On the Smith Normal Form of Weighing Matrices*

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ABSTRACT. The Smith normal forms (SNF) of weighing matrices are studied.

We show that for all orders $n \geq 35$ the *full spectrum* of Smith normal forms (SNF) exists for weighing matrices $W(n,9)$ ie there exists a $W(n,9)$ with SNF $1^{\frac{1}{2}(n-s)}3^s9^{\frac{1}{2}(n-s)}$, for s in a set, which is described, of consecutive integers.

1 Introduction

The general question which gave rise to the present paper is the question of isomorphism of weighing matrices. Theoretically it is enough to check $(n!)^2$

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cases to find if two $n \times n$ given incidence matrices are permutation equivalent. In practice it is not feasible in any interesting case, even despite the fact that the number of comparisons can be reduced to $n!$ as was shown by M.Newman [6]. The Smith normal form may be used for a negative answer : if two incidence matrices have different Smith normal forms then they are not permutation equivalent. It is thus interesting to learn more about the Smith normal form for weighing matrices. First we introduce some notation and summarize results on the Smith normal form and weighing matrices. Let R_n be the ring of $n \times n$ matrices with entries from $\mathbf{Z}$, the ring of integers. The unit (*or* unimodular) elements of R_n are those with determinant ± 1.

Definition 1 *The matrices A, B of R_n are equivalent (or $\mathbf{Z}$- equivalent, written $B \sim A$) if $B = PAQ$ for some unimodular matrices P, Q of R_n.*

The following Theorem is due to Smith [9] and has been reworded from the theorems and proofs in MacDuffee [4, p41] and Marcus and Minc [5, p. 44].

Theorem 1 *If $A = (a_{ij})$ is any integer (or with elements from any Euclidean domain) matrix of order n and rank r, then there is a unique integer (or with elements from any Euclidean domain) matrix*

$$D = diag(a_1,\ a_2,\ \cdots,\ a_r,\ 0,\ \cdots,\ 0),$$

such that $A \sim D$ and $a_i \mid a_{i+1}$ where the a_i are non-negative. The greatest common divisor of the $i \times i$ subdeterminants of A is

$$a_1 a_2 \cdots a_i.$$

If $A \sim E$ where

$$E = diag(a_1,\ a_2,\ \cdots,\ a_i) \oplus F$$

then a_{i+1} is the greatest common divisor of the non-zero elements of F. □

Definition 2 *The a_i of theorem 1 are called the* invariants *of A and the diagonal matrix D is called the* Smith normal form (SNF).

In this paper we will study the Smith normal forms of weighing matrices.

An *orthogonal design A*, of order n, and type $(s_1, s_2, \ldots, s_u)$, denoted $OD\ (n; s_1, s_2, \ldots, s_u)$ on the commuting variables $(\pm x_1, \pm x_2, \ldots, \pm x_u, 0)$ is a square matrix of order n with entries $\pm x_k$ where each x_k occurs s_k times in each row and column such that the distinct rows are pairwise orthogonal. In other words

$$AA^T = (s_1 x_1^2 + \ldots + s_u x_u^2) I_n$$

where I_n is the identity matrix. It is known that the maximum number of variables in an orthogonal design is $\rho(n)$, the Radon number, where for $n = 2^a b$, b odd, set $a = 4c + d$, $0 \le d < 4$, then $\rho(n) = 8c + 2^d$.

A *weighing matrix* $W = W(n,k)$ is a square matrix with entries $0, \pm 1$ having k non-zero entries per row and column and inner product of distinct rows zero. Hence W satisfies $WW^T = kI_n$, and W is equivalent to an orthogonal design $OD(n;\, k)$. The number k is called the *weight* of W.

2 SNF via Kronecker Products

Lemma 1 *Suppose the SNF of a matrix* $A = W(n,k)$ *is* $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *where* $\lambda_i \mid \lambda_{i+1}$. *Then the SNF of one* $W(mn, kl)$ *is derived from*

(i) *where* $l = m = 2$ *and* $W(2n, 2k) = \begin{pmatrix} A & A \\ A & -A \end{pmatrix}$,

$$S = (\lambda_1, \lambda_2, \ldots, \lambda_n, 2\lambda_1, 2\lambda_2, \ldots, 2\lambda_n);$$

(ii) *where* $l = 3$, $m = 4$ *and*

$$W(4n, 3l) = \begin{pmatrix} 0 & A & A & A \\ A & 0 & A & -A \\ A & -A & 0 & A \\ A & A & -A & 0 \end{pmatrix},$$

$$S = (\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_1, \lambda_2, \ldots, \lambda_n, 3\lambda_1, \ldots, 3\lambda_n, 3\lambda_1, \ldots, 3\lambda_n);$$

(iii) *where* $l = 4$, $m = 4$ *and*

$$W(4n, 4l) = \begin{pmatrix} A & A & A & A \\ A & A & -A & -A \\ A & -A & A & -A \\ A & -A & -A & A \end{pmatrix},$$

$$S = (\lambda_1, \ldots, \lambda_n, 2\lambda_1, \ldots, 2\lambda_n, 2\lambda_1, \ldots, 2\lambda_n, 4\lambda_1, \ldots, 4\lambda_n),$$

by using the operations

(a) *order the elements of* S *in ascending order and write as* $(\mu_1, \ldots, \mu_s)$;

(b) *if* $\mu_i \nmid \mu_{i+1}$, $g \mid \mu_i$ *and* $g \mid \mu_{i+1}$, $g \neq 1$, *replace* μ_i *and* μ_{i+1} *by* μ_i/g *and* $g\mu_{i+1}$. *Repeat steps (a) and (b) until the* μ_i *are in ascending order and* $\mu_i \mid \mu_{i+1}$ *for each* $i = 1, \ldots, s-1$. □

Example 1 *SNF*$(W(6,4)) = (1^2\, 2^2\, 4^2)$, *so there are* $W(12,8)$ *and* $W(24,16)$ *with SNFs* $(1^2\, 2^4\, 4^4\, 8^2)$ *and* $(1^2\, 2^6\, 4^8\, 8^6\, 16^2)$ *respectively however the SNF of the* $W(24,12)$ *obtained this way will be* $(1^4\, 2^8\, 6^8\, 12^4)$.

Remark 1 *The previous Lemma can be written more generally to encompass the SNF of the Kronecker product of any two matrices. We are concerned only with small weighing matrices.*

3 The Algorithm

We used a slight variation on the usual norm algorithm which permitted both the number of digits in numbers being handled to be kept close to the number in the determinant and usually far less whilst being about $O(n^3)$ in time.

1. Find if ± 1 occurs. If so permute rows and columns until it exists in the $(1,1)$ position.
2. Use elementary row operations to ensure the first column becomes zeros except for the $(1,1)$ element. Now use elementary column operations to zero the other entries of the first row.
3. Rename the $i+1$st row as the first row. Find the *gcd* d of the elements of the present $(n-i)\times(n-i)$ matrix, A. Replace all elements a_{jk} of the matrix by a_{jk}/d. Note d divides this invariant, λ_{i+1}, and all subsequent invariants.
4. If ± 1, the *gcd* of the elements of A, does not appear in A, use elementary row/column operations to make it appear.
5. If $i < n$ go to 1.

4 Numerical Results for SNF of $W(n,9)$

We note that the negacyclic $W(20,9)$ with first row

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has SNF, $1^6 3^8 9^6$.

We use the sequences $\{x_1, x_2, -x_3, 0\}$, $\{x_1, -x_2, 0, x_4\}$, $\{x_1, 0, x_3, -x_4\}$, $\{x_2, x_3, x_4, 0\}$, which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an $OD(16;3,3,3,3)$. With the variables chosen as $x_1 = x_3 = x_4 = 1, x_2 = 0$, and as $x_1 = 0, x_2 = x_3 = x_4 = 1$ we obtain $W(16,9)$ with SNF, $1^6 3^4 9^6$.

Using these two results, those from Chan, Rodger and Seberry [1, Appendix A] and Geramita and Seberry [3] we have the results in Table 1.

(n)	(9)	$(\#)$	SNF
10	9	unique	$1^4 3^2 9^4$
11	9	not exist	
12	9	≥ 3	$1^6 9^6$
			$1^5 3^2 9^5$
			$1^4 3^4 9^4$
13	9	≥ 2	see Table 2
14	9	≥ 2	$1^6 3^2 9^6$
15	9	≥ 1	$1^7 3^1 9^7$
16	9	≥ 4	$1^8 9^8$
			$1^6 3^4 9^6$
			$1^4 3^8 9^4$
17	9	unknown	
18	9	≥ 1	$1^9 16^9$
19	9	≥ 1	$1^9 3^1 9^9$
20	9	≥ 8	$1^{10} 9^{10}$
			$1^8 3^4 9^8$
			$1^6 3^8 9^6$

Table 1: SNF for small $W(n,9)$

By processing Ohmori"s [7, p. 76] results through our computer program (after correcting a few typographical errors) we found that:

SNF	Type
$1^3 3^7 9^3$	W_1^*
$1^4 3^5 9^4$	W_4^*
$1^5 3^3 9^5$	W_2^*, W_3^*, W_7^*
$1^6 3^1 9^6$	W_5^*, W_6^*, W_8^*

Table 2: SNF for $W(13,9)$

Hence there exist $W(13,9)$ with SNF $1^{\frac{1}{2}(13-b)} 3^b 9^{\frac{1}{2}(13-b)}$ for $b = 1, 3, 5$ and 7.

We now consider how to combine known results to make larger results. By MacDuffee [4] and Marcus and Minc [5] we know that if the SNF of A is $1^a k^b (lk)^a$ and the SNF of B is $1^\alpha k^\beta (lk)^\alpha$ then the SNF of $A \oplus B$ is $1^{a+\alpha} k^{b+\beta} (lk)^{a+\alpha}$.

Example 2 *The SNF, $A_b = W(12,9)$ is $1^a 3^b 9^a$ for $b = 0, 2, 4$. Hence all the combinations, $A_{b_1} \oplus A_{b_2}$, $b_1, b_2 = 0, 2, 4$ give $W(24,9)$ with SNF $1^x 3^y 9^x$ for $x = 0, 2, 4, 6, 8$. Indeed we can ensure that there is a $W(12s,9)$ with SNF, $1^{6s-t} 3^{2t} 9^{6s-t}$, for every $2t = 0, 2, \ldots, 4s$. This shows $W(60,9)$ exist with the required SNF for every $2t = 0, 2, \ldots, 20$. However this is not the best possible.*

Notation 1 *We say* $W(n,9)$ *has a* full spectrum *for* n *if the class of* $W(n,9)$*, for a given* n *contains a member with SNF* $1^{\frac{1}{2}(n-s)}3^s9^{\frac{1}{2}(n-s)}$*, where for*

(i) $n \equiv 0(mod 4)$*, s takes the values* $s = 0, 2, 4, \dots, \sigma$*;*

(ii) $n \equiv 2(mod 4)$*, s takes the values* $s = 2, 4, 6, \dots, \sigma$*;*

(iii) $n \equiv 1, 3(mod 4)$*, s takes the values* $s = 1, 3, 5, \dots, \sigma$*.*

Definition 3 (Full Spectrum). *We will say there exists a full spectrum for a* $W(n,9)$ *if there exists a* $W(n,9)$ *with canonical form where s takes continuous values (depending on the convergence class mod 4 of n) up to the maximum value* σ *(*σ *depends on the convergence class mod 16 of n).*

Lemma 2 *If there exists a full spectrum canonical form* $W(n,9)$*, A, and a full spectrum canonical* $W(m,9)$*, B, then* $A \oplus B$ *is a full spectrum canonical* $W(m+n,9)$ *when*

(a) $n \equiv 0(mod 4)$ *and m is any value;*

(b) *except for* $s = 0$ *when* $n \equiv 2(mod 4)$ *and* $m \equiv 0(mod 4)$*;*

(c) *except for* $s = 1$ *when* $n \equiv 2(mod 4)$ *and* $m \equiv 1, 3(mod 4)$*;*

(d) *except for* $s = 0, 2$ *when* $n \equiv 2(mod 4)$ *and* $m \equiv 2(mod 4)$*;*

(e) *except for* $s = 0$ *when* $n \equiv 1(mod 4)$ *and* $m \equiv 3(mod 4)$*.*

In general if the upper bounds for the spectrum of A and B are a, b respectively the upper bound for $A \oplus B$ *will be* $a + b$*.* □

Theorem 2 (Full Spectrum Theorem). *We suppose there exist* $W(16,9)$ *with canonical SNF for* $s = 0, 1, 2, 3, 4$*. Then*
Case I: *There exist* $W(16k,9)$*,* $W(16k+4,9)$*,* $W(16k+8,9)$*,* $W(16k+12,9)$ *with canonical forms* $s = 0, 2, \dots, \sigma$ *where* $\sigma = 8k, 8k-4, 8k, 8k+4$ *respectively. The case for* $W(20,9)$ *is not yet resolved.*
Case II: *There exist* $W(16k+1,9)$*,* $W(16k+5,9)$*,* $W(16k+9,9)$*,* $W(16k+13,9)$ *with canonical forms* $s = 1, 3, \dots, \sigma$ *where* $\sigma = 8k-5, 8k-1, 8k+3$ *and* $8k+7$ *respectively. The cases for* $W(17,9)$ *and* $W(21,9)$ *are not yet resolved.* $W(13,9)$ *exists with canonical forms* $s = 1, 3, 5, 7$*.*
Case III: *There exist* $W(16k-5,9)$*,* $W(16k-1,9)$*,* $W(16k+3,9)$*,* $W(16k+7,9)$ *with canonical forms* $s = 1, 3, \dots, \sigma$ *where* $\sigma = 8k-7, 8k-7, 8k-3$ *and* $8k+3$ *respectively. The cases for* $W(15,9)$*,* $W(19,9)$ *and* $W(23,9)$ *have not yet been resolved. A* $W(11,9)$ *does not exist.*

Case IV: *There exist* $W(16k-6,9)$, $W(16k-2,9)$, $W(16k+2,9)$, *and* $W(16k+6,9)$ *with canonical forms* $for s = 2,4,6,\dots,\sigma$ *where* $\sigma = 8k-2, 8k-6, 8k-2$ *and* $8k+2$ *respectively.* $W(10,9)$ *is an exception, and the cases* $W(4n+2,9)$, $n = 3,4,5,8$ *are unresolved.*

Proof. Case I: We first note from Chan, Rodger and Seberry [1] as noted in our Table 1 that a $W(12,9)$ with canonical SNF for $s = 0,2,4$ exists. We now use Lemma 2 to establish the following results

label	order	$16k+4i$	method	number
W_{12}	12	$0,2,4$	construction	3
W_{16}	16	$0,2,4,6,8$	construction	5
W_{20}	20	$0,4$	construction	3
W_{24}	24	$0,2,4,6,8$	$W_{12}\oplus W_{12}$	5
W_{28}	28	$0,\dots,12$	$W_{12}\oplus W_{16}$	7
W_{32}	32	$0,\dots,16$	$W_{16}\oplus W_{16}$	9
W_{36}	36	$0,\dots,12$	$W_{12}\oplus W_{24}$	7
W_{40}	40	$0,\dots,16$	$W_{16}\oplus W_{24}$	9
W_{44}	44	$0,\dots,20$	$W_{16}\oplus W_{28}$	11

We use the theorem is true for $k = 2$. We assume it is true for $k = m$ so that there are weighing matrices with orders $16m$, $16m+4$, $16m+8$ and $16m+12$ with canonical SNF's for $s = 0,2,\dots,8m$, $s = 0,2,\dots,8m-4$, $s = 0,2,\dots,8m$ and $s = 0,2,\dots,8m+4$ respectively. We use Lemma 2 and $\oplus W(16,9)$ to obtain the orders $16(m+1)$, $16(m+1)+4$, $16(m+1)+8$ and $16(m+1)+12$ with the required canonical SNF's respectively.

Thus the result is true for $k = 2$ and assuming it is true for $k = m$ allows us to prove the result for $k = m+1$ and thus we have the result by induction.

Case II: As in Case I we first establish some cases . We first use

label	order	$4k+i$	method	number
W_{12}	12	$0,2,4$	construction	3
W_{13}	13	$1,3,5,7$	construction	4
W_{16}	16	$0,2,4,6,8$	construction	5
W_{20}	20	$0,4$	construction	≥ 2
W_{24}	24	$0,2,4,6,8$	above	5
W_{25}	25	$1,3,\dots,11$	$W_{12}\oplus W_{13}$	6
W_{29}	29	$1,3,\dots,15$	$W_{13}\oplus W_{16}$	8
W_{33}	33	$1,3,\dots,11$	$W_{13}\oplus W_{20}$	6
W_{37}	37	$1,3,\dots,15$	$W_{24}\oplus W_{13}$	8
W_{41}	41	$1,3,\dots,19$	$W_{25}\oplus W_{16}$	10
W_{45}	45	$1,3,\dots,23$	$W_{29}\oplus W_{16}$	12

We observe the results for $W_{33}, W_{37}, W_{41}, W_{45}$ fit the enunciation for $k = 2$ and that adding $\oplus\ W_{16}$ to each of these gives the enunciation for $k = 3$. The proof is obtained by induction.

Case III: As in Cases I and II we first establish some results. We first use

label	order	$4k+i$	method	number
W_{22}	22	$2,4,6$	$W_{10} \oplus W_{12}$	≥ 3
W_{27}	27	$1,3,5,7,9$	$W_{15} \oplus W_{12}$	
			and	5
			$W_{14} \oplus W_{13}$	
W_{31}	31	$1,3,\ldots,9$	$W_{15} \oplus W_{16}$	5
W_{35}	35	$1,3,\ldots,13$	$W_{15} \oplus W_{20}$	
			and	7
			$W_{22} \oplus W_{13}$	
W_{39}	39	$1,3,\ldots,21$	$W_{13} \oplus W_{13} \oplus W_{13}$	
			and	10
			$W_{24} \oplus W_{15}$	

As before, we observe the results for W_{27}, W_{31}, W_{35} and W_{39} are sufficient with W_{16} to start the induction process.

Case IV: Again we establish some initial cases

label	order	$4k+i$	method	number
W_{10}	10	2	construction	1
W_{14}	14	2	construction	≥ 1
W_{18}	18	2	construction	≥ 1
W_{22}	22	$2,4,6$	$W_{12} \oplus W_{10}$	≥ 3
W_{26}	26	$2,4,\ldots,14$	$W_{13} \oplus W_{13}$	7
W_{30}	30	$2,4,\ldots,10$	$W_{14} \oplus W_{16}$	5
W_{34}	34	$2,4,\ldots,10$	$W_{10} \oplus W_{24}$	≥ 5
W_{38}	38	$2,4,\ldots,18$	$W_{25} \oplus W_{13}$	9
W_{42}	42	$2,4,\ldots,22$	$W_{26} \oplus W_{16}$	11
W_{46}	46	$2,4,\ldots,18$	$W_{30} \oplus W_{16}$	9
W_{50}	50	$2,4,\ldots,22$	$W_{25} \oplus W_{25}$	11
W_{54}	54		$W_{38} \oplus W_{16}$	13

As before the values for W_{42}, W_{46}, W_{50} and W_{54} together with W_{16} are sufficient to establish the induction for the results of the enunciation. □

Remark 2 *Although the cases for $W(20,9)$ is not yet resolved matrices are known with SNF, $1^{10}9^{10}$ and $1^{8}3^{4}9^{8}$. We conjecture that a $W(20,9)$ with SNF, $1^{9}3^{2}9^{9}$ exists and order 20 also satisfies the theorem.*

Remark 3 *The results of Table 1 for all values except 13 and 16 remain undecided as does the situation in orders 21, 22 and 34. For all the other* *orders (provided all cases for 16 exist) are resolved with a full spectrum of SNF's showing the number of equivalences class approaches ∞ as the order approaches ∞.*

Remark 4 *This theorem ensures there is a $W(60,9)$ with canonical SNF for $s = 0,\ldots,14$, considerably improving our previous estimate of $s = 0,\ldots,10$.*

5 SNF and Projective Planes

In Seberry Wallis [10, p. 411] we note the following theorem:

Theorem 3 *The incidence matrix of a (v,k,λ)-configuration is equivalent to a diagonal matrix with entries*

$$\begin{cases} 1 & \frac{v+1}{2} \text{ times}, \\ (k-\lambda) & \frac{v-3}{2} \text{ times}, \\ k(k-\lambda) & \text{once}, \end{cases}$$

when $k-\lambda$ is squarefree and $(k-\lambda, k) = 1$. □

We apply this theorem to $SBIBD(q^2+q+1, q+1, 1)$. These always exist when q is a prime power.

Lemma 3 *The incidence matrix of a $SBIBD(q^2+q+1, q+1, 1)$ is equivalent to a diagonal matrix with entries*

$$\begin{cases} 1 & \frac{1}{2}(q^2+q+2) \text{ times}, \\ q & \frac{1}{2}(q^2+q-2) \text{ times}, \\ q(q+1) & \text{once}, \end{cases}$$

when q is squarefree. □

We found the following results using cyclic difference sets, D, from Ryser [8, p. 132].

q	SBIBD	D	SNF
2	(7,3,1)	$\{0,1,3\}$	$1^4 2^2 6$
3	(13,4,1)	$\{0,1,3,9\}$	$1^7 3^5 12$
5	(31,6,1)	$\{0,1,3,8,12,18\}$	$1^{16} 5^{14} 30$
7	(57,8,1)	$\{0,1,3,13,32,36,43,52\}$	$1^{29} 7^{27} 56$
$4=2^2$	(21,5,1)	$\{0,1,4,14,16\}$	$1^{10} 2^2 4^8 20$
$8=2^3$	(73,9,1)	$\{0,1,3,7,15,31,36,54,63\}$	$1^{28} 2^9 4^9 8^{26} 72$

Future work will be concerned with SNF of inequivalent projective planes and the cases where q is not squarefree.

6 Circulant Weighing Matrices

It is shown in Geramita and Seberry [3, p. 153] that if Q is the circulant incidence matrix of a projective plane of order q then $W = Q^2 - J$, where J is the matrix of all ones, is a circulant $W(q^2+q+1, q^2)$. In order to search for relationships between the SNF of the $SBIBD(q^2+q+1, q+1, 1)$ and its complement signed, which is a $W(q^2+q+1, q^2)$, we found the SNF of the corresponding weighing matrix.

q	$W(q^2+q+1, q^2)$	SNF
2	$W(7,4)$	$1^3 2 4^3$
3	$W(13,9)$	$1^3 3^7 9^3$
4	$W(21,16)$	$1^9 4^3 16^9$
5	$W(31,25)$	$1^{15} 5^1 25^{15}$
7	$W(57,49)$	$1^{10} 7^{37} 49^{10}$
8	$W(73,64)$	$1^{22} 8^{29} 64^{22}$

In addition we found the circulant $W(31,16)$ with first row

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had SNF, $1^5 2^5 4^{11} 8^5 16^5$ and the negacyclic $W(20,9)$ with first row

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had SNF, $1^6 3^8 9^6$.

As yet we have been unable to establish a theoretical link between them.

7 Further Numerical Results

Remark 5 *We took the* $OD(16; 1,1,1,1,2,2,2,2,2)$ *in Geramita and Seberry [3, p. 361] and used all possible substitutions for the variables to obtain a* $W(16,9)$. *Every one of these* $W(16,9)$ *had SNF,* $1^8 9^8$. □

Remark 6 *We took the same* $OD(16; 1,1,1,1,2,2,2,2,2)$ *to obtain a* $W(16,8)$. *With the variables chosen as* $x_1 = x_2 = x_3 = x_4 = 1$, $x_5 = x_6 = x_7 = 0$ *and* $x_8 = x_9 = 1$ *we obtained a* $W(16,8)$ *with SNF,* $1^6 2^2 4^2 8^6$ *which is new. When the variables were chosen as* $x_1 = x_2 = 1$, $x_3 = x_4 = x_5 = x_6 = 0$, $x_7 = x_8 = x_9 = 1$ *the* $W(16,8)$ *so obtained had SNF,* $1^8 8^8$. *When the variables were chosen as*

$$x_1 = x_2 = x_3 = x_4 = x_5 = x_7 = x_8 = x_9 = 1, \quad x_6 = 0$$

$$x_1 = x_2 = x_3 = x_4 = x_6 = x_7 = x_8 = x_9 = 1, \quad x_5 = 0$$

the $W(16,12)$ *so obtained had SNF,* $1^8 12^8$ *and* $1^6 2^2 6^2 12^6$ *respectively.* □

When the variables of the $OD(16; 1, 1, 1, 1, 1, 1, 5, 5)$ were chosen using all possible combinations which gave a $W(16, 9)$ all yielded the SNF, $1^8 9^8$. Also choosing $x_3 = x_4 = 0$, $x_7 = x_8 = 1$ with $x_1 = 1$, $x_2 = 0$, $x_5 = 0$, $x_6 = 1$ and with $x_1 = 0$, $x_2 = 1$, $x_5 = 1$, $x_6 = 0$ gave $W(16, 12)$ with SNF, $1^8 12^8$.

Working with the $OD(16; 1, 1, 1, 1, 2, 2, 4, 4)$ and choosing $x_1 = x_6 = 1$, $x_2 = 0$ with $x_3 = x_4 = x_8 = 1$, $x_5 = x_7 = 0$ and with $x_3 = x_4 = x_8 = 0$, $x_5 = x_7 = 1$ both gave $W(16, 9)$ with SNF, $1^8 9^8$. Choosing $x_1 = x_4 = x_6 = x_7 = x_8 = 1$, $x_2 = x_3 = x_5 = 0$ gave a $W(16, 12)$ with SNF, $1^8 12^8$.

Remark 7 *We use the sequences* $\{x_1, x_2, x_3, 0\}$, $\{-x_2, x_1, x_4, 0\}$, $\{-x_3, -x_4, x_1, 0\}$, $\{-x_4, x_3, -x_2, 0\}$, *which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an* $OD(16; 3, 3, 3, 3)$. *With the variables chosen as* $x_1 = 0, x_2 = x_3 = x_4 = 1$, *with* $x_2 = 0, x_1 = x_3 = x_4 = 1$, *with* $x_3 = 0, x_1 = x_2 = x_4 = 1$, *and with* $x_4 = 0, x_1 = x_2 = x_3 = 1$ *we obtained a* $W(16, 9)$ *with SNF,* $1^6 3^4 9^6$, $1^8 9^8$, $1^8 9^8$, *and* $1^6 3^4 9^6$ *respectively. When the variables were chosen as* $x_1 = x_2 = x_3 = x_4 = 1$ *the* $W(16, 12)$ *so obtained had SNF,* $1^6 2^2 6^2 12^6$. □

Remark 8 *We use the sequences* $\{x_4, x_1, -x_4, 0\}$, $\{x_4, x_2, -x_4, 0\}$, $\{x_4, x_3, x_4, 0\}$, $\{x_4, -x_3, x_4, 0\}$, *which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an* $OD(16; 1, 1, 2, 8)$. *With the variables chosen as* $x_1 = x_4 = 1, x_2 = x_3 = 0$, *and with* $x_1 = x_3 = 0, x_2 = x_4 = 1$ *we obtained a* $W(16, 9)$ *with the same SNF,* $1^8 9^8$. *When the variables were chosen as* $x_1 = x_2 = x_3 = x_4 = 1$ *the* $W(16, 12)$ *so obtained had SNF,* $1^6 2^2 6^2 12^6$. □

Remark 9 *We use the sequences* $\{x_1, 0, 0, 0\}$, $\{x_2, 0, 0, 0\}$, $\{x_3, x_3, x_4, -x_4\}$, $\{x_4, x_4, -x_3, x_3\}$, *which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an* $OD(16; 1, 1, 4, 4)$. *With the variables chosen as* $x_1 = x_3 = x_4 = 1, x_2 = 0$, *and with* $x_1 = 0, x_2 = x_3 = x_4 = 1$ *we obtained a* $W(16, 9)$ *with the same SNF,* $1^8 9^8$. □

Remark 10 *We use the sequences* $\{x_2, x_1, -x_2, 0\}$, $\{x_2, 0, x_2, 0\}$, $\{x_3, x_4, 0, 0\}$, $\{x_3, -x_4, 0, 0\}$, *which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an* $OD(16; 1, 2, 2, 4)$. *When the variables were chosen as* $x_1 = x_2 = x_3 = x_4 = 1$ *the* $W(16, 9)$ *so obtained had SNF,* $1^8 9^8$. □

Remark 11 *We use the sequences* $\{x_1, x_2, x_3, 0\}$, $\{x_1, x_2, -x_3, 0\}$, $\{x_2, -x_1, x_2, 0\}$, $\{x_2, x_4, -x_2, 0\}$, *which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an* $OD(16; 1, 2, 3, 6)$. *With the variables chosen as*

$x_1 = x_2 = 1, x_3 = x_4 = 0$, *and with* $x_1 = 0, x_2 = x_3 = x_4 = 1$ *we obtained a* $W(16,9)$ *with SNF,* $1^6 3^4 9^6$, *and* $1^8 9^8$ *respectively. When the variables were chosen as* $x_1 = x_4 = 0, x_2 = x_3 = 1$ *the* $W(16,8)$ *so obtained had SNF,* $1^6 2^2 4^2 8^6$. *When the variables were chosen as* $x_1 = x_2 = x_3 = x_4 = 1$ *the* $W(16,12)$ *so obtained had SNF,* $1^6 2^2 6^2 12^6$. □

Remark 12 *We use the sequences* $\{x_2, x_1, -x_2, 0\}$, $\{x_2, 0, x_2, 0\}$, $\{x_3, x_3, x_4, -x_4\}$, $\{x_4, x_4, -x_3, x_3\}$, *which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an* $OD(16; 1, 4, 4, 4)$. *With the variables chosen as* $x_1 = x_2 = x_3 = 1, x_4 = 0$, *with* $x_1 = x_2 = x_4 = 1, x_3 = 0$, *and with* $x_1 = x_3 = x_4 = 1, x_2 = 0$ *we obtained a* $W(16,9)$ *with the same SNF,* $1^8 9^8$. *With the variables chosen as* $x_1 = x_4 = 0, x_2 = x_3 = 1$, *with* $x_1 = x_3 = 0, x_2 = x_4 = 1$, *and with* $x_1 = x_2 = 0, x_3 = x_4 = 1$ *we obtained a* $W(16,8)$ *with SNF,* $1^6 2^2 4^2 8^6$, $1^6 2^2 4^2 8^6$, *and* $1^2 2^6 4^6 8^2$ *respectively. When the variables were chosen as* $x_1 = 0, x_2 = x_3 = x_4 = 1$ *the* $W(16,12)$ *so obtained had SNF,* $1^4 2^4 6^4 12^4$. □

Remark 13 *We use the sequences* $\{x_1, x_2, -x_3, 0\}$, $\{x_1, -x_2, 0, x_4\}$, $\{x_1, 0, x_3, -x_4\}$, $\{x_2, x_3, x_4, 0\}$, *which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an* $OD(16; 3, 3, 3, 3)$. *With the variables chosen as* $x_1 = x_2 = x_3 = 1, x_4 = 0$, *with* $x_1 = x_2 = x_4 = 1, x_3 = 0$, *with* $x_1 = x_3 = x_4 = 1, x_2 = 0$, *and with* $x_1 = 0, x_2 = x_3 = x_4 = 1$ *we obtained a* $W(16,9)$ *with SNF,* $1^8 9^8$, $1^8 9^8$, $1^6 3^4 9^6$, *and* $1^6 3^4 9^6$ *respectively. When the variables were chosen as* $x_1 = x_2 = x_3 = x_4 = 1$ *the* $W(16,12)$ *so obtained had SNF,* $1^8 12^8$. □

Remark 14 *We use the sequences* $\{x_1, x_2, 0, 0\}$, $\{x_2, -x_1, 0, 0\}$, $\{x_3, x_4, -x_4, x_3\}$, $\{x_3, x_4, x_4, -x_3\}$, *which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an* $OD(16; 2, 2, 4, 4)$. *With the variables chosen as* $x_1 = x_2 = 0, x_3 = x_4 = 1$, *with* $x_1 = x_2 = x_3 = 1, x_4 = 0$, *and with* $x_1 = x_2 = x_4 = 1, x_3 = 0$ *we obtained a* $W(16,8)$ *with SNF,* $1^2 2^6 4^6 8^2$, $1^5 2^3 4^3 8^5$, *and* $1^5 2^3 4^3 8^5$ *respectively. When the variables were chosen as* $x_1 = x_2 = x_3 = x_4 = 1$ *the* $W(16,12)$ *so obtained had SNF,* $1^6 2^2 6^2 12^6$. □

Remark 15 *We use the sequences* $\{x_1, x_2, -x_2, x_1\}$, $\{x_1, x_2, x_2, -x_1\}$, $\{x_3, x_4, -x_4, x_3\}$, $\{x_3, x_4, x_4, -x_3\}$, *which have zero non-periodic autocorrelation function, as the first rows of the corresponding circulant matrices in the Goethals-Seidel array to obtain an* $OD(16; 4, 4, 4, 4)$. *With the variables chosen as* $x_1 = x_2 = 1, x_3 = x_4 = 0$, *and with* $x_1 = x_2 = 0, x_3 = x_4 = 1$ *we obtained a* $W(16,8)$ *with the same SNF,* $1^2 2^6 4^6 8^2$. *With the variables chosen as* $x_1 = x_3 = 1, x_2 = x_4 = 0$, *with* $x_1 = x_4 = 1, x_2 = x_3 = 0$, *with* $x_1 = x_4 = 0, x_2 = x_3 = 1$, *and with* $x_1 = x_3 = 0, x_2 = x_4 = 1$ *we obtained a* $W(16,8)$ *with the same SNF,* $1^5 2^3 4^3 8^5$. *With the variables chosen as* $x_1 = x_2 = x_3 = 1, x_4 = 0$, *with* $x_1 = x_2 = x_4 = 1, x_3 = 0$, *with* $x_1 = x_3 = x_4 = 1, x_2 = 0$, *and with* $x_2 = x_3 = x_4 = 1, x_1 = 0$ *we obtained a* $W(16,12)$ *with the same SNF,* $1^6 2^2 6^2 12^6$. □

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