# Improving the Strict Avalanche Characteristics of Cryptographic Functions *

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**Abstract**

This paper presents a simple yet effective method for transforming Boolean functions that do not satisfy the strict avalanche criterion (SAC) into ones that satisfy the criterion. Such a method has a wide range of applications in designing cryptographically strong functions, including substitution boxes (S-boxes) employed by common key block encryption algorithms.

# Key Words

cryptography, security in digital systems, strict avalanche criterion (SAC), substitution boxes (S-boxes).

# 1   The Strict Avalanche Criterion

A (Boolean) function on $V_n$, where $V_n$ denotes the the vector space of $n$-tuples of elements from $GF(2)$, is said to satisfy the strict avalanche criterion (SAC) if complementing a single bit in its input results in the output of the function being complemented half the time over all the input vectors. The SAC is one of the most important requirements for cryptographic functions. The formal definition for the SAC seems to appear first in the open literature in 1985 [11, 12]:

**Definition 1** *Let $f$ be a function on $V_n$. $f$ is said to satisfy the SAC if $f(x) \oplus f(x \oplus \alpha)$ assumes the values zero and one an equal number of times, or simply, $f(x) \oplus f(x \oplus \alpha)$ is balanced, for every $\alpha \in V_n$ with $W(\alpha) = 1$, where $x = (x_1, \ldots, x_n)$ and $W(\alpha)$ denotes the number of ones in (or the Hamming weight of) the vector $\alpha$.*

The SAC has been further generalized in two different directions: high order SAC and propagation criterion. The first direction is represented by [4], while the second by [1, 8, 7]. We shall not pursue further these developments in this paper. Instead we will focus our attention on how to transform functions which do not satisfy the SAC into ones that satisfy the criterion.

# 2   Single Functions

First we introduce the following basic theorem.

**Theorem 1** *Let $f$ be a function on $V_n$, and $A$ be a nondegenerate matrix of order $n$ whose entries are from $GF(2)$. Suppose that $f(x) \oplus f(x \oplus \gamma_i)$ is balanced for each row $\gamma_i$ of $A$, where $i = 1, \ldots, n$ and $x = (x_1, \ldots, x_n)$. Then $\psi(x) = f(xA)$ satisfies the SAC.*

*Proof.* Let $\delta_i$ be a vector in $V_n$ whose entries, *except the ith*, are all zero. Note that $W(\delta_i) = 1$ and $\delta_i A = \gamma_i$, $i = 1, \ldots n$. Then we have $\psi(x) \oplus \psi(x \oplus \delta_i) = f(xA) \oplus f((x \oplus \delta_i)A) = f(u) \oplus f(u \oplus \gamma_i)$, where $u = xA$. Since $A$ is nondegenerate, $u$ runs through $V_n$ while $x$ does. By assumption, $f(u) \oplus f(u \oplus \gamma_i)$ runs through the values zero and one an equal number of times while $u$ runs through $V_n$. Consequently $\psi(x) \oplus \psi(x \oplus \delta_i)$ runs through the values zero and one an equal number of times while $x$ runs through $V_n$. That is, $\psi(x)$ satisfies the SAC. $\square$

Note that the algebraic degree, the nonlinearity and the balancedness of a function is unchanged under a linear transformation of coordinates [13]. In the case of S-boxes (tuples of functions), the profile of its XOR distribution table, which measures the strength against the differential cryptanalysis [2], also remains invariant under such a transformation [10]. Thus Theorem 1 provides us a powerful tool to improve the strict avalanche characteristics of cryptographic functions. In the following we consider two applications of the theorem.

**Application 1** Our first application shows that a SAC-fulfilling function on a higher dimensional space can be easily obtained from a SAC-fulfilling function on a lower dimensional space.

Let $g(y_1, \ldots, y_s)$ be a function on $V_s$ that satisfies the SAC. Adding $t$ pseudo-coordinates $x_1, \ldots, x_t$ into $g$, we obtain a function $f$ on $V_{s+t}$, namely,

$$f(y_1, \ldots, y_s, x_1, \ldots, x_t) = g(y_1, \ldots, y_s)$$

The $t$ newly added coordinates have no influence on the output of $f$. Hence $f$ does not satisfy the SAC.

Let $A$ be a nondegenerate matrix of order $s + t$. Assume that each row $\gamma_i$ of $A$ can be written as $\gamma_i = (\beta_i, \alpha_i)$, where $W(\beta) = 1$, $\beta_i \in V_s$ and $\alpha_i \in V_t$. Let $x = (x_1, \ldots, x_t)$, $y = (y_1, \ldots, y_s)$ and $z = (y, x)$. Then we have $f(z) \oplus f(z \oplus \gamma_i) = g(y) \oplus g(y \oplus \beta_i)$. This shows that $f(z) \oplus f(z \oplus \gamma_i)$ is balanced for $\gamma_i$, $i = 1, \ldots, s + t$. By Theorem 1, $\psi(z) = f(zA)$ satisfies the SAC.

An example of the matrices that satisfy the requirements is as follows:

$$A = \begin{bmatrix} I_s & 0_{s \times t} \\ Q_{t \times s} & I_t \end{bmatrix} \tag{1}$$

where $I$ denotes the identity matrix, $0$ denotes the zero matrix, and $Q$ is defined as

$$Q = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ & & \vdots & \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

**Application 2** For each vector $\delta = (i_1, \ldots, i_s) \in V_s$, we define a function $D_\delta$ on $V_s$ in the following way:

$$D_\delta(y) = (y_1 \oplus \bar{i}_1) \ldots (y_s \oplus \bar{i}_s)$$

where $y = (y_1, \ldots, y_s)$ and $\bar{i}$ denotes the binary complement of $i$, namely, $\bar{i} = 1 \oplus i$.

Using this notation, we define the "concatenation" of $2^s$ functions on $V_t$ as follows:

$$f(y, x) = \bigoplus_{\delta \in V_s} D_\delta(y) g_\delta(x) \oplus r(y) \tag{2}$$

where $x = (x_1, \ldots, x_t)$, each $g_\delta$ is a function on $V_t$, and $r$ is an arbitrary function on $V_s$. Of particular interest is the concatenation of linear functions on $V_t$. In Theorems 4 and 5 of [9], the following result is proved:

**Lemma 1** *When $t \geqq s$ and all $g_\delta$, $\delta \in V_s$, are distinct nonzero linear functions on $V_t$, the function $f$ constructed by (2) is highly nonlinear and balanced. In addition, $f(z) \oplus f(z \oplus \gamma)$ is balanced for all $\gamma = (\beta, \alpha)$ with $\beta \neq 0$, where $z = (y, x)$, $\beta \in V_s$ and $\alpha \in V_t$.*

Let $A$ be a nondegenerate matrix of order $s + t$. Suppose that the $i$th row $\gamma_i$ of $A$ can be written as $\gamma_i = (\beta_i, \alpha_i)$ with $\beta_i \neq 0$, where $\beta_i \in V_s$ and $\alpha_i \in V_t$. Then by Theorem 1, $\psi(z) = f(zA)$ satisfies the SAC. Note that the matrix $A$ defined by (1) satisfies the requirements.

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# 3 A Set of Functions

In computer security practice, such as the design of S-boxes, we often consider a set of functions. It is desirable that all component functions in a set simultaneously satisfy the SAC. ¿From Theorem 1 we can see that given a set of functions on $V_n$, $\{f_1, \ldots, f_m\}$, if $A$ is a nondegenerate matrix of order $n$ such that $f_i(x) \oplus f_i(x \oplus \gamma_j)$ is balanced for every function $f_i$ and every row $\gamma_j$ in $A$, then $g_1(x) = f_1(xA)$, …, $g_m(x) = f_m(xA)$ all satisfy the SAC. The following theorem gives a sufficient condition for the existence of such a nondegenerate matrix.

**Theorem 2** *Let $f_1$, …, $f_m$ be functions on $V_n$. Denote by $B$ the set of vectors $\gamma$ in $V_n$ such that $f_j(x) \oplus f_j(x \oplus \gamma)$ is* not *balanced for some $1 \leq j \leq m$, and by $\#B$ the number of vectors in $B$. If $\#B < 2^{n-1}$, then there exists a nondegenerate matrix $A$ of order $n$ with entries from $GF(2)$ such that each $\psi_j(x) = f_j(xA)$ satisfies the SAC.*

*Proof.* We show how to construct a nondegenerate matrix $A$ of order $n$, under the condition that $\#B < 2^{n-1}$. Denote by $S_{\alpha_1, \ldots, \alpha_k}$ the set of vectors consisting of all the linear combinations of vectors $\alpha_1, \ldots, \alpha_k$.

The first row of $A$, $\gamma_1$, is selected from $V_n$ excluding those in $B$ and the zero vector, i.e., from the vector set $V_n - B - S_0$. There are $2^n - \#B - 2^0$ different choices for $\gamma_1$. The second row of $A$, $\gamma_2$, is selected from the vector set $V_n - B - S_{\gamma_1}$. This guarantees that $\gamma_2$ is linearly independent of $\gamma_1$. We have $2^n - \#B - 2^1$ different choices for $\gamma_2$.

In general, once the first $k - 1$ linearly independent rows $\gamma_1$, …, $\gamma_{k-1}$ of $A$ are selected, the $k$th row $\gamma_k$, $k \leq n$, will be selected from the vector set $V_n - B - S_{\gamma_1, \ldots, \gamma_{k-1}}$. This process ensures that $\gamma_1$, …, $\gamma_k$ are all linearly independent.

The number of choices for the last row $\gamma_n$ is $2^n - \#B - 2^{n-1} = 2^{n-1} - \#B > 0$. Therefore, we can always find a nondegenerate matrix $A$ such that $f_i(x) \oplus f_i(x \oplus \gamma_j)$ is balanced for every $1 \leq i \leq m$ and $1 \leq j \leq n$. By Theorem 1, $\psi_1(x) = f_1(xA)$, …, $\psi_m(x) = f_m(xA)$ all satisfy the SAC. □

Theorem 2 has been applied in [10] to design S-boxes that possess many desirable cryptographic properties, which include the high nonlinearity, the SAC, the balanced-

ness and the robustness against differential cryptanalysis. As is shown below, the transformation technique can also be applied to other approaches to the construction of S-boxes.

**Application 3** With the S-boxes studied in [6, 5, 3] each component function $f_j$ has the following property: $f_j(x) \oplus f_j(x \oplus \alpha)$ is balanced for all but one nonzero vector $\alpha \in V_n$, where $x = (x_1, \ldots, x_n)$ and $n \geqq 3$ is odd. Thus we have $\#B \leqq n$. By Theorem 2 we can use a nondegenerate matrix to transform all the component functions of such an S-box into SAC-fulfilling ones.

# 4   A Final Remark

In [13], we have constructed highly nonlinear balanced functions on $V_{2k+1}$ that satisfy the propagation criterion of degree $2k$ [8], and highly nonlinear balanced functions on $V_{2k}$ that satisfy the propagation criterion of degree $\frac{4}{3}k$. A transformation technique similar to that presented in this paper has played an important role in the constructions.

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