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INFINITE FAMILIES OF GENERALIZED BHASKAR RAO DESIGNS

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We show that $GBRD(p, \frac{1}{2}(p-1), \frac{1}{8}(p-1)(p-3); EA(\frac{1}{2}(p-1))$ exist for all prime powers $p \equiv 3 \pmod 4$. We also show that $GBRD(p, \frac{1}{2}(p-1), \frac{1}{4}(p-1)(p-3); EA(\frac{1}{2}(p-1))$ exist for all prime powers $p \equiv 1 \pmod 4$.

This allows us to give a new proof that a

$$BIBD(f(ef+1), (ef+1)(ef^2+f-1), ef+f-1, f, f-1)$$

exists whenever $p = ef + 1$ is a prime power.

This gives many new GBRDs including a $GBRD(19, 9, 36; EA(9))$, a $GBRD(13, 6, 30; Z_6)$ and a $GBRD(17, 8, 56; EA(8))$.

1 Introduction

Let $G = \{h_1 = e, h_2, \ldots, h_g\}$ be a finite group of order g with identity e. Form the matrix W

$$W = h_1 A_1 + \ldots + h_g A_g$$

where $A_1, \ldots, A_g$ are $v \times b$ (0,1)-matrices such that the Hadamard product $A_k * A_j = 0$ for any $k \neq j$. Let

$$W^+ = (h_1^{-1} A_1 + \ldots + h_g^{-1} A_g)^T$$

and

$$N = A_1 + A_2 + \ldots + A_g.$$

Then we say W is a *generalized Bhaskar Rao design* over G denoted by $GBRD(v,b,r,k,\lambda;G)$, or in abbreviated form $GBRD(v,k,\lambda;G)$, if N satisfies

$$NN^T = (r-\lambda)I + \lambda J :$$

that is, N is the incidence matrix of the $BIBD(v,k,\lambda)$ and

$$WW^+ = reI + (\lambda/g)(h_1 + \ldots + h_g)(J-I).$$

We say that the design W is based on the matrix N or that N is signed over the group G.

$GBRD$ have been studied by the author and others, for example see [3, 4, 5, 6, 7, 10, 11]. We now illustrate the construction of the $GBRD$ with some examples. For further examples the reader is referred to [9].

To check whether a set of blocks $(x_a^i,\ y_b^i,\ z_c^i)$ mod (v,g), each with k elements, are part of a $GBRD$ or a $BIBD$ we check all the differences: $(y-x)^i_{ba^{-1}}$, $(x-y)^i_{ab^{-1}}$, $(z-x)^i_{ca^{-1}}$, $(x-z)^i_{ac^{-1}}$, $(z-y)^i_{cb^{-1}}$ and $(y-z)^i_{bc^{-1}}$, where the subscripts such as ab^{-1} are from a group G of order $|G|=g$ and the subscripted elements such as $(x-y)^i$ come from a group V of order v.

If each non-zero-zero difference, that is differences other than 0_0, including the differences $0_1, \cdots, 0_{g-1}$ occurs the same number of times, λ, we will have a $BIBD(v,k,\lambda)$. If each non-zero difference, that is not including the differences $0_0,\ 0_1,\ \cdots,\ 0_{g-1}$, occurs the same number of times, λ, we will have a $GBRD(v,k,\lambda;G)$. In this case we say the initial sets form a $GBRDSDS(v,k,\lambda;G)$ or *generalized Bhaskar Rao difference sets.*

Example 1 Consider the initial blocks

$$\begin{array}{ccccc} (1_1,\ 6_1,\ 0_2) & (2_1,\ 5_1,\ 0_2) & (3_1,\ 4_1,\ 0_2) & (1_2,\ 6_2,\ 0_3) & (2_2,\ 5_2,\ 0_3) \\ (3_2,\ 4_2,\ 0_3) & (1_3,\ 6_3,\ 0_1) & (2_3,\ 5_3,\ 0_1) & (3_3,\ 4_3,\ 0_1) & (0_1,\ 0_2,\ 0_3) \end{array}$$

mod $(7,3)$ which Bose gives as the initial blocks of a $BIBD(21,3,1)$. Without the last block of $(0_1,\ 0_2,\ 0_3)$ we have a $GBRD(7,3,9;Z_3)$.

As we have each non-zero difference occurring twice we have a $GBRD(6,3,4;Z_2)$. Its incidence matrix is

0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
·	·	·	·	·	·	·	·	·	·	1	1	1	1	1	1	1	1	1	1
0	1	·	·	0	0	1	·	·	0	0	·	0	·	·	1	·	1	·	·
0	0	1	·	·	0	0	1	·	·	·	0	·	0	·	·	1	·	1	·
·	0	0	1	·	·	0	0	1	·	·	·	0	·	0	·	·	1	·	1
·	·	0	0	1	·	·	0	0	1	0	·	·	0	·	1	·	·	1	·
1	·	·	0	0	1	·	·	0	0	·	0	·	·	0	·	1	·	·	1

where 0, 1 are elements of the group, Z_2, and $\cdot$ means the zero of the group ring so $\cdot - 1 = \cdot$, and $\cdot - 2 = \cdot$.

To change these sets into starting blocks for a $BIBD$ we would also need to have the difference 0_1 occur twice. This can be done but not in a straight forward way.

This clearly shows that Bhaskar Rao designs are not another representation of $BIBD$s. □

Example 2 To illustrate how a design developed from blocks using differences can be shown to be a $BIBD$ or $GBRD$ we consider the initial blocks

$$(0_0,1_0,2_0)(0_0,1_1,2_1)(0_1,2_0,3_1)(0_0,2_1,4_1)(0_0,2_0,0_1)(0_0,0_1,4_1)\mathrm{mod}(5,-)$$

So the differences are

$$\begin{array}{cccccc}
1_0 & 2_0 & 1_0 & 4_0 & 3_0 & 4_0 \\
1_0 & 1_1 & 2_1 & 4_0 & 4_1 & 3_1 \\
1_1 & 2_1 & 3_0 & 4_1 & 3_1 & 2_0 \\
2_1 & 4_1 & 2_0 & 3_1 & 1_1 & 3_0 \\
2_0 & 0_1 & 3_1 & 3_0 & 0_1 & 2_1 \\
4_0 & 0_1 & 4_1 & 1_0 & 0_1 & 1_1
\end{array}$$

This means each of the differences $0_1, 1_0, 1_1, 2_0, 2_1, 3_0, 3_1, 4_0, 4_1$ occurs 4 times, that is 2 times corresponding to each difference where the direction it is taken in is considered. Hence we have, after developing the initial blocks modulo 5, exactly Bose's example (iv) [2, p370] for a $BIBD(10, 3, 2)$.

1	1	1	0	0	1	0	0	0	0	0	0	1	0	0
0	1	1	1	0	0	1	0	0	0	0	0	0	1	0
0	0	1	1	1	0	0	1	0	0	0	0	0	0	1
1	0	0	1	1	0	0	0	1	0	1	0	0	0	0
1	1	0	0	1	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	1	1	0	0	1	0	0	1	0
0	0	0	0	0	0	0	1	1	0	0	1	0	0	1
0	0	0	0	0	0	0	0	1	1	1	0	1	0	0
0	0	0	0	0	1	0	0	0	1	0	1	0	1	0
0	0	0	0	0	1	1	0	0	0	0	0	1	0	1

1	0	0	0	0	1	0	1	0	0	1	0	0	0	0
0	1	0	0	0	0	1	0	1	0	0	1	0	0	0
0	0	1	0	0	0	0	1	0	1	0	0	1	0	0
0	0	0	1	0	1	0	0	1	0	0	0	0	1	0
0	0	0	0	1	0	1	0	0	1	0	0	0	0	1
0	0	1	0	1	1	0	0	0	0	1	0	0	0	1
1	0	0	1	0	0	1	0	0	0	1	1	0	0	0
0	1	0	0	1	0	0	1	0	0	0	1	1	0	0
1	0	1	0	0	0	0	0	1	0	0	0	1	1	0
0	1	0	1	0	0	0	0	0	1	0	0	0	1	1

It has GBRD type incidence matrix for the last three sets of

0		1	.	1	01	.	0	.	.	01	.	.	.	1
1	0	.	1	.	.	01	.	0	.	1	01	.	.	.
.	1	0	.	1	.	.	01	.	0	.	1	01	.	.
1	.	1	0	.	0	.	.	01	.	.	.	1	01	.
.	1	.	1	0	.	0	.	.	01	.	.	.	1	01

It is not a $GBRD$ as the difference 0_1 occurs 4 times. This is represented by the "01" in the above incidence matrix. However the first four blocks give a $GBRD(6,3,6;Z_2)$

$$(0_0,\ 1_0,\ 2_0)\ (0_0,\ 1_1,\ 2_1)\ (0_1,\ 2_0,\ 3_1)\ (0_0,\ 2_1,\ 4_1)\ \bmod(5,Z_2),$$

which has incidence matrix:

$$\begin{array}{ccccc@{\quad}ccccc@{\quad}ccccc@{\quad}ccccc}
0&0&0&.&. & 0&1&1&.&. & 0&.&0&1&. & 0&.&1&.&1\\
.&0&0&0&. & .&0&1&1&. & .&0&.&0&1 & 1&0&.&1&.\\
.&.&0&0&0 & .&.&0&1&1 & 1&.&0&.&0 & .&1&0&.&1\\
0&.&.&0&0 & 1&.&.&0&1 & 0&1&.&0&. & 1&.&1&0&.\\
0&0&.&.&0 & 1&1&.&.&0 & .&0&1&.&0 & .&1&.&1&0
\end{array}$$

2 Motivation and Results

Example 3 We note that $(1_1, 2_1, 4_1)$, $(1_1, 2_w, 4_{w^2})$ and $(1_1, 2_{w^2}, 4_w)$ are $3-\{7;3;1;Z_3\}$ $GBRSDS$ as observed in Seberry [8]. So that the incidence matrices formed from these sets form a $GBRD(7,21,9,3,3;Z_3)$, where $Z_3 = \{1,w,w^2;w^3=1\}$:

$$K = [A_1|A_2|A_3]$$

where

$$A_1 = \begin{bmatrix} 0&1&1&0&1&0&0\\ 0&0&1&1&0&1&0\\ 0&0&0&1&1&0&1\\ 1&0&0&0&1&1&0\\ 0&1&0&0&0&1&1\\ 1&0&1&0&0&0&1\\ 1&1&0&1&0&0&0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0&1&w&0&w^2&0&0\\ 0&0&1&w&0&w^2&0\\ 0&0&0&1&w&0&w^2\\ w^2&0&0&0&1&w&0\\ 0&w^2&0&0&0&1&w\\ w&0&w^2&0&0&0&1\\ 1&w&0&w^2&0&0&0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0&1&w^2&0&w&0&0\\ 0&0&1&w^2&0&w&0\\ 0&0&0&1&w^2&0&w\\ w&0&0&0&1&w^2&0\\ 0&w&0&0&0&1&w^2\\ w^2&0&w&0&0&0&1\\ 1&w^2&0&w&0&0&0 \end{bmatrix}$$

We now observe that 3 is a generator of $GF(7)$ and that $3^2 = 2$, $3^4 = 4$ and $3^6 = 1$ as the $GBRSDS$ are obtained by choosing $\{3_1^{2i} : i = 1,2,3\}$, $\{3_{w^i}^{2i} : i = 1,2,3\}$ and $\{3_{w^{-i}}^{2i} : i = 1,2,3\}$. This fact can be generalized.

Theorem 1 *Let p be a prime power with the generator y for Z_p. Let g be the generator of a group G, with $g^{\frac{1}{2}(p-1)} = 1$. Write $\frac{1}{2}(p-1) = n$.*
Let $f(x) = \sum_{i=1}^{n} g^i x^{y^{2i}}$ and $h(x) = \sum_{i=1}^{n} g^{-i} x^{y^{2i+1}}$.
Define

$$f(x^*) = \sum_{i=1}^{n} g^{n-i} x^{p-y^{2i}} \quad and \quad h(x^*) = \sum_{i=1}^{n} g^{n+i} x^{p-y^{2i+1}}.$$

Then

$$f(x)f(x^*) + h(x)h(x^*) = (p-1) + \sum_{s=1}^{2n} (G\{0\}) x^s.$$

Example 4 Let $p = 13$ and $y = 2$ be the generator of Z_{13}. Let g be the generator of G where $g^6 = 1$.

$$\begin{aligned}
f(x) &= gx^4 + g^2x^3 + g^3x^{12} + g^4x^4 + g^5x^{10} + x \\
f(x^*) &= g^5x^9 + g^4x^{10} + g^3x^1 + g^2x^4 + gx^3 + x^{12} \\
f(x)f(x^*) &= 6 + (g + g^5)(x + x^3 + x^4 + x^9 + x^{10} + x^{12}) \\
&\quad + (g^2 + g^3 + g^4)(x^2 + x^5 + x^6 + x^7 + x^8 + x^{11}) \\
h(x) &= g^5x^8 + g^4x^6 + g^3x^{11} + g^2x^5 + gx^7 + x^2 \\
h(x^*) &= gx^5 + g^2x^7 + g^3x^2 + g^4x^8 + g^5x^6 + x^{11} \\
h(x)h(x^*) &= 6 + (g^2 + g^3 + g^4)(x + x^3 + x^4 + x^9 + x^{10} + x^{12}) \\
&\quad + (g + g^5)(x^2 + x^5 + x^6 + x^7 + x^8 + x^{11})
\end{aligned}$$

Hence

$$f(x)f(x^*) + h(x)h(x^*) = 12 + G\backslash\{0\} \sum_{i=1}^{12} x^i.$$

Proof.

$$\begin{aligned}
f(x)f(x^*) + h(x)h(x^*) &= \sum_{i=1}^{n} g^i x^{y^{2i}} \sum_{j=1}^{n} g^{n-j} x^{p-y^{2j}} \\
&\quad + \sum_{i=1}^{n} g^{-i} x^{y^{2i+1}} \sum_{j=1}^{n} g^{n+j} x^{p-y^{2j+1}} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} (g^{i-j} x^{y^{2i}-y^{2j}} + g^{j-i} x^{y^{2i+1}-y^{2j+1}}) \\
&= 2n + G\backslash\{0\} \sum_{i=1}^{p-1} x^i.
\end{aligned}$$

□

Example 5 Since there exists an $SBIBD(13,4,1)$, Q, with F and H the $GBRDSDS$ generated by the theorem, we have $3-\{13,4,6,6,1,Z_3\}$ $GBRDSDS$.

Now if $p \equiv 3 \pmod 4$ is a prime power then there exists a $(p, \frac{1}{2}(p-1), \frac{1}{4}(p-3))$ difference set and thus this difference set with $\frac{1}{4}(p-3)$ copies of each of the sets $X = \{g^i x^{y^{2i}} : i = 1, \ldots, \frac{1}{2}(p-1)\}$ and $Y = \{g^{-i} x^{y^{2i+1}} : i = 1, \ldots, \frac{1}{2}(p-1)\}$ gives $\frac{1}{2}(p-1) - \{p; \frac{1}{2}(p-1); \frac{1}{8}(p-1)(p-3)\}$ mod $(p, Z_{\frac{1}{2}(p-1)})$ $GBRSDS$ and hence

Theorem 2 *Let $p \equiv 3 \pmod 4$ be a prime power then there exists a*

$$GBRD(p, \frac{1}{2}(p-1), \frac{1}{8}(p-1)(p-3); Z_{\frac{1}{2}(p-1)}).$$

Example 6 There exist

$$GBRD(7,3,3;Z_3)$$

$$GBRD(11,5,10;Z_5)$$

$$GBRD(19,9,36;EA(9))$$

Similarly when $p \equiv 1 \pmod 4$ is a prime power then there exists $2-(p, \frac{1}{2}(p-1), \frac{1}{2}(p-3))$ supplementary difference set with $\frac{1}{2}(p-3)$ copies of each of the sets X and Y we have $(p-1) - \{p; \frac{1}{2}(p-1); \frac{1}{4}(p-1)(p-3)\}$ $\mathrm{mod}(p, EA(\frac{1}{2}(p-1))$ $GBRSDS$ and hence

Theorem 3 *Let $p \equiv 1 \pmod 4$ be a prime power then exists a*

$$GBRD(p, \frac{1}{2}(p-1), \frac{1}{2}(p-1), \frac{1}{4}(p-1)(p-3); EA(\frac{1}{2}(p-1))).$$

Example 7 There exists

$$GBRD(9,4,12;EA(4))$$

$$GBRD(13,6,30;EA(6))$$

$$GBRD(17,8,56;EA(8)).$$

But from the example using $p = 13$ and $y = 2$ as the generator of Z_{13} we see that we could have chosen g such that $g^3 = 1$ from $g^2 = 1$ in which case we would have

$$f(x)f(x^*) + h(x)h(x^*) = 12 + \{1, g, g, g^2, g^2\} \sum_{i=1}^{12} x^i.$$

or

$$f(x)f(x^*) + h(x)h(x^*) = 6 + \{1, 1, a, a, a\} \sum_{i=1}^{12} x^i, \quad a^2 = 1$$

respectively.

With these two generators the previous constructions will give us $GBRD(13,6,30;Z_3)$ and $GBRD(13,6,30;Z_2)$ respectively.

This last example indicates that other subgroups and cosets might also prove interesting.

Example 8 $(1_1, 3_1, 9_1)$, $(1_1, 3_\alpha, 9_{\alpha^2})$, and $(1_1, 3_{\alpha^2}, 9_\alpha)$, $(2_1, 5_1, 6_1)$, $(2_1, 5_\alpha, 6_{\alpha^2})$, and $(2_1, 5_{\alpha^2}, 6_\alpha)$ are $6 - \{13; 3, 3; Z_3\}$ $GBRDSDS$ which gives a $GBRD(13, 78, 18, 3, 3; Z_3)$.

Theorem 4 *Let* $p = 6s + 1$ *be a prime power then there exists* $3s - \{6s + 1, 3, 3; Z_3\}$ $GBRDSDS$ *and hence a* $GBRD(6s + 1, 36s + 6, 9s, 3, 3; Z_3)$.

Proof. Let x be a generator of $GF(p)\backslash\{0\}$ and C_0 be the subgroup of index $2s$ and order 3 and $C_i = x^i C_0$ be the corresponding cosets. Suppose C_i has elements $\{a^i, b^i, c^i\}$ then form, for each i, three sets $D_{i1} = \{a^i_1, b^i_1, c^i_1\}$, $D_{i2} = \{a^i_1, b^i_\alpha, c^i_{\alpha^2}\}$, $D_{i3} = \{a^i_1, b^i_{\alpha^2}, c^i_\alpha\}$.

The sets D_{ij}, $i = 1, \ldots, s$, $j = 1, 2, 3$ are the required $GBRDSDS$ which can be developed into the stated $GBRD$. □

Example 9 For $p = 19$, use $C_0 = \{1, 7, 11\}$, $C_1 = \{2, 3, 14\}$, $C_{23} = \{4, 6, 9\}$ to obtain the required result.

Example 10 Suppose (a^i, b^i, c^i, d^i), $i = 1, \ldots, n$ are $n - \{v, 4, \lambda\}$ sets. Then $(a^i_1, b^i_1, c^i_w, d^i_w)$, $(a^i_1, b^i_w, c^i_1, d^i_w)$, $(a^i_1, b^i_w, c^i_w, d^i_1)$, $(a^i_1, b^i_1, c^i_{w^2}, d^i_{w^2})$, $(a^i_1, b^i_{w^2}, c^i_1, d^i_{w^2})$, $(a^i_1, b^i_{w^2}, c^i_{w^2}, d^i_1)$, $i = 1, \ldots, n$ are $6n - \{v, 4, 6\lambda; Z_3\}$ $GBRDSDS$.

The following result is similar to many others found elsewhere for example [12, 13].

Theorem 5 *Let* $p = ef + 1$ *be a prime power. Let* $C_0 = \{x^e, x^{2e}, x^{3e}, \ldots, x^{fe}\} = \{y^1, y^2, y^3, \ldots, y^{f-1}\}$. *Let* g *be the generator of the group of order* f. *Then*

$$\{x^e_1, x^{2e}_2, x^{3e}_3, \ldots, x^{fe}_0\},\ \{x^{e+1}_1, x^{2e+1}_2, \ldots, x^{(f-1)e+1}_{f-1}, x^1_0\},$$
$$\ldots, \{x^{e+j}_1, x^{2e+j}_2, \ldots, x^{(f-1)e+j}_{f-1}, x^j_0\},$$

$j = 0, 1, \ldots, e-1$ *are* e *sets in which each non-zero difference mod* p *occurs with each subscript from* $EA(f)\backslash\{0\}$ *exactly once.* $(f-1)$ *copies of each of these can be combined with the* $e = \{ef + 1; f; f - 1\}$ *sets of Stanton and Sprott to obtain a* $GBRD(ef + 1, f, f(f-1); EA(f))$. *In general this construction gives a* $PBD(f(ef + 1), \{ef + 1\}, 1)$.

Example 11 Consider $p = 19, f = 6$ and the sets

$$\{1_0, 7_4, 8_1, 11_4, 12_5, 18_3\}, \{2_0, 3_4, 5_5, 14_2, 16_1, 17_3\}, \{4_0, 6_4, 9_2, 10_5, 13_1, 15_3\} \bmod(19, Z_6).$$

There is no $BIBD(19, 6, 1)$ but there is a $BIBD(19, 6, 5) = A$ thus A together with 5 copies of the incidence matrices of each of these 3 sets yields a $BRD(19, 6, 30; Z_6)$.

Replacing the elements of $EA(6)$ by their 6×6 matrix representation in the incidence matrix gives a $PBD(114, \{19, 6\}, 1)$.

Remark. The theorem also gives results for subscripts from groups whose orders divide f.

Corollary 1 *Whenever $p = ef + 1$ is a prime power there exists a*

$$BIBD(f(ef+1),(ef+1)(ef^2+f-1),ef+f-1,f,f-1)).$$

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