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COMBINATORIAL DESIGNS

EDITED BY

CHARLES J. COLBOURN
DEPARTMENT OF COMBINATORICS AND OPTIMIZATION
UNIVERSITY OF WATERLOO

JEFFREY H. DINITZ
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF VERMONT

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31 Orthogonal Designs

JENNIFER SEBERRY
R. CRAIGEN

An *orthogonal design* of *order* n and *type* $(s_1, \ldots, s_u)$, s_i positive integers, is an $n \times n$ matrix X, with entries $\{0, \pm x_1, \ldots, \pm x_u\}$ (the x_i commuting indeterminates) satisfying $XX^T = \left(\sum_{i=1}^{u} s_i x_i^2\right) I_n$. This is written as $\mathrm{OD}(n; s_1, s_2, \ldots, s_u)$.

31.1 Remark Alternatively, X has s_i entries of the type $0, \pm x_i$ and the distinct rows are orthogonal under the euclidean inner product. One may view X as a matrix with entries in the field of fractions of the integral domain $\mathbb{Z}[x_1, \ldots, x_u]$, so if $f = (\Sigma_{i=1}^{u} s_i x_i^2)$, then X is an invertible matrix with inverse $\frac{1}{f} X^T$. Thus $XX^T = fI_n$, and so this alternative definition that the row vectors are orthogonal applies equally well to the column vectors of X.

31.2 Remark An orthogonal design with no zeros and in which each of the entries is replaced by $+1$ or -1 is a *Hadamard matrix.* See §IV.24.

31.3 Examples Some small orthogonal designs. OD(4; 1, 1, 1, 1) is the Williamson array. See §IV.24.3.

$$\begin{bmatrix} x & y \\ y & -x \end{bmatrix} \quad \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \quad \begin{bmatrix} a & b & b & d \\ -b & a & d & -b \\ -b & -d & a & b \\ -d & b & -b & a \end{bmatrix} \quad \begin{bmatrix} a & 0 & -c & 0 \\ 0 & a & 0 & c \\ c & 0 & a & 0 \\ 0 & -c & 0 & a \end{bmatrix}$$

OD(2;1,1) OD(4;1,1,1,1) OD(4;1,1,2) OD(4;1,1)

31.4 Lemma (Equating, Killing and Growing [3]) Let D be an $\mathrm{OD}(n; u_1, u_2, \ldots, u_t)$ orthogonal design, on the t commuting variables $x_1, x_2, \ldots, x_t$. Then the following orthogonal designs exist:

1. $\mathrm{OD}(n; u_1, u_2, \ldots, u_i + u_j, \ldots, u_t)$ on $t-1$ variables;
2. $\mathrm{OD}(n; u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_t)$ on $t-1$ variables;
3. $\mathrm{OD}(2n; u_1, u_2, \ldots, u_t)$ on t variables;
4. $\mathrm{OD}(2n; 2u_1, 2u_2, \ldots, 2u_t)$ on t variables;
5. $\mathrm{OD}(2n; u_1, u_1, u_2, \ldots, u_t)$ on $t+1$ variables;
6. $\mathrm{OD}(2n; u_1, u_1, 2u_2, \ldots, 2u_t)$ on $t+1$ variables.

31.5 Example Let D_1 be the OD(4; 1, 1, 1, 1) of Example 31.3; letting $b = c$ as in Lemma 31.4 (1) gives the OD(4; 1, 1, 2) design in Example 31.3; letting $d = 0$ as in Lemma 31.4 gives the first OD(4; 1, 1, 1) design in Example 31.6. Let D_2 be the OD(2; 1, 1) design of Example 31.3; replacing variables by 2×2 matrices as in Lemma 31.4 (3), (4), (5) gives the OD(4; 1, 1), OD(4; 2, 2), and second OD(4; 1, 1, 1) in Example 31.6.

31.6 Examples More small orthogonal designs.

$$\begin{bmatrix} a & -b & -c & 0 \\ b & a & 0 & c \\ c & 0 & a & -b \\ 0 & -c & b & a \end{bmatrix} \quad \begin{bmatrix} x & 0 & y & 0 \\ 0 & x & 0 & y \\ y & 0 & -x & 0 \\ 0 & y & 0 & -x \end{bmatrix} \quad \begin{bmatrix} x & x & y & y \\ x & -x & y & -y \\ y & y & -x & -x \\ y & -y & -x & x \end{bmatrix} \quad \begin{bmatrix} z & x & 0 & y \\ -x & z & y & 0 \\ 0 & y & -z & -x \\ y & 0 & x & -z \end{bmatrix}$$

OD(4; 1, 1, 1) OD(4; 1, 1) OD(4; 2, 2) OD(4; 1, 1, 1)

For n a positive integer, the *Radon number* of n is $\rho(n) = 8c + 2^d$ when $n = 2^a \cdot b$, b odd, $a = 4c + d$, and $0 \le d < 4$.

31.7 Remark Any matrix C with entries $\{0, \pm y_i\}$, where y_i are variables can be written as $C = y_1C_1 + y_2C_2 + \ldots y_uC_u$, where, for C to be orthogonal either (i) $C_iC_j^T + C_jC_i^T = 0$, $i \ne j$ and $y_iy_j = y_jy_i$ for any pair C_i, C_j or (ii) $C_kC_\ell^T = C_\ell C_k^T = 0$, $k \ne \ell$ and $y_ky_\ell = -y_\ell y_k$ for any pair C_k, C_ℓ. Both cases allow us to use matrix representation theory to bound the number of variables that can occur in the orthogonal matrix. In case (i), for the orthogonal design $\mathrm{OD}(n; s_1, s_2, \cdots, s_u)$, the number of variables does not exceed $\rho(n)$.

31.1 Necessary Conditions and More Examples

31.8 Remark There are both combinatorial and algebraic/number-theoretic conditions restricting the existence of orthogonal designs.

Let p be a prime and let $a = p^\alpha u$ and $b = p^\beta v$ where $p \nmid u$ and $p \nmid v$. The *Hilbert norm residue symbol* $(a, b)_p$ for the prime p can be computed as

$$(a,b)_p = \begin{cases} (-1^{\alpha\beta(p-1)/2}) \left(\frac{u}{p}\right)^\beta \left(\frac{v}{p}\right)^\alpha & \text{if } p \ne 2 \\ -1^{(u-1)(v-1)/4 + \alpha(v^2-1)/8 + \beta(u^2-1)/8} & \text{if } p = 2 \end{cases}$$

where $\left(\frac{u}{p}\right)$ is the Legendre symbol.

31.9 Theorems (Geramita and Seberry, [3])

1. If there is an $\mathrm{OD}(2n; s_1, s_2)$, n odd, then $s_1 + s_2 < n - 1$. Also s_1, s_2 and $s_1 + s_2$ are each the sum of two squares.
2. If there is an $\mathrm{OD}(4n; s_1, s_2)$, n odd, then $(-1, s_1s_2)_p(s_1, s_2)_p = 1$ for all primes p and if $s_1 + s_2 = n - 1$, then an $\mathrm{OD}(4n; 1, s_1, s_2)$ exists.
3. If there is an $OD(4n; s_1, s_2, s_3)$, n odd, then for all primes p
$$(-1, s_1s_2s_3)_p(s_1, s_2)_p(s_1, s_3)_p(s_2, s_3)_p = 1.$$
In addition, if $s_1 + s_2 + s_3 = n - 1$, then an $OD(4n; 1, s_1, s_2, s_3)$ exists;
4. If there is an $\mathrm{OD}(4n; s_1, s_2, s_3, s_4)$, n odd, then $\prod_{1 \le i < j \le 4}(s_i, s_j)_p = 1$ for all primes p, $s_1s_2s_3s_4$ is a square, and $s_1 + s_2 + s_3 + s_4 \ne n - 1$.

31.10 Theorem (Eades Sum-Fill Theorem) The Goethals-Seidel construction can only be used to construct an $\mathrm{OD}(4n; s_1, s_2, \cdots, s_\ell))$, n odd, if there exists an integer matrix P, with all entries of modulus $\le n$, which satisfies $PP^T = diag(s_1, s_2, s_3, s_4)$.

31.11 Remark Necessary algebraic conditions for an $\mathrm{OD}(4n; s_1, s_2, \cdots, s_\ell)$, n odd, are given in [3, p71] but introduce considerable new notation outside the scope here.

31.12 Examples Some orthogonal designs of special interest are:

1. The Williamson array – an OD(4;1,1,1,1). The first is the right representation of the quaternions while the second is the left representation of the quaternions.

$$\begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix} \qquad \begin{bmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{bmatrix}$$

2. An OD(8;1,1,1,1,1,1,1,1).

$$\left[\begin{array}{cccc|cccc} A & B & C & D & E & F & G & H \\ -B & A & D & -C & F & -E & -H & G \\ -C & -D & A & B & G & H & -E & -F \\ -D & C & -B & A & H & -G & F & -E \\ \hline -E & -F & -G & -H & A & B & C & D \\ -F & E & -H & G & -B & A & -D & C \\ -G & H & E & -F & -C & D & A & -B \\ -H & -G & F & E & -D & -C & B & A \end{array}\right]$$

3. The Baumert-Hall array, $A(x,y,z,w)$. An OD(12;3,3,3,3).

$$\left[\begin{array}{cccccccccccc} y & x & x & x & -z & z & w & y & -w & w & z & -y \\ -x & y & x & -x & w & -w & z & -y & -z & z & -w & -y \\ -x & -x & y & x & w & -y & -y & w & z & z & w & -z \\ -x & x & -x & y & -w & -w & -z & w & -z & -y & -y & -z \\ & & & & & & & & & & & \\ -y & -y & -z & -w & z & x & x & x & -w & -w & z & -y \\ -w & -w & -z & y & -x & z & x & -x & y & y & -z & -w \\ w & -w & w & -y & -x & -x & z & x & y & -z & -y & -z \\ -w & -z & w & -z & -x & x & -x & z & -y & y & -y & w \\ & & & & & & & & & & & \\ -y & y & -z & -w & -z & -z & w & y & w & x & x & x \\ z & -z & -y & -w & -y & -y & -w & -z & -x & w & x & -x \\ -z & -z & y & z & -y & -w & y & -w & -x & -x & w & x \\ z & -w & -w & z & y & -y & y & z & -x & x & -x & w \end{array}\right]$$

4. The Plotkin array – an OD(24;3,3,3,3,3,3,3,3). Let $A(x,y,z,w)$ be as in part (3) and let $B(x,y,z,w)$ be the array

$$\left[\begin{array}{cccccccccccc} y & x & x & x & -w & w & z & y & -z & z & w & -y \\ -x & y & x & -x & -z & z & -w & -y & w & -w & z & -y \\ -x & -x & y & x & -y & -w & y & -w & -z & -z & w & z \\ -x & x & -x & y & w & w & -z & -w & -y & z & y & z \\ & & & & & & & & & & & \\ -w & -w & -z & -y & z & x & x & x & -y & -y & z & -w \\ y & y & -z & -w & -x & z & x & -x & -w & -w & -z & y \\ -w & w & -w & -y & -x & -x & z & x & z & y & y & z \\ z & -w & -w & z & -x & x & -x & z & y & -y & y & w \\ & & & & & & & & & & & \\ z & -z & y & -w & y & y & w & -z & w & x & x & x \\ y & -y & -z & -w & -z & -z & -w & -y & -x & w & x & -x \\ z & z & y & -z & w & -y & -y & w & -x & -x & w & x \\ -w & -z & w & -z & -y & y & -y & z & -x & x & -x & w \end{array}\right],$$

then the required design is

$$\left[\begin{array}{cc} A(x_1,x_2,x_3,x_4) & B(x_5,x_6,x_7,x_8) \\ B(-x_5,x_6,x_7,x_8) & -A(-x_1,x_2,x_3,x_4) \end{array}\right].$$

5. The Goethals-Seidel array can be found in §IV.24.3.

6. The Welch array – an OD(20;5,5,5,5) is constructed from sixteen block circulant matrices.

$$
\left[\begin{array}{ccccc|ccccc|ccccc|ccccc}
-d & b & -c & -c & -b & c & a & -d & -d & -a & -b & -a & c & -c & -a & a & -b & -d & d & -b \\
-b & -d & b & -c & -c & -a & c & a & -d & -d & -a & -b & -a & c & -c & -b & a & -b & -d & d \\
-c & -b & -d & b & -c & -d & -a & c & a & -d & -c & -a & -b & -a & c & d & -b & a & -b & -d \\
-c & -c & -b & -d & b & -d & -d & -a & c & a & c & -c & -a & -b & -a & -d & d & -b & a & -b \\
b & -c & -c & -b & -d & a & -d & -d & -a & c & -a & c & -c & -a & -b & -b & -d & d & -b & a \\
\hline
-c & a & d & d & -a & -d & -b & -c & -c & b & -a & b & -d & d & b & -b & -a & -c & c & -a \\
-a & -c & a & d & d & b & -d & -b & -c & -c & b & -a & b & -d & d & -a & -b & -a & -c & c \\
d & -a & -c & a & d & -c & b & -d & -b & -c & d & b & -a & b & -d & c & -a & -b & -a & -c \\
d & d & -a & -c & a & -c & -c & b & -d & -b & -d & d & b & -a & b & -c & c & -a & -b & -a \\
a & d & d & -a & -c & -b & -c & -c & b & -d & b & -d & d & b & -a & -a & -c & c & -a & -b \\
\hline
b & -a & -c & c & -a & a & b & -d & d & b & -d & -b & c & c & b & -c & a & -d & -d & -a \\
-a & b & -a & -c & c & b & a & b & -d & d & b & -d & -b & c & c & -a & -c & a & -d & -d \\
c & -a & b & -a & -c & d & b & a & b & -d & c & b & -d & -b & c & -d & -a & -c & a & -d \\
-c & c & -a & b & -a & -d & d & b & a & b & c & c & b & -d & -b & -d & -d & -a & -c & a \\
-a & -c & c & -a & b & b & -d & d & b & a & -b & c & c & b & -d & a & -d & -d & -a & -c \\
\hline
-a & -b & -d & d & -b & b & -a & c & -c & -a & c & a & d & d & -a & -d & b & c & c & -b \\
-b & -a & -b & -d & d & -a & b & -a & c & -c & -a & c & a & d & d & -b & -d & b & c & c \\
d & -b & -a & -b & -d & -c & -a & b & -a & c & d & -a & c & a & d & c & -b & -d & b & c \\
-d & d & -b & -a & -b & c & -c & -a & b & -a & d & d & -a & c & a & c & c & -b & -d & b \\
-b & -d & d & -b & -a & -a & c & -c & -a & b & a & d & d & -a & c & b & c & c & -b & -d
\end{array}\right]
$$

31.13 Example The Ono-Sawade-Yamamoto array – an OD(36; 9, 9, 9, 9).

a	a	a	b	-c	-d	-b	d	c	b	-a	a	b	-c	d	b	-d	c	-c	-a	a	-b	-c	-d	b	d	-c	-d	-a	a	b	c	-d	-b	-d	-c
a	a	a	-d	b	-c	c	-b	d	a	b	-a	d	b	-c	c	b	-d	a	-c	-a	-d	-b	-c	-c	b	d	a	-d	-a	-d	b	c	-c	-b	-d
a	a	a	-c	-d	b	d	c	-b	-a	a	b	-c	d	b	-d	c	b	-a	a	-c	-c	-d	-b	d	-c	b	-a	a	-d	c	-d	b	-d	-c	-b
-b	d	c	a	a	a	b	-c	-d	b	-d	c	b	-a	a	b	-c	d	b	d	-c	-c	-a	a	-b	-c	-d	-b	-d	-c	-d	-a	a	b	c	-d
c	-b	d	a	a	a	-d	b	-c	c	b	-d	a	b	-a	d	b	-c	-c	b	d	a	-c	-a	-d	-b	-c	-c	-b	-d	a	-d	-a	-d	b	c
d	c	-b	a	a	a	-c	-d	b	-d	c	b	-a	a	b	-c	d	b	d	-c	b	-a	a	-c	-c	-d	-b	-d	-c	-b	-a	a	-d	c	-d	b
b	-c	-d	-b	d	c	a	a	a	b	-c	d	b	-d	c	b	-a	a	-b	-c	-d	b	d	-c	-c	-a	a	b	c	-d	-b	-d	-c	-d	-a	a
-d	b	-c	c	-b	d	a	a	a	d	b	-c	c	b	-d	a	b	-a	-d	-b	-c	-c	b	d	a	-c	-a	-d	b	c	-c	-b	-d	a	-d	-a
-c	-d	b	d	c	-b	a	a	a	-c	d	b	-d	c	b	-a	a	b	-c	-d	-b	d	-c	b	-a	a	-c	c	-d	b	-d	-c	-b	-a	a	-d
-b	a	-a	-b	-c	d	-b	-d	c	a	a	a	b	c	d	-b	-d	-c	d	-a	a	b	-c	d	-b	d	c	-c	a	-a	b	-c	-d	-b	d	-c
-a	-b	a	d	-b	-c	c	-b	-d	a	a	a	d	b	c	-c	-b	-d	a	d	-a	d	b	-c	c	-b	d	-a	-c	a	-d	b	-c	-c	-b	d
a	-a	-b	-c	d	-b	-d	c	-b	a	a	a	c	d	b	-d	-c	-b	-a	a	d	-c	d	b	d	c	-b		-a	-c	-c	-d	b	d	-c	-b
-b	-d	c	-b	a	-a	-b	-c	d	-b	-d	-c	a	a	a	b	c	d	-b	d	c	d	-a	a	b	-c	d	-b	d	-c	-c	a	-a	b	-c	-d
c	-b	-d	-a	-b	a	d	-b	-c	-c	-b	-d	a	a	a	d	b	c	c	-b	d	a	d	-a	d	b	-c	-c	-b	d	-a	-c	a	-d	b	-c
-d	c	-b	a	-a	-b	-c	d	-b	-d	-c	-b	a	a	a	c	d	b	d	c	-b	-a	a	d	-c	d	b	d	-c	-b	a	-a	-c	-c	-d	b
-b	-c	d	-b	-d	c	-b	a	-a	b	c	d	-b	-d	-c	a	a	a	b	-c	d	-b	d	c	d	-a	a	b	-c	-d	-b	d	-c	-c	a	-a
d	-b	-c	c	-b	-d	-a	-b	a	d	b	c	-c	-b	-d	a	a	a	d	b	-c	c	-b	d	a	d	-a	-d	b	-c	-c	-b	d	-a	-c	a
-c	d	-b	-d	c	-b	a	-a	-b	c	d	b	-d	-c	-b	a	a	a	-c	d	b	d	c	-b	-a	a	d	-c	-d	b	d	-c	-b	a	-a	-c
c	a	-a	-b	c	-d	b	d	c	-d	a	-a	b	-c	-d	-b	-d	c	a	a	a	-b	-c	d	b	-d	c	-b	-a	a	-b	-c	-d	-b	d	c
-a	c	a	-d	-b	c	c	b	d	-a	-d	a	-d	b	-c	c	-b	-d	a	a	a	d	-b	-c	c	b	-d	a	-b	-a	-d	-b	-c	c	-b	d
a	-a	c	c	-d	-b	d	c	b	a	-a	-d	-c	-d	b	-d	c	-b	a	a	a	-c	d	-b	-d	c	b	-a	a	-b	-c	-d	-b	d	c	-b
b	d	c	c	a	-a	-b	c	-d	-b	-d	c	-d	a	-a	b	-c	-d	b	-d	c	a	a	a	-b	-c	d	-b	d	c	-b	-a	a	-b	-c	-d
c	b	d	-a	c	a	-d	-b	c	c	-b	-d	-a	-d	a	-d	b	-c	c	b	-d	a	a	a	d	-b	-c	c	-b	d	a	-b	-a	-d	-b	-c
d	c	b	a	-a	c	c	-d	-b	-d	c	-b	a	-a	-d	-c	-d	b	-d	c	b	a	a	a	-c	d	-b	d	c	-b	-a	a	-b	-c	-d	-b
-b	c	-d	b	d	c	c	a	-a	b	-c	-d	-b	-d	c	-d	a	-a	-b	-c	d	b	-d	c	a	a	a	-b	-c	-d	-b	d	c	-b	-a	a
-d	-b	c	c	b	d	-a	c	a	-d	b	-c	c	-b	-d	-a	-d	a	d	-b	-c	c	b	-d	a	a	a	-d	-b	-c	c	-b	d	a	-b	-a
c	-d	-b	d	c	b	a	-a	c	-c	-d	b	-d	c	-b	a	-a	-d	-c	d	-b	-d	c	b	a	a	a	-c	-d	-b	d	c	-b	-a	a	-b
d	a	-a	b	c	d	-b	d	-c	c	-a	a	b	c	-d	-b	d	c	b	a	-a	b	-c	-d	b	d	c	a	a	a	-b	c	-d	b	d	-c
-a	d	a	d	b	c	-c	-b	d	a	c	-a	-d	b	c	c	-b	d	-a	b	a	-d	b	-c	c	b	d	a	a	a	-d	-b	c	-c	b	d
a	-a	d	c	d	b	d	-c	-b	-a	a	c	c	-d	b	d	c	-b	a	-a	b	-c	-d	b	d	c	b	a	a	a	c	-d	-b	d	-c	b
-b	d	-c	d	a	-a	b	c	d	-b	d	c	c	-a	a	b	c	-d	b	d	c	b	a	-a	b	-c	-d	b	d	-c	a	a	a	-b	c	-d
-c	-b	d	-a	d	a	d	b	c	c	-b	d	a	c	-a	-d	b	c	c	b	d	-a	b	a	-d	b	-c	-c	b	d	a	a	a	-d	-b	c
d	-c	-b	a	-a	d	c	d	b	d	c	-b	-a	a	c	c	-d	b	d	c	b	a	-a	b	-c	-d	b	d	-c	b	a	a	a	c	-d	-b
b	c	d	-b	d	-c	d	a	-a	b	c	-d	-b	d	c	c	-a	a	b	-c	-d	b	d	c	b	a	-a	-b	c	-d	b	d	-c	a	a	a
d	b	c	-c	-b	d	-a	d	a	-d	b	c	c	-b	d	a	c	-a	-d	b	-c	c	b	d	-a	b	a	-d	-b	c	-c	b	d	a	a	a
c	d	b	d	-c	-b	a	-a	d	c	-d	b	d	c	-b	-a	a	c	-c	-d	b	d	c	b	a	-a	b	c	-d	-b	d	-c	b	a	a	a

31.2 Asymptotic Existence Results

31.14 Conjecture When $n \equiv 4 \pmod 8$, there exists an $OD(n; 1, k)$ when $k \leq n-1$, k the sum of three or fewer integer squares and $n-2$ the sum of two or fewer integer squares.

31.15 Conjecture When $n \equiv 0 \pmod 8$, there exists an $OD(n; 1, k)$ for all $k \leq n-1$.

31.16 Remarks Conjecture 31.14 has been verified for all $n \leq 84$. Conjecture 31.15 is true for $2^t \cdot 3, 2^t \cdot 5, 2^t \cdot 7, 2^t \cdot 9$ and $2^t \cdot 13$ for $t \geq 3$, $2^t \cdot 15$ and $2^t \cdot 21$, for $t \geq 4$. Lemma 31.4 yields Theorem 31.17.

31.17 Theorem Let a, b, c, x, y, z be nonnegative integers such that $a + b + c = n$ and $x + y + z = 2n$. Suppose that for all possible choices of a, b, c an $OD(n; a, b, c)$ exists. Then for all possible choices of x, y, z an $OD(2n; x, y, z)$ exists. Hence for all possible choices of x, y, z, $x+y+z = 2^m$, an $OD(2^m; x, y, z)$ and an $OD(2^m; x, y, z)$, $x+y \leq m$, exists.

31.18 Theorem (Robinson [3]) If $n > 40$, there is no $OD(n : 1, 1, 1, 1, 1, n-5)$.

31.19 Theorem (Robinson [3]) The following orthogonal designs exist:

1. $OD(2^t; 1, 1, 1, 1, 2, 2, 4, 4, \cdots, 2^{t-2}, 2^{t-2})$,
2. $OD(2^t; 1, 1, 2, 1, 2, 4, 8, \cdots, 2^{t-3}, 3, 6, 12, \cdots, 3 \cdot 2^{t-3})$,
3. $OD(2^t; 1, 1, 2, 4, 8, \cdots, 2^{t-3}, 2^{t-3}, 2^{t-2}, 3, 3, 6, \cdots, 3 \cdot 2^{t-4})$,
4. $OD(2^t; 1, 1, 2, 4, 8, \cdots, 2^{t-3}, 3, 6, 9, 18, \cdots, 9 \cdot 2^{t-5}, 3 \cdot 2^{t-4})$,
5. $OD(2^t; 1, 2, 3, 2^{t-4}, 3 \cdot 2^{t-3}, 3, 3, 6, 6, 12, 12, \cdots, 3 \cdot 2^{t-5}, 3 \cdot 2^{t-5})$,

A set of 4 *T-matrices*, T_i, $i = 1, \ldots, 4$ of order t are four circulant matrices which have entries 0, +1 or −1 and which satisfy

1. $T_i * T_j = 0$, $i \neq j$, ($*$ denotes the Hadamard product)
2. $\sum_{i=1}^{4} T_i$ is a $(1, -1)$ matrix,
3. $\sum_{i=1}^{4} T_i T_i^T = tI_t$,
4. $t = t_1^2 + t_2^2 + t_3^2 + t_4^2$ where t_i is the row(column) sum of T_i.

31.20 Theorem (Seberry–Yamada–Turyn [5]) Suppose there are T-matrices of order t. Further suppose there is an $OD(4s; u_1, \ldots, u_n)$ constructed of sixteen circulant (or type one) $s \times s$ blocks on the variables $x_1, \ldots, x_n$. Then there is an $OD(4st; tu_1, \ldots, tu_n)$. In particular if there is an $OD(4s; s, s, s, s)$ constructed of sixteen circulant $s \times s$ blocks, then there is an $OD(4st; st, st, st, st)$. This means there exist orthogonal designs $OD(20t; 5t, 5t, 5t, 5t)$ and $OD(36t; 9t, 9t, 9t, 9t)$.

31.21 Remark T-matrices of order t give Hadamard matrices of order $4t$. T-matrices are known for many orders including (see [5]):

1,...,72,74,...,78,80,...,82,84,...,88,90,..., 96,98,...,102,104,...,106,108,110,...,112, 114,..., 126,128,...,130,132,136,138,140,...,148,150,152,..., 156,158,...,162,164,165, 166,168,...,172,174,..., 178,180,182,184,...,190,192,194,...,196,198,200,..., 210.

31.22 Conjecture There exists an $OD(4t; t, t, t, t)$ for every positive integer t.

31.3 Existence Results for Small Orders

31.23 Table Orthogonal designs OD($2n : a_1, a_2$), n odd. (E denotes exists, U means existence is unresolved).

$2n$	(a_1, a_2)	Status
2	(1,1)	E
6	(1,1), (1,4), (2,2)	E
10	(1,1), (1,4), (2,2), (4,4)	E
14	(1,1),(1,4),(1,9),(2,2),(2,8),(4,4),(4,9),(5,5)	E
18	(1,1),(1,4),(2,2),(2,8),(4,4),(5,5)	E
	(1,9), (4,9)	U
22	(1,1),(1,4),(1,16),(2,2),(2,8),(4,4),(4,16),(5,5),(8,8)	E
	(1,9),(2,18), (4,9),(9,9)	U
26	(1,1),(1,4),(1,9),(1,16),(2,2),(2,8),(2,18),(4,4),(4,16),(5,5),(8,8),(9,9),(10,10)	E
	(4,9),(5,20)	U
>26	(1,1),(1,4),(1,9),(2,2),(2,8),(4,4),(4,16),(5,5),(8,8),(10,10),(13,13)	E
>38	(1,16)	E

31.24 Table Orthogonal designs OD($4n : a_1, a_2, \ldots, a_u$), n odd.

$4n$	u	Cases	$\exists$	Comment
4	2	4	4	Existence completely settled
4	3	2	2	Existence completely settled
4	4	1	1	Existence completely settled
12	2	36	33	Existence completely settled
12	3	53	31	Existence completely settled
12	4	53	12	Existence completely settled
20	2	100	89	Existence completely settled
20	3	237	97	(3,7,8) unresolved
20	4	359	35	(1,3,6,8),(1,4,4,9), (2,2,5,5) unresolved
28	2	196	169	Existence completely settled
28	3	640	217	(1,5,20) unresolved

31.25 Table Orthogonal designs OD($8n : a_1, a_2, \cdots, a_u$)

$8n$	u	Comment
8	$1, \ldots, 8$	All exist
16	$1, \ldots, 5$	All exist
16	6	(1,1,1,1,4,7) and (1,1,2,2,2,7) unresolved
16	7	94 possible tuples; 13 do not exist; 81 are known
16	8	67 possible tuples; 30 do not exist; 37 are known
16	9	45 possible tuples; 36 do not exist; 9 are known
24	1,2	All exist
24	3	(4,4,15), (7,7,7) and (7,8,8) unresolved
24	4	717 possible tuples
24	6	996 possible tuples
24	8	eight designs known

31.4 See Also

§IV.24	There is a very close connection between Hadamard matrices and ODs.
§IV.52	Weighing matrices are a special class of ODs.
§IV.42	Zero autocorrelation sequences are related to ODs.
[5]	An extensive survey of Hadamard matrices which also includes much of the material in this section and numerous references.
[2, 6]	Contain theorems dealing with the number of variables in an OD.
[3]	The foundational reference work for ODs.
[4]	Contains many new results on existence of ODs.

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