 can uniquely fill this cell. This forces element 0 into cell $(0,1)$, and element 1 into cell $(2,1)$. The result is an F-square of type $F(n; 2, n-2)$. That is, $D/\{(2, n-2; 0)\}$ has at least two completions. Similar arguments can be made for each of the sets $D/\{(3,n-3)\}, \ldots, D/\{(n-2,2)\}$, as well as the deletion of any of the entries in the lower small diagonal. Thus D' is a critical set.

The size of the critical set is $cr = n-1+n-2 = 2n-3$. □

The above theorem is for any integer n. For n even, we have:

Theorem 4 *If $n \geq 4$ is even, then each F-square of type $F = F(n; 2, n-2)$ has a critical set D' of size $cr = 2n - 4$. Furthermore D' is minimal.*

Proof. Let D' be the set $\{(0,0;0), (0,1;0), (1,0;0), (1,1;0), (2,2;0), (2,3;0), (3,2;0), (3,3;0), \ldots, (n-4, n-4; 0), (n-4, n-3; 0), (n-3, n-4; 0), (n-3, n-3; 0)\}$. Let D be the shape of D'. Then 0 occurs twice each in rows $0, \ldots, n-3$, and twice each in columns $0, \ldots, n-3$. Thus each of the remaining cells in these rows and columns can be filled uniquely with element 1. Having filled these, only four cells remained unfilled, namely: $(n-2, n-2), (n-2, n-1), (n-1, n-2), (n-1, n-1)$. But these can be filled uniquely with element 0. Thus D' is uniquely completable.

Suppose $(0,0;0)$ is deleted from D'. Then rows $1, \ldots, n-3$ contains 0 twice, and columns $0, 2, \ldots, n-3$. Every empty cell in these rows and columns can be uniquely filled with element 1. So far cells $(0,0), (0, n-2), (0, n-1); (n-2, 0), (n-2, n-2), (n-2, n-1); (n-1, 0), (n-1, n-2), (n-1, n-1)$ are not filled. None of these cells can be filled uniquely. Similar arguments can be made for any other proper subset of D'. Thus D' is a critical set.

Consider D' again. Each non-empty row and column contains 0 exactly twice. Any row and column isotope of D' will again have this property. Consider row 0. Suppose we remove the entry 0 in cell $(0,1)$ and place it elsewhere in the same row. Call the new set of triples F. There are only two cells that we can place this 0 element in, without destroying the form $F(n; 2, n-2)$, and these are cells $(0, n-2)$ and $(0, n-1)$. Suppose 0 is placed in cell $(0, n-2)$. Then as above we can fill all the rows and columns containing the element 0 twice. These are rows $0, \ldots, n-3$, and columns $0, 2, \ldots, n-3$. Since in column $n-1$ cells $(0, n-1), \ldots, (n-3, n-1)$ each contain the element 1 once each, therefore cells $(n-2, n-1)$ and $(n-1, n-1)$ can be filled uniquely with entry 0. Thus column $n-1$ too can be uniquely filled.

So far, rows $n-2, n-1$ and columns 1 and $n-2$ haven't been filled completely. The intersections of these rows and columns are cells $(n-2, 1), (n-2, n-2), (n-1, 1)$ and $(n-1, n-2)$. None of these can be uniquely filled. Similar arguments can be made if entry 0 is removed from cell $(0, n-1)$. Similarly, no isotope of F will be uniquely completable.

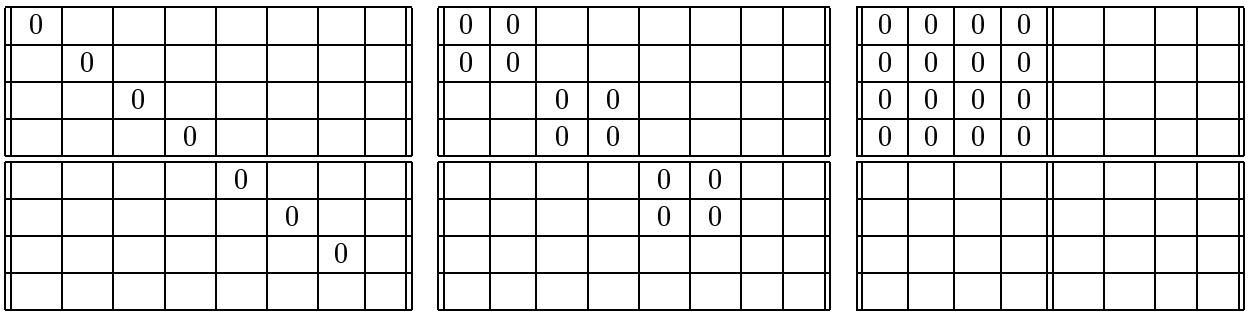
The difference between D' and F is in the set D', every non-empty row and column contains 0 twice, whereas in F, columns 1 and $n-1$ contained 0 only once each. It can be easily checked that if an extra 0 is placed either in column 1 or $n-1$ without destroying the form $F(n; 2, n-2)$, then the new set will be uniquely completable to an F-square of the form $F(n; 2, n-2)$. That is, no non-isotope of D' that has the same size as D' or that has a smaller size can uniquely complete. That is, D' is minimal.

The size of the critical set is $cr = 2.(n-2) = 2n - 4$. □

By modifying the above proofs we can generalise the above result as follows:

Theorem 5 *Let $t|n, t \neq n$. Then there is an F-square of type $F(n; t, n-t)$ with critical set of size $cr = tn - t^2$.* □

Example 1 *Let $n = 8$. Then for $t = 1, 2, 4$ we have the following critical sets:*



□

Theorem 6 *If $t \leq k$ then $F(tk+1; t, t(k-1)+1)$ has a critical set of size $cr = (k-1)t^2 + \frac{1}{2}t(t+1)$.*

Proof. Construct a set D' of cells as follows:

1. From rows $0, \ldots, u, \ldots, t-1$, pick cells (u, v), where $0 \leq v \leq t-1$.
2. From rows $t, \ldots, u, \ldots, 2t-1$, pick cells (u, v), where $t \leq v \leq 2t-1$.

   .....
3. From rows $t(k-2), \ldots, u, \ldots, t(k-1)-1$, pick cells (u, v), where $t \leq v \leq t(k-1)-1$.
4. (a) From row $t(k-1)$, pick cells $(t(k-1), v)$, where $t(k-1) \leq v \leq tk-1$.
   (b) From row $t(k-1)+1$, pick cells $(t(k-1), v)$, where $t(k-1) \leq v \leq tk-2$.

   .....
   (c) From row $tk-1$, pick cell $(tk-2, t(k-1))$.

We claim that the set containing all of the above cells, each filled with entry 0, is a critical set.

Each of the rows $0, \ldots, t(k-1)$ contains the element 0 t times. Thus the other entries in these rows can be filled with entry 1. Similarly for columns $0, \ldots, t(k-1)$. Row tk and column tk now each contain element 1 $t(k-1)+1$ times. So the other cells in these row and column can be filled with element 0. Row $tk+1$ and column $tk+1$ now each contains the element 0 t times, and so the empty cells in these rows and columns can be filled with element 1. Continuing this way, we can uniquely fill the F-square.

The proof that the above set is critical is similar to the proofs of the above theorems.

On inspection $cr = (k-1)t^2 + \frac{1}{2}t(t+1)$. □

Example 2 *Consider $n = 13$. $13 = 2 \times 6 + 1$ and also $13 = 3 \times 4 + 1$. Then for types $F(13; 2, 11)$ and $F(13; 3, 10)$ we have the following critical sets respectively:*

0	0											
0	0											
		0	0									
		0	0									
				0	0							
				0	0							
						0	0					
						0	0					
								0	0			
								0	0			
										0	0	
										0		

0	0	0										
0	0	0										
0	0	0										
			0	0	0							
			0	0	0							
			0	0	0							
						0	0	0				
						0	0	0				
						0	0	0				
									0	0	0	
									0	0		
									0			

Similarly, each subarray above can be replaced by a latin square on the symbols $0, 1, \ldots, t-1$, giving:

Theorem 7 *If $t \leq k$ then $F(tk+1; 1, 1, \ldots, 1, t(k-1)+1)$ has a critical set of size $cr = (k-1)t^2 + \frac{1}{2}t(t+1)$.*

Example 3 *Thus for example the above two critical sets become:*

0	1											
1	0											
		0	1									
		1	0									
				0	1							
				1	0							
						0	1					
						1	0					
								0	1			
								1	0			
										0	1	
										1		

0	1	2										
1	2	0										
2	0	1										
			0	1	2							
			1	2	0							
			2	0	1							
						0	1	2				
						1	2	0				
						2	0	1				
									0	1	2	
									1	2		
									2			

3 Type $F(2t+1; 1, 1, 2t-1)$

Let $t = 2$. Consider now the following partial F-square:

0				1
		2		
	1			
1				0

This is a critical set for an F-square of type $F(5; 1, 1, 3)$. Here cr $= 6$. We can embed this in a 7×7 partial F-square as follows:

0						1
	0				1	
			2			
		1				
	1				0	
1						0

This is a critical set for a 7×7 F-square of type $F(7; 1, 1, 5)$. This critical set can in turn be embedded in a partial F-square of order 9, with extra entries $(0, 0; 0), (8, 0; 1), (8, 8; 0), (0, 8; 1)$, giving:

0								1
	0						1	
		0				1		
				2				
			1					
		1				0		
	1						0	
1								0

This is a critical set of size 14 for an F-square of type $F(9;1,1,7)$

Similarly we can construct a critical set of order 11, 13, etc. In general, given a critical set of the above type of odd order $n \geq 5$, a partial F-square of the same type, of order $n+2$ can be obtained by the above process. We call this process *embedding to a higher order* . The resulting higher order partial F-square is said to be an *embedding from* the lower order F-square.

Theorem 8 *For $t \geq 2$ there is an F-square of type $F(2t+1;1,1,2t-1)$ having critical set of size cr(F) = 4t-2.*

Proof. We will show that the 5×5 partial F-square is a critical set: In the first place rows/columns 0 and 4 contain each of the elements 0 and 1 once each. Thus each empty cell in these rows/columns can be filled with element 2. Row 2 contains element 2 three times. The other two (empty) cells must be filled with either 0 or 1. Since 1 is already contained once in column 2, cell $(1,1)$ must be filled with element 0. Cell $(1,3)$ must therefore be uniquely filled with element 1. Cell $(2,1)$ in column 1 must now be filled with element 2. Of the empty cells, element 0 is now forced into cell $(2,3)$, element 2 into cell $(3,3)$, element 0 into cell $(3,2)$, and element 1 into $(2,2)$. Thus every cell in the partial F-square can be uniquely filled.

We now show that the 5×5 partial F-square is a critical set, by exhibiting a second completion for any deletion of any element from this set. (See table below)

Thus the original 5×5 partial F-square is a critical set.

Suppose H is a critical set of type $F(2t+1;1,1,2t-1)$ and odd order n. Suppose G is an embedding from H. We claim that H is a critical set for an F-square of the above type, but of odd order $2t+3$. Fill H to obtain F. Now only rows 0, $2t+2$, and columns 0 and $2t+2$ remain unfilled in the new partial F-square G. Since each of these rows and columns contain 0 and 1 exactly once each, therefore each empty cell in each of these rows/columns can be uniquely filled with the element 2. But then we obtain an F-square of type $F(2t+3;1,1,2t+1)$. That is, G is uniquely completable.

If an entry is deleted from H then we cannot complete since H is a critical set of order $2t+1$. Suppose entry $(0,0;0)$ is deleted. Fill in all the rows and columns that contain each of the elements 0 and 1 once each. Now fill in the entries in the partial square H. Then only row 0 and column 0 now have empty entries each. In row 0, only cells $(0, \frac{n-3}{2})$, $(0, \frac{n-1}{2})$, and $(0, \frac{n+1}{2})$ have not been filled in. Since every column except column 0 contains each of the elements 0 and 1 once each, therefore element 2 must go into each of these cells. But now row 0 contains 1 once (in cell $(0, 2t+2)$), and the element 2 in every other cell. Clearly this is a contradiction. Deleting any of the other corner entries will give a similar result. Thus H is a critical set.

Remark. Thus for n odd our conjecture becomes

$$n-1 \leq 4t-2 \leq scr(F(n;1,1,1,n-3)) \leq \ldots \leq scr(F(n;1,1,\ldots,2))$$

$$\leq scr(F(n;1,1,\ldots,1)) = lcs(n)$$

.

Table of alternative completions

Element	Completion after deletion
(1,2;2)	0 2 2 2 1 2 2 0 1 2 2 0 1 2 2 2 1 2 0 2 1 2 2 2 0
(3,1;1)	0 2 2 2 1 2 1 2 0 2 2 2 0 1 2 2 0 1 2 2 1 2 2 2 0
(0,0;0)	2 2 2 0 1 2 0 2 1 2 0 2 1 2 2 2 1 0 2 2 1 2 2 2 0
(4,0;1)	0 2 2 2 1 2 0 2 1 2 1 2 2 0 2 2 1 0 2 2 2 2 1 2 0
(0,4;1)	0 2 2 1 2 2 0 2 2 1 2 2 1 0 2 2 1 0 2 2 1 2 2 2 0
(4,4;0)	0 2 2 2 1 2 2 2 1 0 2 2 1 0 2 2 1 0 2 2 1 0 2 2 2

□

4 F-squares of type $F = F(n; 2, 2, \ldots, 2)$

Let n be a natural number, $n \geq 2$. In this section we construct some classes of F-squares ¿from latin squares.

Theorem 9 *There are two (upto isomorphism) F-squares of order* 4 *of type* $F(4; 2, 2)$. *We list them below with their critical sets:*

0	**0**	1	1
0	**0**	1	1
1	1	0	0
1	1	0	0

$TypeF(4; 2, 2)$

0	0		
0	0		

critical set

0	**0**	1	1
0	1	1	0
1	1	0	0
1	0	0	**1**

Type F(4;2,2)

0	0		
0			
			1

critical set

□

Several authors have used direct product constructions to create larger latin squares from smaller ones, such that the critical-set property of the smaller set carries over to the bigger set. See Stinson and van Rees[13] for example. They in fact constructed critical sets from 2-critical sets. In the next few sections we show that F-squares of type $F(n; 2, 2 \ldots, 2)$ can be constructed from latin squares, such that the critical sets of the F-squares retain some of the properties of critical sets of the latin squares.

Theorem 10 *Let L be any latin square formed on any finite set, and let L' be a strong (see[1]) critical set of L. Let F be the F-square:*

L	L
L	L

Then F is an F-square of type $F(n; 2, 2, \ldots, 2)$, having critical set F':

L'	L'
L'	L'

Proof. Let (i, j) be a position in L' that can be uniquely filled, say, with element k, $0 \leq k \leq n-1$. Then each of the elements of the set $\{0, 1, \ldots, n-1\}$, except k, must occur at least once, in either row i or column j of L'. Hence they must also occur at least twice in either row i or column j, row i or column $j+n$, row $i+n$ or column j, and row $i+n$ or column $j+n$, of F'. That is positions $(i, j), (i, j+n), (i+n, j)$ and $(i+n, j+n)$ can be uniquely filled, with the element k. Fill these positions in L' and F', and label the new squares L'' and F'' respectively. Do similarly for L'' and F'' as we did for L' and F', etc. Since the original squares are finite, we will eventually completely fill every position, and get uniquely completed squares. Thus F is uniquely completable from F'.

Let $(i, j; k) \in F'$. Without loss of generality let $i = j = 0$. Consider the partial F-square $F\dagger = F' \setminus \{(0, 0; k)\}$, obtained by deleting $(0, 0; k)$ from F'. We need to show that F is not completable from F'. Consider the following partial F-square:

$L\dagger$	L
L	L

where $L\dagger$ is the result of deleting $(0, 0; k)$. We saw above that the information needed to complete the F-square is the same information needed to complete $L\dagger$. Since L' is a critical set, therefore L cannot be completable from *any* proper subset of L'. Thus L cannot be completable from $L\dagger$. Consequently F is not completable from $F\dagger$. Thus F has critical set F'.

□

Corollary 1 *For the F and L as in the above theorem, $cr(F) = 4.cr(L)$.*

Notation 1 *Given a $m \times m$ square or matrix M, we can obtain isomorphic matrices by permuting the rows and/or columns of M. The notation*

$$\begin{pmatrix} 0 & 1 & \ldots & m \\ \alpha_0 & \alpha_1 & \ldots & \alpha_m \end{pmatrix}$$

is meant to mean that row/column α_i replaces row/column i to form the new matrix. (As such we will often refer to this as the rule for obtaining a new matrix from M.) We will also denote by ρ_w the row permutation $\begin{pmatrix} 0 & 1 & \ldots & m \\ \alpha_w & \alpha_{w+1} & \ldots & \alpha_{w-1} \end{pmatrix}$. Then $\rho_w(L)$ is that matrix obtained from L by applying the rule ρ_w to its rows.

Lemma 2 *Let I denote the latin square*

0	1	2	...	n-2	n-1
1	2	3	...	n-1	0
2	3	4	...	0	1
...	...	...	...	...	...
n-2	n-1	0	...	n-4	n-3
n-1	0	1	...	n-3	n-2

That is, $I = \{(i, j; i+j) : 0 \leq i, j \leq n-1\}$, where addition is reduced modulo n. Then $\rho_w(I)$ is the symmetric latin square given by $L = (i, j; i+j+w)$ with addition reduced modulo n. By abuse of notation we may write $\rho_w(i, j; i+j) = (i, j; i+j+w)$ with addition reduced modulo n, to indicate the effect ρ_w has on the elements of L.

Proof. That $\rho_w(i, j; i+j) = (i, j; i+j+w)$ follows easily from the definition of ρ_w. Since $i+j+w = j+i+w$ for every i, j therefore $\rho_w(I)$ is symmetrical. □

Theorem 11 *Let n be a natural number, $n \geq 2$. Then the square E below:*

I	I
I	$\rho_1(I)$

is an F-square in the elements 0,1,...,n-1 of type $E = E(2n;2,2,\ldots,2)$, and is isotopic to the following F-square F:

0	0	1	1	2	2	...	n-1	n-1
0	1	1	2	2	3	...	n-1	0
1	1	2	2	3	3	...	0	0
1	2	2	3	3	4	...	0	1
...	...	...	...	...	...	...	...	...
n-1	0	0	1	1	2	...	n-2	n-1

which is given by $F = \{(i, j; \lfloor i+j)/2 \rfloor)\}$ where $\lfloor m/2 \rfloor$ is $m/2$ when m is even, and is $(m-1)/2$ when m is odd.

Proof. We will give a sketch of the proof as follows: Consider the F-square F in the theorem. Permute the *columns* using the following rule:

$$\begin{pmatrix} 0 & 1 & 2 & \ldots & n-1 & n & n+1\ldots & 2n-1 \\ 0 & 2 & 4 & \ldots & 2n-2 & 1 & 3\ldots & 2n-1 \end{pmatrix}.$$

We get the following F-square:

0	1	2	...	n-1	0	1	2	...	n-1
0	1	2	...	n-1	1	2	3	...	0
1	2	3	...	0	1	2	3	...	0
1	2	3	...	0	2	3	4	...	1
...	...	...	...	...	...	...	...	...	...
n-1	0	1	...	n-2	n-1	0	1	...	n-2
n-1	0	1	...	n-2	0	1	2	...	n-1

We now permute the *rows* with the same rule and we're done.

We will show in section 3 that $cr = \frac{n^2}{4}$.

□

Theorem 12 *Let I be of even order n. Let $2 \leq m \leq n-2$. Then for each of these m's the square F below:*

I	I
I	$\rho_{n-1}(I)$

has a critical set of size $cr = M(M+m)+(N-1)(2N-1)$, where $M = m+1$, and $N = n-m$. This F-square is also described by

$$F = \{(i,j;i+j \bmod n) : 0 \leq i \leq n-1, 0 \leq j \leq 2n-1\}$$

$$\cup\{(i,j;i+j \bmod n) : n \leq i \leq 2n-1, 0 \leq j \leq n-1\}$$

$$\cup\{(i,j;i+j-1 \bmod n) : n \leq i,j \leq 2n-1\}.$$

Proof. We prove two cases, (a) $m = n-2$ and (b) $2 \leq m < n-2$.
Case (a): When $m = n-2$ the general construction of the partial set F' is:

0	1	...	$n-3$			0	1	...	$n-3$		
1	...	$n-3$				1	...	$n-3$			
...	...					...	...				
$n-3$						$n-3$					
					$n-2$						$n-2$

0	1	...	$n-3$			$n-1$	0	1	...	$n-3$	
1	...	$n-3$				0	1	...	$n-3$		
...	...					1	...	$n-3$			
$n-3$						...	...				
						$n-3$					
					$n-2$						

We first remark that given any triple $(i,j;k)$ in the F-square the coordinates i and j belong to the set $\{0,1,\ldots,2n-1\}$, and addition in these coordinates is reduced modulo $2n$. On the other hand $k \in \{0,1,\ldots,n-1\}$ and addition in k is reduced modulo n. Because of symmetry what we prove for the r^{th} column will also be true for the r^{th} row. We further remark that each of the elements $0,1,\ldots,n-1$ occurs *twice* in each row and in each column in F.

We will prove (a) in two stages; (i) We will show that F' had unique completion (to F), and (ii) that any proper subset of F' cannot have any completion.

(i): Cells $(0,n-1),(n-1,0),(n-1,n)$ and $(n,n-1)$ can be uniquely filled. Also only element $n-1$ can fill each of these. Fill these. Now between them, row 1 and column $n-1$ contain each of the elements $1,\ldots,n-2$ and $n-1$ twice. Thus only element 0 can be placed in cell $(1,n-1)$. Similarly, it can be shown that element $i+j$ mod n can be placed in cell (i,j), for $0 \leq i \leq n-1$, and $0 \leq j \leq 2n-1$, as well as in cells (i,j) for $n \leq i \leq 2n-1$ and $0 \leq j \leq n-1$.

Element $i+j-1$ mod n can be placed in cell (i,j), for $n \leq i,j \leq 2n-1$. That is, F' is uniquely completable.

(ii): Delete entry $(0,0;0)$ from F'. It is easy to check that both row n and column n can be uniquely filled. Consider element $n-3$. We want to show that $n-3$ can uniquely fill cells $(n-2,n-2)$ and $(2n-2,n-2)$. Now $n-3$ is contained twice each in rows $0,\ldots,n-3$ and twice each in rows $n,\ldots,2n-3$. The non-empty cells in column $n-1$ are $(n-1,n-1),(n,n-1)$ and $(2n-1,n-1)$. Thus

$$\cup i = \{0,\ldots,n-3,n,\ldots,2n-3,n-1,n-3\}$$

so that

$$[2n]/\cup i = \{n-2,2n-2\}.$$

Also

$$[2n]/\cup j = \{n-2,n-2\}.$$

That is, $k=n-3$ can uniquely fill cells $(n-2,n-1)$ and $(2n-2,n-1)$. Continuing in this way, the following partial F-square can be obtained:

	1	...	$n-3$			0	1	...	$n-3$		
1	...	$n-3$				1	...	$n-3$			
...	...				1	...	...				1
$n-3$				...	...	$n-3$				...	...
			1	...	$n-3$	$n-2$			1	...	$n-3$
		1	...	$n-3$	$n-2$	$n-1$		1	...	$n-3$	$n-2$

0	1	...	$n-3$	$n-2$	$n-1$	$n-1$	0	1	...	$n-3$	$n-2$
1	...	$n-3$				0	1	...	$n-3$		
...	...				1	1	...	$n-3$			
$n-3$				...	...	...	...				1
			1	...	$n-3$	$n-3$				...	...
		1	...	$n-3$	$n-2$				1	...	$n-3$

We will show now that no other cell can be filled uniquely. We note first that replacing cell $(0,0)$ with any element other than the element 0 does not lead to any new useful information, since that element will either have already occured twice in row 0 or column 0, or occurs once only. Of the empty cells, cells $(0,n-1),(0,2n-1),(n-1,0),(n-1,n+1),(n+1,n-1),(n+1,2n-1),(2n-1,0),(2n-1,n+1)$ have most information pertaining to them, and thus are the most likely to be able to be filled. But on inspection, none of these cells satisfy the criteria for unique completion. For example, consider cell $(0,n-1)$. Row 0 contains each of the elements $1,\ldots,n-3$ twice each,and element 0 once. Column $n-1$ contains each of the elements $1,\ldots,n-2$ twice each, and element $n-1$ once. The union of these sets is the multiset $U=\{0,0,\ldots,n-2,n-2,n-1\}$, giving $[2n]/U=\{0,n-1\}$, and thus cell $(0,n-1)$ cannot be filled by any element. Similarly none of the other empty cells given above can be filled. Throughout the whole partial F-square, elements $1,\ldots n-3$ each occurs twice in each row and in each column. Element $n-2$ occurs twice each in rows $n-1$ and n, and columns $n-1$ and n. Row(column) 0 has only three empty cells. Using the criteria we can show that neither element 0 nor $n-1$ can be forced uniquely into any of these three cells. Similarly, neither element 0 nor $n-1$ can be forced into any empty cell in row $2n-1$ or column $2n-1$. Any other row or column is either filled completely, or has less filled entries and therefore less likelyhood of being filled uniquely by any of the elements $0,n-1$ or $n-2$. That is, the partial F-square cannot be uniquely filled.

We observe that removing the entry $(0,0;0)$ is no worse than replacing 0 by k. This is because we need the replacement element to occur twice in the respective row and column in order for it to be of any use in the completion process. As long as there is at most one occurance of an element in the union of row i and column j in F' we cannot complete to a full F-square. Thus the position from which an entry is removed is immaterial. From this observation it follows that the F' is a critical set.

Since $|I| = \frac{1}{2}(n^2 - 3n + 4)$ and $|A| = \frac{1}{2}(n^2 - n)$ therefore the size of the partial F-square is $2n^2 - 5n + 6 = (n-1)(2n-3) + 3$.

(b): The partial set F' is represented as follows:

0	1	...	$m-1$						0	1	...	$m-1$						
1	...	$m-1$							1	...	$m-1$							
...	$m-1$								...	$m-1$								
$m-1$									$m-1$									
								m										m
								...										...
						m	...	$n-3$								$m-1$	...	$n-3$
					m	...	$n-3$	$n-2$							m	...	$n-3$	$n-2$
0	1	...	$m-1$						$n-1$	0	1	...	$m-1$					
1	...	$m-1$							0	1	...	$m-1$						
...									1	...	$m-1$							
$m-1$									...									
									$m-1$									
								m										
								...										m
						m	...	$n-3$									...	...
					m	...	$n-3$	$n-2$								m	...	$n-3$

For ease of notation we will denote row i by R_i and column j by C_j. Consider column $n-1$. We will show that we can uniquely fill column $n-1$. Now $R_0 = \{0, 0, \ldots, m-1, m-1\}$ and $C_{n-1} = \{m, m, \ldots, n-2, n-2\}$. Thus $[2n]/\{R_0 \cup C_{n-1} = [2n]/\{0, 0, \ldots, n-2, n-2\} = \{n-1, n-1\}$ and so element $n-1$ can uniquely fill cell $(0, n-1)$. Also, $R_n = \{0, 0, \ldots, m-1, m-1\}$ and so $[2n]/R_n \cup C_{n-1} = \{n-1, n-1\}$, and so cell $(n, n-1)$ can also be uniquely filled with element $n-1$. Fill these cells. $R_2 = \{1, 1, \ldots, m-1, m-1\}$ and $C_{n-1} = \{m, m, \ldots, n-1, n-1\}$, so cell can be filled with element 0. Similarly, cell $(n+1, n+1)$ can also be filled by element 0. Continuing in this way, we can eventually uniquely fill all of column $n-2$.

So far, $R_n = \{0, 0, \ldots, m-1, m-1, n-1, n-1\}$ and $C_{n-1} = \{m, m, \ldots, n-3, n-3\}$, and so cell $(n, n-1)$ can be uniquely filled with element $n-2$. Similarly, cell $(n, 2n-1)$ can be filled with element $n-2$. Continuing in this way, we can uniquely fill row n.

Having filled the above column and row, row $2n-1$ can now be filled, then column $n-2$, row $n+1$, column $2n-2$, etc. That is, the partial F-square F' is uniquely completable to the $F - square$.

We need now to show that no subset of F' is completable. Suppose $(0,0;0)$ is deleted from F'. That is $R_0 = \{1, 1, \ldots, m-1, m-1\} = C_0$. Consider the $(n-1)^{th}$ column. To fill say cell $(0, n-1)$ we need to have $[2n]/R_0 \cup C_{n-1} = \{k, k\}$ for some $k \in [n]$. As above we can still fill row n. We can also use the criteria for unique completion to fill cells $(2, n-1), \ldots, (m, n-1)$ with elements $1, \ldots, m-1$, respectively. Similarly, cells $(n+2 \ mod \ 2n, n-1), \ldots, (n+m \ mod \ 2n, n-1)$

can be filled with elements $1, \ldots, m-1$ respectively. Thus $C_{n-1} = \{1, 1, \ldots, n-2\}$, giving $[2n]/R_0 \cup C_{n-1} = \{0, 0, n-1, n-1\}$. That is, $(0, n-1)$ cannot be uniquely filled. Similarly, no other cell in column $n-1$ can be filled uniquely. That is, $F'/(0,0;0)$ is not completable. Again, it is not so much the position of the deleted entry that matters, as the information that the deleted entry does not provide. That is, F' is a critical set.

On inspection, the size of the partial set is

$$cr(F') = 3[(1/2)(m(m+1)+(n-m-1)(n-m)]+(1/2)[(m+1)(m+2)+(n-m-2)(n-m-1)]$$

$$= (1/2)[(m+1)(3m+m+2)+(n-m-1)(3n-3m+n-m-2)]$$

$$= (1/2)[2(m+1)(2m+1)+2(n-m-1)(2n-2n-1)]$$

$$= [(m+1)(2m+1)+(n-m-1)(2(n-m)-1)].$$

Putting $M = m+1$ and $N = n-m$, we get

$$cr(F') = M(M+m)+(N-1)(2N-1)$$

as required.

4.1 A general construction for type F = F(n;2,2,. . . ,2)

We give a general construction for F-squares, of type $F(n; 2, 2, \ldots, 2)$. As above let S be a finite set, say $S = \{0, 1, \ldots, m-1\}$. Let $\Pi = \{C_1, C_2, \ldots, C_m\}$ be any ordered collection of m subsets of S, each of size 2, such that *each* element $k \in S$ occurs in precisely two sets in Π. Now form the collection $\{L_1, L_2, \ldots, L_m\}$ of 2×2 latin squares, where L_i is formed from the elements of the set $C_i, 1 \leq i \leq m$. *Then a latin square (or latin structure) in the symbols $L_1, L_2, \ldots, L_m$ respectively is also an F-square of type $F(n; 2, 2, \ldots, 2)$, in the elements 0,1,. . . ,m-1.*

Example 4 Let S be as given above, and let $\Pi = \{C_1, C_2, \ldots, C_m\}$ be the collection

$$\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \ldots, \{m-1, m\}, \{m, 1\}\}.$$

Now form the latin squares:

0	1
1	0

1	2
2	1

2	3
3	2

$\ldots$

m-2	m-1
m-1	m-2

m-1	0
0	m-1

Denote these by $L_1, L_2, \ldots, L_m$. Then the latin square F below, in the elements $L_1, L_2, \ldots, L_m$ is also an F-structure in the elements $0, 1, \ldots, m-1$.

L_1	L_2	L_3	$\ldots$	L_{m-1}	L_m
L_2	L_3	L_4	$\ldots$	L_m	L_1
L_3	L_4	L_5	$\ldots$	L_1	L_2
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
L_m	L_1	L_2	$\ldots$	L_{m-2}	L_{m-1}

Theorem 13 *The above F-structure is isomorphic to the F-structure :*

I	I
I	$\rho_2(I)$

Proof. Permute the rows of the F-square according to the rule:

$$\begin{pmatrix} 0 & 1 & 2 & \dots & n-1 & n & n+1 & n+2 & \dots & 2n-1 \\ 0 & 2 & 4 & \dots & 2n-2 & 2n-1 & 1 & 3 & \dots & 2n-3 \end{pmatrix}.$$

Then permute the columns using the same rule. □

Theorem 14 *Let $n = 2m, m \geq 2$. Let I be of order n. Then the F-square above has a critical set of size $7m^2 - 2$. When I is of order $n+1$, then there is a critical set of of size $7m^2 + 7m - 1$.*

Proof.

0	1	...	$m-1$					0	1	...	$m-1$				
1	...	$m-1$						1	...	$m-1$					
...	$m-1$							...	$m-1$						
$m-1$								$m-1$							
							m								m
						...	...							...	...
					m	...	$n-2$						m	...	$n-2$
0	1	...	$m-1$					2		4	5	...	...	0	1
1	...	$m-1$							4	5	6	...	...	1	2
...	$m-1$							4	5	6	7	...	...	2	3
$m-1$								...	...	...	...	...	...	...	...
								...	...	...	...	...	...	...	...
							m	$n-1$	0	1	2	...	...	$n-3$	$n-2$
						...	...	0	1	2	3	...	...	$n-2$	$n-1$
					m	...	$n-2$	1	2	3	4	...	...	$n-1$	0

We will show that we can complete row n and column n. $R_n = \{0, 0, \dots, m-1, m-1\}$, and $C_{n-1} = \{m, m, \dots, n-2\}$. Thus element $n-1$ can uniquely fill cell $(n, n-1)$. Having filled this cell, we now have $R_n = \{0, 0, \dots, m-1, m-1, n-1, n-1\}$, and $C_{n-1} = \{m, m, \dots, n-3\}$. That is, cell $(n, n-2)$ can be uniquely filled by element $n-2$. Continuing in this way, we can uniquely complete row n. Similarly, column n can also be uniquely completed. That is, we have completed the lower right sub-square. Clearly the rest of the partial F-square can also be uniquely completed, since the partial latin square is completable.

Delete the entry $(0, 0; 0)$. Then row n and column n can be uniquely completed still, thus the lower right subsquare can be filled. Also some sub diagonals can be filled in the top let, top right and bottom right partial sub-squares. However these entries do not include either the elements 0 or $n-1$ which are lacking, so they do not provide any more information than if they were not filled. Similar sort of argument can be made if any element was deleted from the top left and right, and bottom left subsquares.

Suppose now that entry $(n, n+1; 4)$ was deleted. Then we can fill cells $(n, m+1), \dots, (n, n-1)$ with elements $m+1, \dots, n-1$ respectively, giving $R_n = \{0, 0, \dots, m-1, m-1, m+2, m+2, \dots, n-1, n-1\}$. Thus we cannot fill, for example, any cell in column $n+1$, since the elements occuring twice in this column already exist twice in row n. Again, deleting another element in this subsquare will lead to a non-completion for the partial F-square. That is, F' is a critical set.

The size of the critical set is

$$cr(F) = (3/2)(m(m+1) + (m-1)(m)) + n^2 - 2$$

$$= 7m^2 - 2$$

For $n = 2m + 1$ then

$$cr(F) = (3/2)(m(m+1) + m(m+1)) + n^2 - 2$$

$$= 3m^2 + 3m + 4m^2 + 4m - 1$$

$$= 7m^2 + 7m - 1.$$

□

5 A general construction for $F = F(n; \alpha_1, \alpha_2, \ldots, \alpha_v)$.

Theorem 15 *Let $n = r + s$ where r and s are non-zero, positive integers; $(n-3)/2 \leq r \leq n-2$, $r = \alpha_1 + \alpha_2 + \ldots + \alpha_u$, and $s = \alpha_{u+1} + \alpha_{u+2} + \ldots + \alpha_v$. Then there is an F-square of type $F(n; \alpha_1, \alpha_2, \ldots, \alpha_v)$ with critical set of size $cr_1(F) = \frac{1}{2}((r-1)r + s(s+1))$.*

Proof.

Consider the F-square $F = F(n; 1, 1, \ldots, 1)$ below:

0	1	2	...	n-1
1	2	3	...	0
2	3	4	...	1
...	...	...	...	...
n-1	0	1	...	n-2

In the latin square, replace the first α_1 elements of the ordered set $\{0, 1, \ldots, n-1\}$ by the element 0, the next α_2 elements by the element 1, etc. Then we get a new F-square of type $F(n; \alpha_1, \alpha_2, \ldots, \alpha_v)$.

If $n = r + s$ then we construct the partial F-square F' as follows: Delete every element in the F-square except the first r diagonals going from upper right to lower left, and the last $s-1$ diagonals going from the upper right to the lower left. See the examples below with $n = 8$.

If $\alpha_i = 1$ for every i then F is a latin square, and thus F' is a critical set ([7]). Suppose F is not a latin square. Consider cell $(0, n-1)$. The union of R_0 and C_{n-1} is the multiset containing each of the elements $i \in \{0, \ldots, s-1\}$ α_i times, and each of the elements $j \in \{s, \ldots, v-2\}$ α_j times. That is, the only element that does occur a sufficient number of times is $v-1$, which occurs only $\alpha_{v-1} - 1$ times. Thus cell $(0, n-1)$ can be uniquely filled by element $v-1$. Fill this. Row two so far contains element 0, $\alpha_0 - 1$, and each of the elements $i \in \{1, \ldots, s\}, \alpha_i$ times. Column $n-1$ now contains each of the elements $j \in \{s, \ldots, v-1\}$ α_j times. Thus element 0 can uniquely fill cell $(1, n-1)$. Continuing this way, column $n-1$ can be entirely filled. Similarly, column $(n-2)$, etc, until the entire partial square is filled.

Delete the entry $(0, 0; 0)$ from the partial F-square. Then modifying the arguments we made for F-squares of type $F(n; 2, \ldots, 2)$, we can show that no entry in row 0 can ever be filled, and similarly for column 0.

Thus we have a critical set. The critical set consists of two triangles, the upper triangle having size $\frac{1}{2}m(m-1)$, and the lower triangle has size $\frac{1}{2}(n-m)(n-m+1)$. Thus on substituting, we get : $cr(F) = \frac{1}{2}((s-1)s + r(r+1))$.

□

Example 5 Let $n = 8$. Now $8 = 1 + 2 + 2 + 3$. (There are of course other ways of writing 8 as a sum). Further, we can write (i) $8 = (1 + 2 + 2) + 3$ or (ii) $8 = (1 + 3) + (2 + 2)$. The F-squares corresponding to these two cases are as follows:

$Case(i) : 8 = (1 + 2 + 2) + 3$

1	**2**	**2**	**3**	**3**	4	4	4
2	**2**	**3**	**3**	4	4	4	1
2	**3**	**3**	4	4	4	1	2
3	**3**	4	4	4	1	2	2
3	4	4	4	1	2	2	3
4	4	4	1	2	2	3	3
4	4	1	2	2	3	3	**4**
4	1	2	2	3	3	**4**	**4**

Type F(8;1,2,2,3)

1	2	2	3	3				18
2	2	3	3					
2	3	3						
3	3							
3								
							4	
						4	4	64

critical set

$Case(ii) : 8 = (1 + 3) + (2 + 2)$

1	**2**	**2**	**2**	3	3	4	4
2	**2**	**2**	3	3	4	4	1
2	**2**	3	3	4	4	1	2
2	3	3	4	4	1	2	2
3	3	4	4	1	2	2	2
3	4	4	1	2	2	2	**3**
4	4	1	2	2	2	**3**	**3**
4	1	2	2	2	**3**	**3**	**4**

Type F(8;1,2,2,3)

1	2	2	2					16
2	2	2						
2	2							
2								
							3	
						3	3	
					3	3	4	64

critical set

Corollary 2 *If n is even and $r = s$, then $cr(F) = \frac{n^2}{4}$. If n is odd and $s = r + 1$, then $cr(F) = \frac{n^2-1}{4}$.*

Proof. If n is even and $r = s$, then $cr(F) = \frac{1}{2}(r(r-1) + r(r+1)) = \frac{n^2}{4}$. If n is odd and $s = r + 1$, then $cr(F) = \frac{1}{2}(r(r+1) + r(r+1)) = r^2 + r = (\frac{n-1}{2})^2 + \frac{n-1}{2} = \frac{n^2-1}{4}$. □

References

[1] J.Cooper, D.Donovan and J.Seberry. Latin squares and critical sets of minimal size. Aust. J. Combinatorics, 4, (1991), 113-120.

[2] J.Cooper, D.Donovan and J.Seberry. *Secret sharing schemes arising from Latin squares*, Bulletin of the Institute of Combinatorics and its Applications, September, (1994), 33-43.

[3] J. Cooper, D. Donovan and R. Gower, *Critical sets in direct products of back circulant squares*, Uti. Math., 50 (1996)127-162.

[4] C.J.Colbourn and J.H.Dinitz, Latin squares, The CRC Handbook of Combinatorial designs, CRC Press, Boca Raton, (1996), 97-110.

[5] D.Curran and G.H.J.Van Rees. *Critical sets in Latin squares,* in Proc. Eighth Manitoba Conference on Numer. Math. and Computing, (1978), 165-168.

[6] D.Donovan, J.Cooper, D.J.Nott and J.Seberry, *Latin squares: critical sets and their lower bounds*, Ars Combinatoria, 39, (1995), 33-48.

[7] D.Donovan and Joan Cooper, *Critical sets in back circulant latin squares*, Aequationes Mathematicae 52 (1996) 157-179.

[8] R.A.H.Gower, *Critical Sets in Latin Squares and their Applications to Minimal Defining Sets of Designs*, PhD Thesis, University of Queensland, Australia, July 1995.

[9] Adelle Howse. *Minimal Critical Sets For Some Small Latin Squares*, Australasian Journal of Combinatorics, 17(1988), pp.275-288.

[10] A.Khodkar, *On smallest critical sets for the elementary abelian 2-group,* (manuscript).

[11] J.Nelder. *Critical sets in Latin squares*, CSIRO Div. of Math. and Stats, Newsletter, 38, (1977).

[12] B.Smetaniuk. *On the minimal critical set of a Latin square*, Utili. Math., 16, (1979), 97-100.

[13] D.R.Stinson and G.H.J. Van Rees. *Some large critical sets*, Congr. Numer., 34, (1982), 441-456.

[14] D.R.Stinson and S.A.Vanstone, *A combinatorial approach to threshold schemes*, SIAM Journal of Discrete Mathematics, 1, (1988), 230–236.

[15] A.P.Street. *Defining sets for t-designs and critical sets for Latin squares*, New Zealand Journal of Maths., 21, (1992), 133-144.

[16] S V M Raju Peddada, Jennifer Seberry and Ghulam R Chaudhry. *Bounds on the Size of Near Critical Sets in Latin Squares with Subsquares of Order Two*, (manuscript).

[17] S V M Raju Peddada, Jennifer Seberry and Ghulam R Chaudhry. *Near Critical Sets in Latin Squares with Maximum Numbers of Subsquares*, (manuscript).