Bose's Method of Differences Applied to Construct Bhaskar Rao Designs

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Abstract

In this paper we show that $BIBD(v, b, r, k, \lambda)$, where $v = pq$ or $pq + 1$, when written in the notation of Bose's method of differences may often be used to find generalized Bhaskar Rao designs $GBRD(p, b', r', k, \lambda; G)$ where G is a group of order q and vice versa.

This gives many new GBRDs including a $GBRD(9, 5, 5; Z_5)$ and a $GBRD(13, 7, 7; Z_7)$.

1 Introduction

Let $G = \{h_1 = e,\ h_2,\ \ldots\ ,\ h_g\}$ be a finite group of order g with identity e. Form the matrix W

$$W = h_1 A_1 + \ \ldots\ + \ h_g A_g$$

where $A_1, \ \ldots\ , A_g$ are $v \times b$ (0,1)-matrices such that the Hadamard product $A_k * A_j = 0$ for any $k \neq j$. Let

$$W^+ = (h_1^{-1} A_1 + \ \ldots\ + \ h_g^{-1} A_g)^T$$

and

$$N = A_1 + A_2 + \ \ldots\ + \ A_g.$$

Then we say W is a *generalized Bhaskar Rao design* over G denoted by $GBRD(v, b, r, k, \lambda; G)$, or in abbreviated form $GBRD(v, k, \lambda; G)$, if N satisfies

$$NN^T = (r - \lambda)I \ + \ \lambda J :$$

that is, N is the incidence matrix of the $BIBD(v, k, \lambda)$ and

$$WW^+ = reI \ + \ (\lambda/g)(h_1 \ + \ \ldots\ + \ h_g)(J - I).$$

We say that the design W is based on the matrix N or that N is signed over the group G.

$GBRD$ have been studied by the author and others, for example see [7, 8, 9, 14, 16, 21, 22]. We now illustrate the construction of the $GBRD$ with some examples. For further examples the reader is referred to [20].

To check whether a set of blocks $(x_a^i,\ y_b^i,\ z_c^i)$ mod (v,g), each with k elements, are part of a $GBRD$ or a $BIBD$ we check all the differences: $(y-x)^i_{ba^{-1}}$, $(x-y)^i_{ab^{-1}}$, $(z-x)^i_{ca^{-1}}$, $(x-z)^i_{ac^{-1}}$, $(z-y)^i_{cb^{-1}}$ and $(y-z)^i_{bc^{-1}}$, where the subscripts such as ab^{-1} are from a group G of order $|G|=g$ and the subscripted elements such as $(x-y)^i$ come from a group V of order v.

If each non-zero-zero difference, that is differences other than 0_0, including the differences 0_1, $\cdots$, 0_{g-1} occurs the same number of times, λ, we will have a $BIBD(v,k,\lambda)$. If each non-zero difference, that is not including the differences 0_0, 0_1, $\cdots$, 0_{g-1}, occurs the same number of times, λ, we will have a $GBRD(v,k,\lambda;G)$.

We also use a distinguished element ∞ which has the properties that

$$\infty - a_i = \infty_i^{-1}, \text{ and } a_i - \infty = -\infty_i,$$

for any $a_i \in V$.

For a Bhaskar Rao design ∞_j and $-\infty_j$ must occur λ times. For a $BIBD$ it is merely necessary that the number of elements in the sets with ∞ is $k-1$.

Example 1 Consider the initial blocks

$$\begin{array}{ccccc} (1_1,\ 6_1,\ 0_2) & (2_1,\ 5_1,\ 0_2) & (3_1,\ 4_1,\ 0_2) & (1_2,\ 6_2,\ 0_3) & (2_2,\ 5_2,\ 0_3) \\ (3_2,\ 4_2,\ 0_3) & (1_3,\ 6_3,\ 0_1) & (2_3,\ 5_3,\ 0_1) & (3_3,\ 4_3,\ 0_1) & (0_1,\ 0_2,\ 0_3) \end{array}$$

mod $(7,3)$ which Bose gives as the initial blocks of a $BIBD(21,3,1)$. Without the last block of $(0_1,\ 0_2,\ 0_3)$ we have a $GBRD(7,3,9;Z_3)$.

Example 2 The subscripts from Bose's [4, p373, example (i)] $BIBD(9,3,1)$ give a $GBRD(3,3,9;Z_3)$ (the $GBRD$ is the 3×9 array):

$$\begin{bmatrix} 2 & 1 & 1 & 3 & 2 & 2 & 1 & 3 & 3 \\ 1 & 2 & 1 & 2 & 3 & 2 & 3 & 1 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 1 \end{bmatrix}$$

Clearly, the differences from a single starting block also suffice to give all the differences, and hence we have a $GBRD(3,3,3;Z_3)$.

Example 3 Consider the initial sets

$$(0_0,\ 1_0,\ 4_1) \text{ twice},\quad (\infty,\ 0_0,\ 3_0),\quad (\infty,\ 0_1,\ 3_1) \ \text{mod}(5,2)$$

To develop the subscripts we note

$$0^{-1}=0,\quad 1^{-1}=1,\quad 0\cdot 1^{-1} = 1\cdot 0^{-1} = 1,\quad 0\cdot 0^{-1} = 1\cdot 1^{-1} = 0.$$

We now note that the set $(\infty,\ 0_0,\ 3_0)$ means the differences ∞_0 and $-\infty_0$ each occur twice. The set $(\infty,\ 0_1,\ 3_1)$ means the differences ∞_1 and $-\infty_1$ also each occur twice.

When we check the non-∞ differences we get

$$\begin{array}{lllll} & 0_0-1_0=4_0, & 0_0-4_1=1_1, & 1_0-4_1=2_1, & \text{twice}, \\ & 1_0-0_0=1_0, & 4_1-0_0=4_1, & 4_1-1_0=3_1, & \text{twice}, \\ \text{and} & 3_0-0_0=3_0, & 0_0-3_0=2_0, & 3_1-0_1=3_0, & 0_1-3_1=2_0. \end{array}$$

As we have each non-zero difference occurring twice we have a $GBRD(6,3,4;Z_2)$. Its incidence matrix is

0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
.	.	.	.	.	.	.	.	.	.	1	1	1	1	1	1	1	1	1	1
0	1	.	.	0	0	1	.	.	0	0	.	0	.	.	1	.	1	.	.
0	0	1	.	.	0	0	1	.	.	.	0	.	0	.	.	1	.	1	.
.	0	0	1	.	.	0	0	1	.	.	.	0	.	0	.	.	1	.	1
.	.	0	0	1	.	.	0	0	1	0	.	.	0	.	1	.	.	1	.
1	.	.	0	0	1	.	.	0	0	.	0	.	.	0	.	1	.	.	1

where 0, 1 are elements of the group, Z_2, and $\cdot$ means the zero of the group ring so $\cdot - 1 = \cdot$, and $\cdot - 2 = \cdot$.

To change these sets into starting blocks for a $BIBD$ we would also need to have the difference 0_1 occur twice. This can be done but not in a straight forward way.

This clearly shows that Bhaskar Rao designs are not another representation of $BIBD$s. □

Example 4 To illustrate how a design developed from blocks using differences can be shown to be a $BIBD$ or $GBRD$ we consider the initial blocks

$$(0_0,\ 1_0,\ 2_0)\ (0_0,\ 1_1,\ 2_1)\ (0_1,\ 2_0,\ 3_1)\ (0_0,\ 2_1,\ 4_1)\ (0_0,\ 2_0,\ 0_1)\ (0_0,\ 0_1,\ 4_1)\ \bmod(5,-)$$

So the differences are

$$\begin{array}{cccccc}
1_0 & 2_0 & 1_0 & 4_0 & 3_0 & 4_0 \\
1_0 & 1_1 & 2_1 & 4_0 & 4_1 & 3_1 \\
1_1 & 2_1 & 3_0 & 4_1 & 3_1 & 2_0 \\
2_1 & 4_1 & 2_0 & 3_1 & 1_1 & 3_0 \\
2_0 & 0_1 & 3_1 & 3_0 & 0_1 & 2_1 \\
4_0 & 0_1 & 4_1 & 1_0 & 0_1 & 1_1
\end{array}$$

This means each of the differences $0_1, 1_0, 1_1, 2_0, 2_1, 3_0, 3_1, 4_0, 4_1$ occurs 4 times, that is 2 times corresponding to each difference where the direction it is taken in is considered. Hence we have, after developing the initial blocks modulo 5, exactly Bose's example (iv) [4, p370] for a $BIBD(10,3,2)$.

1	1	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0	0	0
0	1	1	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0	0
0	0	1	1	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0	0	0	1	0	1	0	0	1	0	0
1	0	0	1	1	0	0	0	1	0	1	0	0	0	0	0	0	0	1	0	1	0	0	1	0	0	0	0	1	0
1	1	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0	1	0	1	0	0	1	0	0	0	0	1

0	0	0	0	0	0	1	1	0	0	1	0	0	1	0	0	0	1	0	1	1	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	1	1	0	0	1	0	0	1	1	0	0	1	0	0	1	0	0	0	1	1	0	0	0
0	0	0	0	0	0	0	0	1	1	1	0	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	1	0	0
0	0	0	0	0	1	0	0	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	1	0	0	0	1	1	0
0	0	0	0	0	1	1	0	0	0	0	0	1	0	1	0	1	0	1	0	0	0	0	0	1	0	0	0	1	1

It has GBRD type incidence matrix for the last three sets of

0		1	.	1	01	.	0	.	.	01	.	.	.	1
1	0	.	1	.	.	01	.	0	.	1	01	.	.	.
.	1	0	.	1	.	.	01	.	0	.	1	01	.	.
1	.	1	0	.	0	.	.	01	.	.	.	1	01	.
.	1	.	1	0	.	0	.	.	01	.	.	.	1	01

It is not a $GBRD$ as the difference 0_1 occurs 4 times. This is represented by the "01" in the above incidence matrix. However the first four blocks give a $GBRD(6,3,6;Z_2)$

$$(0_0,\ 1_0,\ 2_0)\ \ (0_0,\ 1_1,\ 2_1)\ \ (0_1,\ 2_0,\ 3_1)\ \ (0_0,\ 2_1,\ 4_1)\ \ \mathrm{mod}(5, Z_2),$$

which has incidence matrix:

$$\begin{array}{ccccc}
0 & 0 & 0 & . & . \\
. & 0 & 0 & 0 & . \\
. & . & 0 & 0 & 0 \\
0 & . & . & 0 & 0 \\
0 & 0 & . & . & 0
\end{array}
\quad
\begin{array}{ccccc}
0 & 1 & 1 & . & . \\
. & 0 & 1 & 1 & . \\
. & . & 0 & 1 & 1 \\
1 & . & . & 0 & 1 \\
1 & 1 & . & . & 0
\end{array}
\quad
\begin{array}{ccccc}
0 & . & 0 & 1 & . \\
. & 0 & . & 0 & 1 \\
1 & . & 0 & . & 0 \\
0 & 1 & . & 0 & . \\
. & 0 & 1 & . & 0
\end{array}
\quad
\begin{array}{ccccc}
0 & . & 1 & . & 1 \\
1 & 0 & . & 1 & . \\
. & 1 & 0 & . & 1 \\
1 & . & 1 & 0 & . \\
. & 1 & . & 1 & 0
\end{array}$$

Example 5 David Glynn [12] has found the only $GW(v,k,G)$ known to the author where G is not an abelian group. Consider the multiplication table for S_3

	1	2	3	4	5	6	
1	1	2	3	4	5	6	$1 \leftrightarrow e$
2	2	3	1	5	6	4	$2 \leftrightarrow (123)(456)$
3	3	1	2	6	4	5	$3 \leftrightarrow (132)(465)$
4	4	6	5	1	3	2	$4 \leftrightarrow (14)(26)(35)$
5	5	4	6	2	1	3	$5 \leftrightarrow (15)(24)(36)$
6	6	5	4	3	2	1	$6 \leftrightarrow (16)(25)(34)$

Then the circulant matrix

$$\left[\begin{array}{ccccccccccccc}
0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 \\
0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 \\
4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 \\
0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 \\
6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 \\
5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 & 6 \\
6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 & 1 \\
1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 & 1 \\
1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 & 0 \\
0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 & 4 \\
4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 & 1 \\
1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0 & 5 \\
5 & 1 & 4 & 0 & 1 & 1 & 6 & 5 & 6 & 0 & 4 & 0 & 0
\end{array}\right]$$

is a generalized weighing matrix $GW(13,9;S_3)$.

In set notation this can be written as

$$(1_5\ 2_1\ 3_4\ 6_1\ 7_1\ 8_6\ 9_5\ 10_6\ 12_4)\ \ \mathrm{mod}\ (13, S_3)$$

There are two inequivalent circulant $GW(13,9;Z_2)$ (see Seberry and Wehrhahn [23]) and a total of eight inequivalent $GW(13,9;Z_2)$ (Ohmori [15]) but the existence of $GW(13,9;G)$ for $G = Z_3$ or Z_6 is not yet resolved.

Another interesting possibility occurs if there are parallel classes associated with a subdesign. For example the $GBRD(13,9,6;S_3)$ given by developing the above block mod (13) can be embedded in an $SBIBD(91,10,1)$ which also has an $SBIBD(13,4,1)$ embedded (see the construction in [19]). □

In this paper we show that $BIBD(v, b, r, k, \lambda)$, where $v = pq$ or $pq + 1$, when written in the notation of Bose's method of differences may often be used to find generalized Bhaskar Rao designs $GBRD(p, b', r', k, \lambda; G)$ where G is a group of order q.

Theorem 1 *Let G be a group. A $GBRD(v, k, \lambda; G)$, where $\mid G \mid$ is λ, is equivalent to a $BIBD(v\lambda, k, 1)$ when both are written using starting blocks developed* $\mathrm{mod}(\lambda)$. *(The most common case has G a cyclic group but it is possible to develop a theory with other groups.)*

Proof. Suppose we have initial blocks

$$(x_{a_i}^{(i)}\ y_{b_i}^{(i)}\ \cdots\ z_{c_i}^{(i)})$$

for the $GBRD$. Then the equivalent starting blocks for the $BIBD$ are:

$$(x_{a_i}^{(i)},\ \cdots\ z_{c_i}^{(i)})\ (x_{a_i+1}^{(i)},\ \cdots\ z_{c_i+1}^{(i)})\ (x_{a_i+k-1}^{(i)},\ \cdots\ z_{c_i+k-1}^{(i)})\ (0_0,\ 0_1,\ \cdots\ 0_{k-1})\ \ \mathrm{mod}(v, \lambda).$$

The differences of the type $(x_{a_i}^{(i)} - y_{a_i}^{(i)})$, $a_i = 0, \ \cdots, \ \lambda$ yield the product elements of type $(x^{(i)} - y^{(i)})_{a_i a_i^{-1}})$ which must occur once each for the $GBRD$ and once each arising from each a_i for the $BIBD$.

The differences of type $(x_{a_i}^{(i)} - y_{b_i}^{(i)})$, $a_i = 0, \ \cdots, \ \lambda$, $a_i \neq b_i$, yield the product elements of type $(x^{(i)} - y^{(i)})_{a_i b_i^{-1}}$ which must occur equally often for the $GBRD$ and hence occur equally often for the $BIBD$. The block with zeros completes the differences for the $BIBD$.

□

See the case $k = 3$ in the next section for an example of how this theorem works.

When we have a $BIBD((k-1)v+1, k, k-1)$ developed mod $(v, k-1)$ with a special element ∞ we seek to form a $GBRD(v, k, (k-1)^2; G)$ where G is a group of order $k-1$. Again the converse, that the $GBRD$ in this case gives a $BIBD$, has been observed previously (see [19]).

Theorem 2 *Let G be a group. If the initial blocks*

$$(x_{a_1}^{(1)},\ x_{a_2}^{(2)},\ \ldots,\ x_{a_k}^{(k)})\ \ (y_{b_1}^{(1)},\ y_{b_2}^{(2)},\ \ldots,\ y_{b_k}^{(k)})\ldots(z_{c_1}^{(1)},\ z_{c_2}^{(2)},\ \ldots,\ z_{c_k}^{(k)})$$

give a $GBRD(v, k, k-1; G)$ where $\mid G \mid = k-1$, then the initial blocks

$$(x_{a_1+i}^{(1)},\ x_{a_2+i}^{(2)}, \ldots, x_{a_k+i}^{(k)})\ (y_{b_1+i}^{(1)},\ y_{b_2+i}^{(2)}, \ldots y_{b_k+i}^{(k)})\ldots(z_{c_1+i}^{(1)},\ z_{c_2+i}^{(2)}, \ldots, z_{c_k+i}^{(k)})\ (\infty,\ 0_1,\ 0_2,\ \ldots,\ 0_{k-1})$$

$i = 0, \ \cdots, \ k-2$, *give a $BIBD((k-1)v+1, k, 1)$, where $i = 0, 1, \ldots, k-1$. The converse is also true. (The most common case has G a cyclic group but it is possible to develop a theory with other groups.)*

Proof. The proof follows by noticing that the mixed non-zero differences occur the same number of times. The extra block makes sure the zero elements occur the appropriate number of times. □

See the case $k = 4$ in the next section for an example of how this theorem works.

Comment. In some sense Bose's theorem implies that the subscripts are from a cyclic group. However the formulation in terms of $GBRD$ indicates clearly that any group will suffice.

The construction with non-cyclic groups can be visualized by replacing the 0 element of the $GBRD$ by a $(k-1) \times (k-1)$ zero matrix and the other elements by their right (left) regular matrix representation. Finally, the extra $k-1$ blocks corresponding to $(\infty\ 0_1\ 0_2 \ldots 0_{k-1})$ are added.

2 Examples and Constructions

We now give some examples of these methods. The $GBRD$ are new.

k=3: To illustrate

$$(0_1, 1_0, 4_0), (0_1, 2_0, 3_0) \bmod(5, Z_3)$$

give a $GBRD(5,3,3;Z_3)$. The blocks

$$\begin{array}{llll} (0_1, 1_0, 4_0) & (0_1, 2_0, 3_0) & (0_0, 0_2, 0_3) & \bmod\ (5,3) \\ (0_2, 1_1, 4_1) & (0_2, 2_1, 3_1) & & \\ (0_0, 1_2, 4_2) & (0_0, 2_2, 3_2) & & \end{array}$$

give a $BIBD(15,3,1)$. That the $GBRD$ in this case gives a $BIBD$, has been observed previously (see [19]).

In Lemma 3 this method is applied to $BIBD(3v,3,1)$ and $GBRD(v,3,3;Z_3)$ for all odd v.

k=4: The initial blocks for the $BIBD(28,4,1)$, $i = 0,\ 1,\ 2$, are

$$[(21)_i, (01)_i, (12)_{i+1}, (10)_{i+1}],\ \ [(20)_i, (02)_i, (22)_{i+1}, (00)_{i+1}]\ \text{ and }\ [\infty, (00)_0, (00)_1, (00)_2].$$

The first two blocks, with $i = 0$, give a $GBRD(9,4,3;Z_3)$. The elements work modulo (3,3) and suffixes modulo 3.

k=5: The initial blocks of a $BIBD(45,5,1)$ are

$$\begin{array}{rl} & [(01)_0, (02)_0, (10)_2, (20)_2, (00)_1], \\ & [(21)_0, (12)_0, (22)_2, (11)_2, (00)_1] \\ \text{and} & [(00)_0, (00)_1, (00)_2, (00)_3, (00)_4]. \end{array}$$

The first two blocks give a $GBRD(9,5,5;Z_5)$ which is new. The elements work modulo (3,3) and suffixes modulo 5.

k=6: Denniston [10] has given a $BIBD(66,6,1)$ design. This design is rewritten in a convenient way as follows:

$$\begin{array}{ll} (0_3,\ 2_3,\ 3_0,\ 5_2,\ 6_2,\ 10_0), & (0_3,\ 1_4,\ 4_0,\ 6_0,\ 7_1,\ 11_2), \\ (0_4,\ 1_4,\ 3_2,\ 9_2,\ 11_1,\ 12_2), & (0_3,\ 3_4,\ 5_1,\ 7_0,\ 8_2,\ 12_1), \\ (0_4,\ 1_0,\ 3_4,\ 7_2,\ 9_0,\ 10_0), & (2_3,\ 4_4,\ 5_3,\ 6_3,\ 10_4,\ 12_4), \\ (0_4,\ 1_1,\ 3_1,\ 4_1,\ 8_0,\ 9_4), & (0_3,\ 2_0,\ 4_1,\ 5_0,\ 6_3,\ 9_1), \\ (0_3,\ 1_2,\ 2_1,\ 5_3,\ 6_1,\ 12_2), & (0_3,\ 2_2,\ 8_1,\ 9_4,\ 10_2,\ 11_0), \\ (\infty,\ 0_0,\ 0_1,\ 0_2,\ 0_3,\ 0_4), & \end{array}$$

all blocks developed $\bmod(13,5)$. We note that the first 10 of these blocks also give a $GBRD(13,6,25; Z_5)$. If we knew how to sign the blocks $\{2\ 5\ 6\ 7\ 8\ 11\}$ and $\{1\ 3\ 4\ 9\ 10\ 12\}$, which develop mod(13) to give a $BIBD(13,6,5)$, we would have a $GBRD(13,6,5;Z_5)$.

k=7: The following three sets of initial blocks from Bhat-Nayak and Kane [1, 2] give initial blocks for a $BIBD(91,7,1)$:

Solution 1:

$$\begin{array}{rl} & [6_0, 7_0, 11_1, 2_1, 8_5, 5_5, 0_2], \\ & [10_1, 3_1, 9_0, 4_0, 12_5, 1_5, 0_2] \\ \text{and} & [0_0, 0_1, 0_2, 0_3, 0_4, 0_5, 0_6]. \end{array}$$

Solution 2:

$$\begin{array}{rl} & [6_4, 7_4, 11_3, 2_3, 8_6, 5_6, 0_2], \\ & [10_3, 3_3, 9_4, 4_4, 12_6, 1_6, 0_2] \\ \text{and} & [0_0, 0_1, 0_2, 0_3, 0_4, 0_5, 0_6]. \end{array}$$

Solution 3:

$$\begin{array}{rl} & [6_4, 7_4, 11_3, 2_3, 8_6, 5_6, 0_2], \\ & [10_1, 3_1, 9_0, 4_0, 12_5, 1_5, 0_2] \\ \text{and} & [0_0, 0_1, 0_2, 0_3, 0_4, 0_5, 0_6]. \end{array}$$

In each case the first two blocks mod $(13, 7)$ are initial blocks for a $GBRD(13, 7, 7; Z_7)$. These are new.

One typical result illustrates many possible theorems arising from constructions for $BIBD$ using Bose's method of differences.

Lemma 3 *The starting blocks*

$$(1_0,\ 2t_0,\ 0_1)\ \ (2_0,\ (2t-1)_0,\ 0_1)\ \ \ldots\ \ (t_0,\ (t+1)_0,\ 0_1)\ \ (0_0,\ 0_1,\ 0_2)\ \ \text{mod } (2t+1, 3)$$

were used by Bose [4, p373, equation 4.22] *to form a* $BIBD(3(2t+1), 3, 1)$. *The starting blocks*

$$(1_0,\ 2t_0,\ 0_1)\ \ (2_0,\ (2t-1)_0,\ 0_1)\ \ \ldots\ \ (t_0,\ (t+1)_0,\ 0_1)\ \ \text{mod } (2t+1, Z_3)$$

form a $GBRD(2t+1, 3, 3; Z_3)$.

Proof. A simple check shows all the differences for the $GBRD$, $\{1,\ \cdots,\ 2t\}$, are obtained with each subscript 0, 1, 2. This immediately gives the $GBRD(2t+1, 3, 3; Z_3)$.

It remains to notice that the starting blocks for the $BIBD$ are the same as for the $GBRD$ with the exception of the last. This means we have the required number of pairs both with the subscripts i and i, $i = 0,\ 1,\ 2$ and with cross subscripts i and j, $i \neq j$, $i,\ j = 0,\ 1,\ 2$. This gives the result after noting the block $(0_0,\ 0_1,\ 0_2)$ guarantees the remaining differences which are needed.

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