Minimal and Maximal Critical Sets in Room Squares

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Abstract

In this paper we introduce critical sets in Room squares. We give the cardinality of the minimal critical sets (*min. cs*) and maximal critical sets (*max. cs*) for inequivalence classes of Room squares of side 7, 9 and 11. We also describe algorithms to compute *min. cs* and *max. cs* and conjecture the lower and upper bounds for *min. cs* and *max. cs*.

1 Introduction

A number of authors have studied the minimum amount of information needed to recreate combinatorial structures. Critical sets in Latin squares have been studied by (Nelder [15]), (Curran and Van Rees [5]), (Smetaniuk [18]), (Stinson and Van Rees [20]) and (Cooper, Donovon and Seberry [4]). Minimum defining sets of combinatorial designs (see for example [21]) have been studied by Street, Sarvate, Kunkle, Seberry et al. Critical sets have a number of applications in both agriculture and cryptography. This research has been motivated by studies of secret sharing schemes by Cooper et al., key distribution schemes by Merkle (PhD thesis) and some problems in the design of experiments. In this paper we widen the structures considered to include critical sets in Room squares.

To find the true minimum information needed to reconstruct a Room square, we need to have the minimum of structure possible imposed on the Room square. However, for comparative purposes, it is valuable to compare the size of the least critical set under a number of scenarios.

A *Room square* R of side r is an $r \times r$ array each of whose cells may either be empty or contain an unordered pair of objects $0, 1, 2, ..., r$, subject to the following conditions :

(i) each of the objects $0, 1, 2, ..., r$ occurs precisely once in each row of R and precisely once in each column of R, and

(ii) every possible unordered pair of objects occurs precisely once in the whole array.

A *partial Room square* P of side r is an $r \times r$ array each of whose cells may either be empty or contain an unordered pair of objects $0, 1, 2, ..., r$, subject to the following conditions :

(i) each of the objects $0, 1, 2, ..., r$ occurs at most once in each row of R and at most once in each column of R, and

(ii) every possible unordered pair of objects occurs at most once in the whole array.

Theorem 1 (Mullin and Wallis [14]) *There exists a Room square of every odd integer side r greater than or equal to 7.*

Theorem 2 (Dinitz and Stinson [8]) *A Room square of side r is equivalent to a pair of orthogonal one-factorizations of a complete graph.*

We denote $N_r = 0, 1, 2, ...r$. A Room square of side r based on N_r is *standardized* if the ith diagonal cell, cell(i, i), contains $\{0, i\}$ for $1 \leq i \leq r$.

Theorem 3 (Wallis [23]) *If there is a Room square of side r, then there is a standardized Room square of side r.*

A *skew* Room square R [22] is a Room square with the property that when $i \neq j$, either the (i, j) or the (j, i) cell of R is occupied, but not both.

Theorem 4 (Dinitz and Stinson [8]) *There is a skew Room square of side r, if and only if r is odd and $r \neq 3$ or 5.*

Two Room squares are said to be *isomorphic* [23] if one can be obtained from the other by permuting the rows and columns and relabeling the elements. Two Room squares R and S are *equivalent* if R is isomorphic to S or to the transpose of S. All inequivalent Room squares of size 7 and 9 have been found from one-factorizations of the complete graph. There are exactly 6 inequivalent Room squares of side 7, see [11, 12, 23] and 257,630 Room squares for side 9, see [1, 8, 9]. No exact number is known for higher order Room squares. Although there are :

526,915,620 non-isomorphic one-factorizations of order 11 ($K12$)
9.876×10^{28} distinct one-factorizations of order 13 ($K14$)
1.148×10^{44} distinct one-factorizations of order 15 ($K16$)
1.520×10^{63} distinct one-factorizations of order 17 ($K18$)

and the number of possible Room squares for each of these orders should be even larger, see [6] for further details.

2 Critical Sets in Room Squares

There are many ways to define *critical sets* in Room squares. One of the definitions is :

A *critical set* $Q = [Q_1, Q_2, Q_3, ..., Q_c]$, $|Q| = c$, in a Room square R of side r, is a set of quadruples $Q_a = [i, j; k, l]$ such that if any Q_a is removed from the set, it can no longer be uniquely completed, see [18, 5]. In Q_a, (i, j) shows the position of the pair (k, l) in the square. That is, Q provides minimal information from which R can be reconstructed uniquely. In this paper, we consider that empty positions in the square are given.

The set S is said to be a *strong critical set* of a Room square R of side r if there exists a set $\{P_1, P_2, P_3, ..., P_m\}$ of $m = r^2 - |S|$ partial Room squares, of order r, such that:

(1) $S \subseteq P_1 \subseteq P_2 \subseteq ...P_m \subseteq R$.

(2) for any s, with $2 \leq s \leq m$, where $P_s = P_{s-1} \cup \{(i,j;k,l)\}$, the set $P_{i-1} \cup \{(i,j;\acute{k},\acute{l})\}$ is not a partial Room square for any pair $(\acute{k},\acute{l}) \in N_r \backslash (k,l)$, where $N_r = 0,1,2,3,...r$.

A *minimal critical set (min. cs)* of a Room square R of side r is a critical set of minimum cardinality whereas a *maximal critical set (max. cs)* is a critical set of maximum cardinality.

Q^* is the *smallest critical set (scs)* if $|Q^*| = c$ is minimum for all critical sets Q of the Room squares of order r whereas Q^* is the *largest critical set (lcs)* if $|Q^*| = c$ is the largest for all critical sets Q of the Room squares of order r, see [4, 5, 20].

Throughout this paper, we will deal with strong critical sets only. So we will use the term "critical set" to represent the "strong critical set", min. cs to min. strong cs and max. cs to max. strong cs.

Lemma 1 *A Room square R of side r is written using $r+1$ elements. If any row i and column j have $r-1$ elements already known, then if the (i,j)th entry is non-empty, it is known.*

Lemma 2 *A Room square R of side r is written using $r+1$ elements. If in row i and column j, $(r+1-k)$ elements are known and $\binom{k}{2} - 1$ of the pairs, on the k unknown entries, already exist in the square, then if the (i,j)th entry is non-empty, it is known.*

Lemma 3 *In large critical sets of Room squares, every large critical set has at least one row or one column with $(\frac{r-1}{2})$ entries known.*

Remark Above three lemmas are easy to prove and are useful for computing critical sets of Room squares.

Example A Room square of side 7 and its critical set.

A Room square of side 7 :

01	-	45	67	-	-	23
57	02	-	-	-	13	46
-	56	03	12	-	47	-
-	37	-	04	26	-	15
36	14	27	-	05	-	-
24	-	-	35	17	06	-
-	-	16	-	34	25	07

Critical set *(cs)* of this square is:

*	-	45	*	-	-	23
*	*	-	-	-	13	46
-	*	03	12	-	47	-
-	*	-	*	26	-	15
*	*	*	-	*	-	-
*	-	-	*	*	*	-
-	-	*	-	*	*	*

where "$*$" shows the unknown pair positions and "$-$" shows empty positions in the square.

So the critical set for the above Room square is: $Q = \{(1,3;45), (1,7;23), (2,6;13), (2,7;46), (3,3;03), (3,4;12), (3,6;47), (4,5;26), (4,7;15)\}$. That is, Q is the only Room square of side 7 with a pair 45 at position (1,3), pair 23 at position (1,7), pair 13 at position (2,6), pair 46 at position (2,7), pair 03 at position (3,3), pair 12 at position (3,4), pair 47 at position (3,6), pair 26 at position (4,5) and pair 15 at position (4,7).

2.1 Cardinality of Critical Sets

We have computed the minimal *(min. cs)* and maximal *(max. cs)* critical sets for all inequivalence classes of Room squares of side 7 and some for sides 9 and 11. A few examples of *min. cs* and *max. cs* are given in Appendix I. We summarize our results in table 1 below. In this table, all the Room squares of order 7 have been taken from [23] and the Room squares of order 9 and 11 are given in Appendix II.

Table 1

Order of Room square	Class Representative	*Cardinality*	
		max. cs	min. cs
7	R11	10	8
	R14	10	8
	R15	10	8
	R16	10	8
	R44	10	8
	R45	10	8
7 Skew	R11	10	8
	R15$\tilde{6}$	10	8
	R45$\tilde{5}$	11	8
9	R9_1	17	15
	R9_2	16	13
	R9_3	16	14
	R9_4	15	14
	R9_5	14	12
	R9_6	16	15
9 Skew	R9_1	17	15
11	R11_1	24	20
	R11_2	26	22
	R11_3	25	21

3 Algorithms for constructing *min. cs* and *max. cs*

3.1 Algorithm for *min. cs*

Suppose we have a Room square R of side r, take an empty square Q corresponding to R.

1) for each row of R, mask off (delete) one non-zero pair randomly.

2) for each column of R, mask off (delete) one non-zero pair randomly.

3) for each row of R, process the following steps:

  4) for each column of the row, process the following steps:

    5) if a pair exists at the intersection of the current column and row, then mask off that pair.

    6) create set of known pairs.

    7) call *check_crit_set* routine.

    8) if the known pair set is complete, then continue else put that pair back in R and also move that pair into Q.

9) Q is the small critical set for this square.

10 call *min_crit_set* routine to obtain the minimal critical set.

3.2 Algorithm for *max. cs*

1) Suppose we have a Room square R of side r, take empty squares S and Q corresponding to R.

2) take $[\frac{1}{2}(r-1)-1]$ pairs from a row of R, move these pairs to Q, construct that row of S by calling *check_crit_set* routine.

  3) take another pair from R randomly, which is still unknown in S, to compute more unknown pairs of S, move that pair to Q too and

    4) call *check_crit_set* routine to find more unknown pairs of S.

5) repeat steps 3 and 4 until S is complete, then Q is the large critical set.

6 call *max_crit_set* routine to compute the maximal critical set.

3.3 *check_crit_set* routine

1) initialize known and unknown pairs.

2) for each row of R from top, process steps 3-8:

  3) check for any known pairs, if there are none, process the next row and go to step 2.

  4) form known numbers.

  5) for each unknown pair position, construct known numbers for its column.

  6) take the union of the row numbers with each of the column numbers (call it set Z)

  7) take the intersection for each element of Z with all the numbers from 0 to r.

  8) if after taking the intersection only two numbers are left, then add them to the known set and go to step 2 otherwise make all pairs of the remaining numbers and discard the pairs which are in the known set, after discarding, if one pair is left, add it to the known set and go to step 2 otherwise go to step 5.

3.4 *min_crit_set* routine

1) If Q is a small critical set of a Room square R of side r, we consider all possible combinations each consisting of $(Q-1)$ non-zero entries from the Room square.

2) for each partial Room square consisting of $(Q-1)$ entries:

   3) call *chk_crit_set* routine to try to complete the partial square.

4) If any of above partial squares is uniquely completed, then $(Q-1)$ is the new small critical set, replace the original Q set by the new set containing $|Q-1|$ entries, rename it as Q and now repeat the process from step 1.

5) If none can be completed, then the minimal critical set for that square is the current Q.

3.5 *max_crit_set* routine

1) If Q is a large critical set of a Room square R of side r, we consider all possible combinations each consisting of $(Q+1)$ non-zero entries from the square.

2) for each combination set consisting of $(Q+1)$ entries, process following steps:

   3) for each partial Room square consisting of Q entries from the $(Q+1)$ entries in step (2), process the following steps:

      4) call *chk_crit_set* routine to try to complete the partial square.

      5) If the square is not completed, then repeat steps 3 and 4 otherwise discard this set containing $(Q+1)$ entries, and go to step 2.

   6) If none of the Q subsets is a critical set, then call *chk_crit_set* routine to complete the partial Room square consisting of $(Q+1)$ entries in step 3.

7) If any $(Q+1)$ entries complete the square uniquely, then $(Q+1)$ is the new large critical set for that square, replace the original Q set by the new set containing $|Q+1|$ entries, rename it as Q and now repeat the process from step 1.

8) If none can be completed, then the maximal critical set for that square is the current Q.

4 Complexity

All the examples of critical sets mentioned in this paper are strong critical sets, and therefore convenient to use, in the sense that the completion to a Room square is forced. But these are by no means the only critical sets. There may exist critical sets which cannot be completed by forcing. To consider this further, it is convenient to use the *defect graph*, $G(P)$, of a partial Room square P, defined as follows. $G(P)$ has one vertex for each incomplete row of P, one for each incomplete column, and one for each pair(entry) which is still available for placement in the completion of P. If the (i,j) position of P is unknown, the $G(P)$ contains the edge (r_i, c_j); if the pair (k,l) does not appear in row i of P, the $G(P)$ contains the edge (r_i, e_{kl}) or if the pair (k,l) does not appear in column j of P, then $G(P)$ contains the edge (c_j, e_{kl}).

Finding a completion of P is equivalent to finding a decomposition of $G(P)$ into edge-disjoint triangles. Using this observation, it can be shown that, in general, completing partial Room squares is NP-complete, even if both the partial square and its completion are required to be symmetric. Further, given a partial Room square and one completion, deciding whether a second completion exists is also NP-complete. These results have been deduced from Colbourn

[2], Colbourn, Colbourn and Stinson [3], and Street [21] which apply to the complexity of critical sets in Latin squares. For general background in this area, see Garey and Johnson's book [10].

5 Conjecture for Lower and Upper Bounds

On the basis of our results, we conjecture that for a Room square R of side r, the lower and upper bounds for the minimal and maximal critical sets are :

(*i*) $\quad min.\ cs(r) \leq 8 + \frac{1}{8}(r-7)(r+15)$

(*ii*) $\quad max.\ cs(r) \geq 7 + \frac{3}{8}(r-3)(r-5)$

6 Further Research

- Our algorithms use exhaustive search techniques to look for minimal and maximal critical sets in Room squares. The algorithms are not efficient for larger Room squares. The algorithms need to be optimized or more efficient algorithms invented.
- There is still not much known about critical sets in Room squares, so conjectures for their bounds are rough and research is needed to obtain theoretical bounds.
- Algorithm to find infinite families of critical sets as well as the smallest and largest ones need to be devised.
- There is no algorithm to find critical sets of a Room square when there are no a priori constraints on the square.

Appendix I

Following are some of the examples of the minimal and maximal critical sets of Room squares. In the critical sets, '**' represents unknown pair positions and '–' shows the empty positions in the squares.

Example 1 : A skew Room square of side 7 (R11)

81	-	45	67	-	-	23
57	82	-	-	-	13	46
-	56	83	12	-	47	-
-	37	-	84	26	-	15
36	14	27	-	85	-	-
24	-	-	35	17	86	-
-	-	16	-	34	25	87

Minimal and Maximal critical sets for this square are :

**	–	**	**	–	–	**
**	8,2	–	–	–	**	**
–	**	8,3	1,2	–	**	–
–	3,7	–	8,4	**	–	**
**	**	**	–	8,5	–	–
**	–	–	**	1,7	8,6	–
–	–	**	–	**	**	**

8,1	–	4,5	6,7	–	–	**
5,7	**	–	–	–	**	4,6
–	**	8,3	1,2	–	**	–
–	**	–	**	**	–	1,5
3,6	**	**	–	**	–	–
**	–	–	**	**	8,6	–
–	–	**	–	**	**	**

Example 2 : A symmetrical Room square of side 7

8,1	3,4	6,7	2,5	–	–	–
4,6	1,5	–	–	–	3,7	8,2
2,7	–	1,4	–	8,3	–	5,6
3,5	–	–	1,7	2,4	8,6	–
–	–	8,5	3,6	–	1,2	4,7
–	2,6	–	8,4	5,7	–	1,3
–	8,7	2,3	–	1,6	4,5	–

Minimal and maximal critical sets for this square are :

**	3,4	**	2,5	–	–	–
**	1,5	–	–	–	3,7	8,2
**	–	1,4	–	8,3	–	**
**	–	–	**	**	8,6	–
–	–	**	**	–	**	4,7
–	**	–	**	**	–	**
–	**	**	–	**	**	–

8,1	3,4	6,7	**	–	–	–
4,6	1,5	–	–	–	**	**
2,7	–	1,4	–	**	–	**
**	–	–	1,7	**	**	–
–	–	**	3,6	–	**	4,7
–	2,6	–	**	**	–	**
–	**	**	–	**	**	–

Example 3 : Room Square of side 9 (R9_1)

10,1	–	4,9	3,7	2,8	–	5,6	–	–
8,9	10,2	–	–	–	5,7	3,4	–	1,6
–	5,8	10,3	–	6,9	2,4	–	1,7	–
–	3,6	7,8	10,4	–	1,9	–	2,5	–
–	7,9	–	1,2	10,5	3,8	–	4,6	–
4,5	–	–	–	–	10,6	1,8	3,9	2,7
–	–	2,6	5,9	1,3	–	10,7	–	4,8
6,7	1,4	–	–	–	–	2,9	10,8	3,5
2,3	–	1,5	6,8	4,7	–	–	–	10,9

Minimal and maximal critical sets for this square are :

**	–	**	**	**	–	**	–	–
**	10,2	–	–	–	**	**	–	**
–	5,8	10,3	–	6,9	2,4	–	**	–
–	**	7,8	10,4	–	1,9	–	**	–
–	7,9	–	**	**	3,8	–	**	–
**	–	–	–	–	10,6	1,8	**	**
–	–	**	5,9	1,3	–	**	–	4,8
**	**	–	–	–	–	**	**	**
**	–	**	**	**	–	–	–	**

10,1	–	4,9	3,7	2,8	–	**	–	–
8,9	**	–	–	–	5,7	3,4	–	**
–	**	10,3	–	6,9	**	–	**	–
–	3,6	7,8	10,4	–	**	–	**	–
–	**	–	1,2	10,5	**	–	**	–
4,5	–	–	–	–	**	1,8	**	**
–	–	2,6	**	**	–	**	–	**
**	**	–	–	–	–	**	**	**
**	–	**	**	**	–	–	–	**

Appendix II

The following are the Room squares of order 9 (R9_1...R9_6) and order 11 (R11_1...R11_3) as listed in table 1.

10 1	–	4 9	3 7	2 8	–	5 6	–	–
8 9	10 2	–	–	–	5 7	3 4	–	1 6
–	5 8	10 3	–	6 9	2 4	–	1 7	–
–	3 6	7 8	10 4	–	1 9	–	2 5	–
–	7 9	–	1 2	10 5	3 8	–	4 6	–
4 5	–	–	–	–	10 6	1 8	3 9	2 7
–	–	2 6	5 9	1 3	–	10 7	–	4 8
6 7	1 4	–	–	–	–	2 9	10 8	3 5
2 3	–	1 5	6 8	4 7	–	–	–	10 9

10 1	–	4 5	8 9	–	–	–	6 7	2 3
–	10 2	7 9	–	–	5 8	4 6	1 3	–
–	–	10 3	–	6 8	1 2	–	5 9	4 7
–	7 8	1 6	10 4	3 9	–	2 5	–	–
–	6 9	–	–	10 5	3 7	–	2 4	1 8
4 8	–	–	3 5	2 7	10 6	1 9	–	–
3 6	1 5	2 8	–	–	4 9	10 7	–	–
2 9	3 4	–	1 7	–	–	–	10 8	5 6
5 7	–	–	2 6	1 4	–	3 8	–	10 9

10 1	8 9	–	–	6 7	–	4 5	2 3	–
5 8	10 2	–	–	–	7 9	1 3	–	4 6
4 7	–	10 3	5 9	–	–	6 8	–	1 2
3 9	–	2 5	10 4	–	–	–	1 6	7 8
–	3 7	6 9	1 8	10 5	2 4	–	–	–
–	–	4 8	2 7	1 9	10 6	–	–	3 5
–	–	–	3 6	2 8	1 5	10 7	4 9	–
–	5 6	1 7	–	3 4	–	2 9	10 8	–
2 6	1 4	–	–	–	3 8	–	5 7	10 9

10 1	–	–	–	4 7	–	3 9	2 6	5 8
–	10 2	8 9	–	–	1 4	5 6	3 7	–
2 5	–	10 3	6 9	–	–	4 8	–	1 7
–	1 8	2 7	10 4	–	5 9	–	–	3 6
3 4	6 7	–	2 8	10 5	–	–	1 9	–
7 9	–	1 5	–	3 8	10 6	–	–	2 4
6 8	–	–	1 3	2 9	–	10 7	4 5	–
–	4 9	–	5 7	1 6	2 3	–	10 8	–
–	3 5	4 6	–	–	7 8	1 2	–	10 9

10 1	–	–	5 7	2 9	–	4 8	–	3 6
3 4	10 2	1 5	6 9	–	7 8	–	–	–
7 9	–	10 3	2 8	1 6	–	–	4 5	–
–	3 5	8 9	10 4	–	–	–	2 6	1 7
6 8	–	2 7	–	10 5	1 4	3 9	–	–
–	4 9	–	–	–	10 6	1 2	3 7	5 8
2 5	–	4 6	–	3 8	–	10 7	1 9	–
–	6 7	–	1 3	–	5 9	–	10 8	2 4
–	1 8	–	–	4 7	2 3	5 6	–	10 9

10 1	5 9	7 8	3 6	–	2 4	–	–	–
5 6	10 2	–	8 9	–	–	1 4	–	3 7
–	–	10 3	–	4 8	5 7	6 9	1 2	–
–	–	–	10 4	1 3	–	5 8	7 9	2 6
3 9	–	–	2 7	10 5	–	–	4 6	1 8
2 8	1 7	4 9	–	–	10 6	–	3 5	–
–	–	1 6	–	2 9	3 8	10 7	–	4 5
–	3 4	2 5	–	6 7	1 9	–	10 8	–
4 7	6 8	–	1 5	–	–	2 3	–	10 9

12 1	–	9 6	–	–	–	3 5	7 2	11 10	–	8 4
9 5	12 2	–	10 7	–	–	–	4 6	8 3	1 11	–
–	10 6	12 3	–	11 8	–	–	–	5 7	9 4	2 1
3 2	–	11 7	12 4	–	1 9	–	–	–	6 8	10 5
11 6	4 3	–	1 8	12 5	–	2 10	–	–	–	7 9
8 10	1 7	5 4	–	2 9	12 6	–	3 11	–	–	–
–	9 11	2 8	6 5	–	3 10	12 7	–	4 1	–	–
–	–	10 1	3 9	7 6	–	4 11	12 8	–	5 2	–
–	–	–	11 2	4 10	8 7	–	5 1	12 9	–	6 3
7 4	–	–	–	1 3	5 11	9 8	–	6 2	12 10	–
–	8 5	–	–	–	2 4	6 1	10 9	–	7 3	12 11

12 1	9 5	–	3 2	11 6	8 10	–	–	–	7 4	–
–	12 2	10 6	–	4 3	1 7	9 11	–	–	–	8 5
9 6	–	12 3	11 7	–	5 4	2 8	10 1	–	–	–
–	10 7	–	12 4	1 8	–	6 5	3 9	11 2	–	–
–	–	11 8	–	12 5	2 9	–	7 6	4 10	1 3	–
–	–	–	1 9	–	12 6	3 10	–	8 7	5 11	2 4
3 5	–	–	–	2 10	–	12 7	4 11	–	9 8	6 1
7 2	4 6	–	–	–	3 11	–	12 8	5 1	–	10 9
11 10	8 3	5 7	–	–	–	4 1	–	12 9	6 2	–
–	1 11	9 4	6 8	–	–	–	5 2	–	12 10	7 3
8 4	–	2 1	10 5	7 9	–	–	–	6 3	–	12 11

12 1	8 6	–	11 5	7 10	3 4	–	–	–	9 2	–
–	12 2	9 7	–	1 6	8 11	4 5	–	–	–	10 3
11 4	–	12 3	10 8	–	2 7	9 1	5 6	–	–	–
–	1 5	–	12 4	11 9	–	3 8	10 2	6 7	–	–
–	–	2 6	–	12 5	1 10	–	4 9	11 3	7 8	–
–	–	–	3 7	–	12 6	2 11	–	5 10	1 4	8 9
9 10	–	–	–	4 8	–	12 7	3 1	–	6 11	2 5
3 6	10 11	–	–	–	5 9	–	12 8	4 2	–	7 1
8 2	4 7	11 1	–	–	–	6 10	–	12 9	5 3	–
–	9 3	5 8	1 2	–	–	–	7 11	–	12 10	6 4
7 5	–	10 4	6 9	2 3	–	–	–	8 1	–	12 11

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