On the (10, 5, λ)-Family of Bhaskar Rao Designs

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Abstract

We prove a theorem for BRD(10,5,λ)s and give thirteen (13) inequivalent BRD(10,5,4)s.

1 Introduction

A *balanced incomplete block (BIBD) design* is an arrangement of v symbols in b blocks each containing ($k < v$) symbols, satisfying the following conditions : (i) every symbol occurs at most once in a block, (ii) every symbol occurs in exactly r blocks, (iii) every pair of treatments or symbols occur together in exactly λ blocks.

A *Bhaskar Rao design* [BRD(v, b, r, k, λ)] is a matrix of order $v \times b$ with (0, $\pm$ 1 entries), satisfying the following conditions : (i) it has k non-zero [$(v-k)$ zero] entries/column, (ii) it has r non-zero [$(b-r)$ zero] entries/row, (iii) λ non-zero entries are in the same column as λ other non-zero entries for every pair of rows, (iv) the inner product of any pair of distinct rows is zero, (v) when it's -1s are changed to $+1$s, the resulting matrix becomes the incidence matrix of a BIBD. The necessary conditions for the existence of a BRD(v, b, r, k, λ) are :

(i) $vr = bk$,

(ii) $\lambda(v-1) = r(k-1)$,

(iii) the inner product of distinct rows is zero,

(iv) $2|\lambda$ and $2|b$.

Theorem 1 (Chaudhry and Seberry [2]) *The conditions*

i) $\lambda(v-1) \equiv 0 \pmod 4$

ii) $\lambda v(v-1) \equiv 0 \pmod{20}$

iii) $b \equiv 0 \pmod 2$

are necessary for the existence of $BRD(v, 5, 2\lambda)$ *where* λ *may take values 1, 2, 3, ...*

Lemma 1 (Chaudhry and Seberry [2]) *Suppose in the signing of the rows of a BIBD, it happens that the rows* $\{1, ..., j\} \in A$ *are mutually orthogonal and the rows* $\{j+1, ..., v\} \in B$ *are mutually orthogonal but no row of A is orthogonal to any row of B and vice versa. Then if we take another copy of the BIBD and negate the orthogonal rows in B, we obtain a* $BRD(v, k, 2\lambda)$.

2 BRDs(10,5, λ)

There exist twenty one (21) inequivalent BIBDs with parameters (10,5,4), these are given in [3], appendix B. In this paper, we have constructed thirteen (13) BRD(10, 5, 4)s from the BIBD(10,5,4)s: call this set A, these are given in the appendix in the same order as in [3]. Gibbons BIB designs numbers 9, 12, 13, 14, 15, 16, 19 and 21 were found, by an exhaustive computer search, not to give BRDs: call this set B. Hence thirteen of the twenty one inequivalent BIBD(10,5,4)s can be signed to BRD(10,5,4)s and eight cannot.

We write $D_i||D_j = [D_i D_j]$ for the matrix of order $v \times 2b$ with parameters $(v, 2b, 2r, k, 2\lambda)$ when D_i is the matrix of order v with parameters (v, b, r, k, λ). We note though that the eight BIBD(10, 5, 4)s, which can not be signed to BRD(10, 5, 4)s, satisfy Lemma 1. We use $D_i||\overline{D_i}$ for the BRDs constructed using Lemma 1. Now we construct the BRD(10,5,8)s in two ways :

i) if D_i, $D_j \in A$, there are 78 different BIBD(10,5,8)s for $D_i||D_j$, so we obtain 78 inequivalent BRD(10,5,8)s;

ii) if $E_k \in B$, then $E_k||E_k$ is a BIBD(10,5,8), and so we obtain 8 BRD(10,5,8)s.

Thus we have constructed 86 BRD(10,5,8)s of the ≥ 135922 possible cases.

Theorem 2 *The conditions*

i) $\lambda(v-1) \equiv 0 \pmod 4$

ii) $\lambda v(v-1) \equiv 0 \pmod{20}$

iii) $b \equiv 0 \pmod 2$

iv) $\lambda \equiv 0 \pmod 4$

are necessary and sufficient for the existence of $BRD(10, 5, \lambda)$.

Remark The inequivalent BRD(10,5,4)s are given in the appendix.

References

[1] Bhaskar Rao M, *Group divisible family of PBIB designs*, J. Indian Stat. Assoc. 4 (1966), pp. 14-28.

[2] Chaudhry G R and Seberry J, *On Bhaskar Rao designs of block size five*, 22ACCMCC, Sydney Australia, July 15-19, 1996.

[3] Gibbons P B, Computing Techniques for the Construction and Analysis of Block Designs, PhD Thesis, University of Toronto, Technical Report No. 92, May 1976.

[4] Gibbons P B and Mathon R, *Construction methods for Bhaskar Rao and related designs*, J Austral. Math. Soc. (Series A) 42 (1987), pp. 5-30.

[5] Hall M Jr, Combinatorial Theory, Blaisdell Publishing Company, Waltham, Mass. USA, 1967.

Appendix Thirteen inequivalent BRD(10,18,9,5,4)s are given below :

BRD #	I					II					III					IV				
Gibbons #	I					II					III					IV				
1	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
2	1	2	$\overline{3}$	6	7	1	2	$\overline{3}$	6	7	1	$\overline{2}$	3	6	7	1	2	$\overline{3}$	6	7
3	1	$\overline{2}$	$\overline{3}$	8	9	1	$\overline{2}$	3	8	9	1	2	$\overline{3}$	$\overline{8}$	9	1	$\overline{2}$	3	8	9
4	1	$\overline{2}$	$\overline{4}$	$\overline{5}$	10	1	$\overline{2}$	4	5	10	1	$\overline{2}$	$\overline{4}$	5	10	1	$\overline{2}$	4	$\overline{5}$	10
5	1	3	6	7	10	1	$\overline{3}$	$\overline{6}$	$\overline{7}$	10	1	$\overline{3}$	$\overline{6}$	$\overline{8}$	$\overline{10}$	1	$\overline{3}$	$\overline{6}$	$\overline{8}$	10
6	1	4	$\overline{6}$	8	$\overline{9}$	1	$\overline{4}$	6	8	$\overline{9}$	1	4	$\overline{6}$	7	8	1	$\overline{4}$	$\overline{6}$	7	$\overline{9}$
7	1	$\overline{4}$	$\overline{6}$	$\overline{8}$	$\overline{10}$	1	$\overline{4}$	$\overline{6}$	$\overline{8}$	$\overline{10}$	1	$\overline{4}$	6	$\overline{9}$	$\overline{10}$	1	$\overline{4}$	6	$\overline{9}$	$\overline{10}$
8	1	5	$\overline{7}$	$\overline{8}$	9	1	$\overline{5}$	7	$\overline{8}$	9	1	$\overline{5}$	$\overline{7}$	$\overline{8}$	9	1	5	$\overline{7}$	$\overline{8}$	9
9	1	$\overline{5}$	$\overline{7}$	$\overline{9}$	$\overline{10}$	1	$\overline{5}$	$\overline{7}$	$\overline{9}$	$\overline{10}$	1	$\overline{5}$	$\overline{7}$	$\overline{9}$	10	1	$\overline{5}$	$\overline{7}$	8	$\overline{10}$
10	2	$\overline{3}$	8	9	$\overline{10}$	2	3	8	9	$\overline{10}$	2	3	$\overline{7}$	$\overline{9}$	$\overline{10}$	2	3	$\overline{7}$	$\overline{9}$	10
11	2	$\overline{4}$	$\overline{6}$	7	$\overline{9}$	2	4	$\overline{6}$	7	$\overline{9}$	2	$\overline{4}$	$\overline{6}$	7	$\overline{9}$	2	4	$\overline{6}$	7	8
12	2	$\overline{4}$	$\overline{7}$	8	10	2	$\overline{4}$	$\overline{7}$	9	10	2	$\overline{4}$	7	$\overline{8}$	10	2	$\overline{4}$	$\overline{7}$	8	10
13	2	$\overline{5}$	6	$\overline{7}$	$\overline{8}$	2	5	6	$\overline{7}$	$\overline{8}$	2	5	6	$\overline{8}$	9	2	$\overline{5}$	6	$\overline{8}$	9
14	2	$\overline{5}$	$\overline{6}$	9	10	2	$\overline{5}$	$\overline{6}$	8	10	2	$\overline{5}$	6	8	10	2	$\overline{5}$	$\overline{6}$	9	$\overline{10}$
15	3	4	$\overline{5}$	$\overline{6}$	9	3	$\overline{4}$	5	$\overline{6}$	$\overline{9}$	3	$\overline{4}$	$\overline{5}$	$\overline{6}$	9	3	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{8}$
16	3	$\overline{4}$	5	$\overline{7}$	8	3	4	$\overline{5}$	$\overline{7}$	$\overline{8}$	3	4	$\overline{5}$	7	$\overline{8}$	3	$\overline{4}$	5	7	9
17	3	$\overline{4}$	7	9	$\overline{10}$	3	$\overline{4}$	7	$\overline{8}$	10	3	$\overline{4}$	8	9	$\overline{10}$	3	4	$\overline{8}$	$\overline{9}$	$\overline{10}$
18	3	$\overline{5}$	6	8	$\overline{10}$	3	$\overline{5}$	6	$\overline{9}$	10	3	5	$\overline{6}$	$\overline{7}$	10	3	$\overline{5}$	6	7	10

BRD #	V					VI					VII					VIII				
Gibbons #	V					VI					VII					VIII				
1	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
2	1	2	$\overline{3}$	6	7	1	$\overline{2}$	3	6	7	1	$\overline{2}$	3	6	7	1	2	$\overline{3}$	6	7
3	1	$\overline{2}$	3	8	9	1	$\overline{2}$	$\overline{3}$	8	9	1	$\overline{2}$	$\overline{3}$	8	9	1	$\overline{2}$	3	8	9
4	1	$\overline{2}$	4	5	10	1	2	$\overline{4}$	5	10	1	2	$\overline{4}$	$\overline{5}$	10	1	$\overline{2}$	$\overline{4}$	5	10
5	1	$\overline{3}$	$\overline{6}$	$\overline{7}$	10	1	$\overline{3}$	$\overline{6}$	$\overline{8}$	$\overline{10}$	1	$\overline{3}$	6	$\overline{8}$	10	1	$\overline{3}$	$\overline{6}$	$\overline{8}$	$\overline{10}$
6	1	$\overline{4}$	$\overline{6}$	$\overline{8}$	$\overline{10}$	1	4	6	$\overline{7}$	$\overline{8}$	1	4	$\overline{6}$	7	9	1	4	6	$\overline{7}$	10
7	1	$\overline{4}$	7	8	$\overline{9}$	1	$\overline{4}$	$\overline{7}$	$\overline{9}$	$\overline{10}$	1	$\overline{4}$	$\overline{7}$	$\overline{8}$	$\overline{10}$	1	$\overline{4}$	8	$\overline{9}$	$\overline{10}$
8	1	$\overline{5}$	6	$\overline{8}$	9	1	$\overline{5}$	$\overline{6}$	9	10	1	$\overline{5}$	$\overline{6}$	$\overline{9}$	$\overline{10}$	1	$\overline{5}$	$\overline{6}$	$\overline{7}$	$\overline{9}$
9	1	$\overline{5}$	$\overline{7}$	$\overline{9}$	$\overline{10}$	1	$\overline{5}$	7	8	$\overline{9}$	1	5	$\overline{7}$	8	$\overline{9}$	1	$\overline{5}$	7	$\overline{8}$	9
10	2	3	8	9	$\overline{10}$	2	$\overline{3}$	7	9	$\overline{10}$	2	$\overline{3}$	7	$\overline{9}$	10	2	3	$\overline{7}$	9	$\overline{10}$
11	2	4	$\overline{6}$	7	$\overline{8}$	2	4	$\overline{6}$	7	$\overline{9}$	2	4	6	$\overline{7}$	8	2	$\overline{4}$	6	$\overline{7}$	$\overline{8}$
12	2	$\overline{4}$	$\overline{6}$	9	10	2	$\overline{4}$	6	8	$\overline{10}$	2	$\overline{4}$	6	9	$\overline{10}$	2	$\overline{4}$	$\overline{6}$	8	9
13	2	5	6	$\overline{7}$	$\overline{9}$	2	$\overline{5}$	6	$\overline{8}$	9	2	5	$\overline{6}$	$\overline{8}$	9	2	5	$\overline{6}$	$\overline{9}$	10
14	2	$\overline{5}$	$\overline{7}$	8	10	2	$\overline{5}$	$\overline{7}$	8	10	2	$\overline{5}$	7	8	$\overline{10}$	2	$\overline{5}$	7	8	10
15	3	4	$\overline{5}$	$\overline{6}$	$\overline{9}$	3	$\overline{4}$	5	$\overline{6}$	9	3	4	$\overline{5}$	6	$\overline{8}$	3	$\overline{4}$	$\overline{5}$	6	$\overline{9}$
16	3	$\overline{4}$	5	$\overline{7}$	$\overline{8}$	3	$\overline{4}$	$\overline{5}$	7	$\overline{8}$	3	$\overline{4}$	5	7	$\overline{9}$	3	$\overline{4}$	5	7	$\overline{8}$
17	3	$\overline{4}$	7	$\overline{9}$	10	3	4	8	9	$\overline{10}$	3	$\overline{4}$	8	9	10	3	4	7	$\overline{9}$	$\overline{10}$
18	3	$\overline{5}$	6	$\overline{8}$	10	3	$\overline{5}$	$\overline{6}$	$\overline{7}$	$\overline{10}$	3	$\overline{5}$	$\overline{6}$	$\overline{7}$	10	3	$\overline{5}$	$\overline{6}$	$\overline{8}$	10

BRD #	X					XI					XVII					XVIII				
Gibbons #	X					XI					XVII					XVIII				
1	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
2	1	2	$\overline{3}$	6	7	1	$\overline{2}$	$\overline{3}$	6	7	1	$\overline{2}$	3	6	7	1	$\overline{2}$	$\overline{3}$	6	7
3	1	$\overline{2}$	3	8	9	1	$\overline{2}$	3	8	9	1	$\overline{2}$	$\overline{3}$	8	9	1	2	$\overline{5}$	7	8
4	1	$\overline{2}$	$\overline{4}$	5	10	1	2	$\overline{4}$	$\overline{6}$	10	1	2	$\overline{4}$	$\overline{6}$	10	1	$\overline{2}$	$\overline{6}$	$\overline{7}$	9
5	1	$\overline{3}$	$\overline{6}$	$\overline{8}$	10	1	$\overline{3}$	5	$\overline{8}$	10	1	$\overline{3}$	$\overline{5}$	$\overline{8}$	10	1	3	$\overline{5}$	9	10
6	1	4	6	$\overline{9}$	$\overline{10}$	1	4	$\overline{5}$	6	9	1	$\overline{4}$	5	$\overline{9}$	$\overline{10}$	1	$\overline{3}$	$\overline{6}$	8	$\overline{10}$
7	1	$\overline{4}$	$\overline{7}$	8	$\overline{10}$	1	$\overline{4}$	$\overline{7}$	$\overline{8}$	$\overline{10}$	1	4	$\overline{6}$	7	$\overline{9}$	1	$\overline{4}$	7	$\overline{8}$	10
8	1	$\overline{5}$	$\overline{6}$	7	$\overline{9}$	1	$\overline{5}$	$\overline{7}$	8	$\overline{9}$	1	$\overline{5}$	6	$\overline{7}$	$\overline{8}$	1	$\overline{4}$	$\overline{7}$	$\overline{9}$	$\overline{10}$
9	1	$\overline{5}$	$\overline{7}$	$\overline{8}$	9	1	$\overline{6}$	7	$\overline{9}$	$\overline{10}$	1	$\overline{7}$	8	9	$\overline{10}$	1	5	6	$\overline{8}$	$\overline{9}$
10	2	3	7	9	10	2	$\overline{3}$	7	9	$\overline{10}$	2	$\overline{3}$	7	9	$\overline{10}$	2	$\overline{3}$	$\overline{7}$	$\overline{8}$	9
11	2	$\overline{4}$	6	$\overline{7}$	$\overline{8}$	2	4	$\overline{5}$	7	$\overline{9}$	2	$\overline{4}$	$\overline{5}$	7	8	2	$\overline{3}$	8	$\overline{9}$	10
12	2	$\overline{4}$	$\overline{6}$	8	$\overline{9}$	2	$\overline{4}$	8	9	$\overline{10}$	2	4	$\overline{7}$	8	10	2	4	$\overline{5}$	$\overline{8}$	$\overline{10}$
13	2	5	$\overline{6}$	9	$\overline{10}$	2	$\overline{5}$	6	$\overline{7}$	$\overline{8}$	2	5	6	$\overline{8}$	9	2	$\overline{4}$	6	9	$\overline{10}$
14	2	$\overline{5}$	$\overline{7}$	8	10	2	5	6	8	10	2	$\overline{5}$	6	$\overline{9}$	$\overline{10}$	2	5	$\overline{6}$	$\overline{7}$	10
15	3	$\overline{4}$	$\overline{5}$	6	7	3	$\overline{4}$	$\overline{5}$	7	10	3	4	$\overline{5}$	7	9	3	4	$\overline{5}$	$\overline{7}$	$\overline{9}$
16	3	$\overline{4}$	5	$\overline{8}$	$\overline{9}$	3	$\overline{4}$	6	7	$\overline{8}$	3	$\overline{4}$	$\overline{6}$	$\overline{8}$	9	3	$\overline{4}$	6	$\overline{7}$	8
17	3	4	$\overline{7}$	$\overline{9}$	10	3	4	$\overline{6}$	$\overline{8}$	9	3	$\overline{4}$	6	8	10	3	5	$\overline{6}$	7	$\overline{10}$
18	3	$\overline{5}$	$\overline{6}$	$\overline{8}$	$\overline{10}$	3	5	6	$\overline{9}$	$\overline{10}$	3	$\overline{5}$	$\overline{6}$	$\overline{7}$	$\overline{10}$	4	5	6	8	9

BRD #	XX				
Gibbons #	XX				
1	1	2	3	4	5
2	1	2	$\overline{3}$	$\overline{4}$	6
3	1	$\overline{2}$	5	7	8
4	1	$\overline{2}$	$\overline{6}$	$\overline{7}$	9
5	1	3	$\overline{5}$	$\overline{9}$	10
6	1	$\overline{3}$	7	9	10
7	1	4	6	$\overline{8}$	$\overline{9}$
8	1	$\overline{4}$	$\overline{7}$	8	$\overline{10}$
9	1	$\overline{5}$	$\overline{6}$	$\overline{8}$	$\overline{10}$
10	2	3	$\overline{6}$	8	10
11	2	$\overline{3}$	$\overline{7}$	8	$\overline{9}$
12	2	4	5	9	$\overline{10}$
13	2	$\overline{4}$	$\overline{8}$	9	10
14	2	$\overline{5}$	$\overline{6}$	7	$\overline{10}$
15	3	$\overline{4}$	5	$\overline{7}$	$\overline{8}$
16	3	$\overline{4}$	6	7	$\overline{10}$
17	3	$\overline{5}$	6	8	9
18	4	$\overline{5}$	6	$\overline{7}$	9