

Multiplication of sequences with zero autocorrelation

C. Koukouvinos

Department of Mathematics, National Technical University of Athens,
Zografou 157 73, Athens, Greece

S. Kounias

Department of Mathematics, University of Athens,
Panepistemiopolis 15784, Greece

J. Seberry

Department of Computer Science, University of Wollongong,
Wollongong, NSW, 2522, Australia

C.H. Yang

Department of Mathematical Sciences, State University of New York,
Oneonta, New York 13829, USA

J. Yang

Dept. of Computer Science and Computer Science Division of
Electrical Engineering, University of California, Berkeley, CA 94720, USA

Abstract

Near normal sequences of new lengths $n = 4m + 1 = 49, 53, 57$ are constructed. The relation between a special set of near normal sequences and Golay sequences is discussed. A reformulation of Yang's powerful theorems on T-sequences is also given.

We give base sequences for lengths $m + p, m + p, m, m$ for $p = 1$ and $m \in \{19, \dots, 30\}$. Some of these are new lengths, or new decompositions into four squares for n and constructed here for the first time.

1 Introduction

We use the notations and definitions of Koukouvinos, Kounias, Seberry, C.H. Yang and J. Yang [9] except for those few noted below.

If $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$ are sequences of length n , we use the notation $A^* = \{a_n, \dots, a_1\}$ for the reversed sequence, A/B for the sequence $\{a_1, b_1, \dots, a_n, b_n\}$.

If $A = \{a_1, \dots, a_{n+1}\}$, $B = \{b_1, \dots, b_n\}$ we use the notation A/B for the sequence $\{a_1, b_1, \dots, a_n, b_n, a_{n+1}\}$ and

$$N_{AB}(s) = \sum_{i=1}^{n-s} a_i b_{i+s}, \quad s = 0, 1, \dots, n-1$$

for the cross-correlations.

Definition 1 A quadruple $(E, F; G, H)$ of $(0, \pm 1)$ sequences is said to be a set of *near normal sequences* for length $n = 4m + 1$ (abbreviated as $NN(n)$) if the following conditions are satisfied.

- (i) $E = (X/O_{m-1}, 1)$, $F = (Y/O_{m-1}, 0)$ where X and Y are ± 1 sequences of length m , i.e. E and F are of length $2m$. G and H are $(0, \pm 1)$ sequences of length $2m$, such that $G + H$ is a ± 1 sequence of length $2m$.
- (ii) $N_E(s) + N_F(s) + N_G(s) + N_H(s) = 0$, $s = 1, \dots, 2m - 1$.

Condition (ii) is also equivalent to

$$E(z)E(z^{-1}) + F(z)F(z^{-1}) + G(z)G(z^{-1}) + H(z)H(z^{-1}) = 4m + 1, \quad z \neq 0. \quad (1)$$

Remark 1 It is easy to see that the sequences G and H of Definition 1 are quasi-symmetric, i.e. if $g_k = 0$, then $g_{2m+1-k} = 0$ and also if $h_k = 0$, then $h_{2m+1-k} = 0$ (see [9]).

Let A, B, C, D be base sequences of lengths $n + 1, n + 1, n, n$ with $a_0 = b_0 = 1$, and let $X = (A + B)/2 = (1, U, 0)$, $Y = (A - B)/2 = (0, V, 1)$, $Z = (C + D)/2$ and $W = (C - D)/2$, then pairs U, V and Z, W are both quasi-symmetric supplementary sequences i.e. $U \pm V$ and $Z \pm W$ are both sequences of $1, -1$. If $V = 0_{n-1}$, the zero sequence of length $n - 1$, then $F = (1, U)$, $G = Z$ and $H = W$ are $NS(n)$. Also if $n = 2m$ and $U = O_m/u$, $V = v/O_{m-1}$, then $E = (V, 1)$, $F = (1, U')$, $G = Z$ and $H = W$, where $U' = O_{m-1}/u$, are $NN(4m + 1)$.

It is easy to see that $NS(n), F; G, H$ are equivalent to base sequences $F, F, G + H, G - H$ of lengths n, n, n, n . Consequently quasi-symmetry follows from zero autocorrelation.

Therefore both NS and NN are obtainable from the special sets of base sequences stated above. Thus the quasi-symmetry of NN is a consequence of zero autocorrelation. These special base sequences are essential for the composition of four complementary sequences.

In this paper we construct near normal sequences $NN(n)$, of new lengths $n = 4m + 1 = 49, 53, 57$ and base sequences of lengths $n + 1, n + 1, n, n$ for $n = 19, 20, \dots, 30$. We discuss the relation between a special set of near normal sequences and Golay sequences. Finally we give a reformulation of Yang's powerful theorems on T -sequences and summarize the known results on Yang numbers, base sequences and T -sequences.

2 On Golay sequences and near normal sequences

One of us (C.Yang [15]) has noted the following result. However Eliahou, Kervaire and Saffari [3] have shown that this construction cannot produce previously unknown Golay sequences.

Theorem 1 *A special set of near normal sequences $NN(n)$, $n = 4m + 1$, $(E, F; G, O_{2m})$ with "symmetric" F and skew G are basic sequences from which Golay sequences $GS(2n)$ can be built.*

Proof. By letting $C(z) = G(z^2) + zF(z^2)$ and $D(z) = E(z^2)$, we have

$$\begin{aligned} C(z)C(z^{-1}) + D(z)D(z^{-1}) &= G(z^2)G(z^{-2}) + F(z^2)F(z^{-2}) + E(z^2)E(z^{-2}) \\ &= n, \quad z \neq 0 \end{aligned}$$

and

$$\begin{aligned} A(z) &= C(z) + z^3D(z) \\ B(z) &= C(z) - z^3D(z) \end{aligned}$$

satisfy

$$A(z)A(z^{-1}) + B(z)B(z^{-1}) = 2n, \quad z \neq 0.$$

Consequently

$$\begin{aligned} L(z) &= A(z) + z^{n-1}B^*(z) \quad \text{and} \\ M(z) &= A(z) - z^{n-1}B^*(z) \end{aligned}$$

are $GS(2n)$; i.e.

$$L(z)L(z^{-1}) + M(z)M(z^{-1}) = 2(2n), \quad z \neq 0.$$

□

Example 1 (i) $n = 5$: $E = (++)$, $F = (+0)$; $G = (+-)$, $H = (00)$. From Theorem 1 we obtain: $C = (++ -)$, $D = (+0+)$, $A = (++ - +0+)$, $B = (++ - - 0-)$ and so the sequences $L = (++ - + - + - - ++)$, $M = (++ - + + + + - -)$ are $GS(10)$.

(ii) $n = 13$:

$$\begin{aligned} E &= (+0 - 0 + +), & F &= (+0 + 0 + 0); \\ G &= (+ + - + - -), & H &= (000000). \end{aligned}$$

From Theorem 1 we obtain:

$$\begin{aligned} C &= (++ + 0 - + + 0 - + -) \\ D &= (+000 - 000 + 0 +) \end{aligned}$$

and

$$\begin{aligned} A &= (+ + + + - + + - - + - + 0 +) \\ B &= (+ + + - - + + + - + - - 0 -) \end{aligned}$$

and so the sequences

$$\begin{aligned} L &= (+ + + + - + + - - + - + - + - - + - + + - - + + +) \\ M &= (+ + + + - + + - - + - + + + + - + - - - + + - - -) \end{aligned}$$

are GS(26).

3 On base sequences and near normal sequences

We formally prove the result of Yang [14]:

Theorem 2 ($E, F; G, H$), $n = 4m+1$ where $E = (X/O_{m-1}, 1)$ and $F = (Y/O_{m-1}, 0)$, being near normal sequences $NN(n)$, is equivalent to the ± 1 sequences $A = (Y/X, 1)$, $B = (Y/(-X), -1)$, $C = G + H$, $D = G - H$ of lengths $2m + 1, 2m + 1, 2m, 2m$ respectively being base sequences $BS(4m + 1)$.

Proof. Write

$$\begin{aligned} A(z) &= F(z^2) + zE(z^2) \\ B(z) &= F(z^2) - zE(z^2) \\ C(z) &= G(z) + H(z) \\ D(z) &= G(z) - H(z). \end{aligned} \tag{2}$$

From (2) we obtain

$$\begin{aligned} &A(z)A(z^{-1}) + B(z)B(z^{-1}) + C(z)C(z^{-1}) + D(z)D(z^{-1}) \\ &= (F(z^2) + zE(z^2))(F(z^{-2}) + z^{-1}E(z^{-2})) + (F(z^2) - zE(z^2))(F(z^{-2}) - z^{-1}E(z^{-2})) \\ &+ (G(z) + H(z))(G(z^{-1}) + H(z^{-1})) + (G(z) - H(z))(G(z^{-1}) - H(z^{-1})) \\ &= 2(E(z^2)E(z^{-2}) + F(z^2)F(z^{-2}) + G(z)G(z^{-1}) + H(z)H(z^{-1})) \\ &= 8m + 2 \end{aligned}$$

i.e. A, B, C, D are $BS(4m + 1)$.

The proof of the converse is straightforward. \square

The algorithm described in [8] can now be modified to give a special set of base sequences A, B, C, D of lengths $n+1, n+1, n, n$ as indicated in Theorem 2 to find near normal sequences $NN(2n+1)$. This modified algorithm was used to find the results in Tables 1 and 2. One of us (Joel Yang) has found a different algorithm (unpublished) for computing base sequences which gives us a method of independently verifying our results.

The near normal sequences given in Table 1 for $n = 61$ are due to C. Yang [15]. The base sequences of length 61 constructed from these are given in Table 2.

Length n	Sums of squares $a^2 + b^2 + c^2 + d^2$	Sequences
49	$7^2 + 0^2 + 0^2 + 0^2$	There are no near normal sequences.
49	$3^2 + 2^2 + 6^2 + 0^2$	$E = (-0 - 0 - 0 - 0 - 0 - 0 + 0 + 0 - 0 + 0 + 0 - +)$ $F = (+0 + 0 + 0 - 0 - 0 - 0 + 0 + 0 - 0 + 0 - 0 + 0)$ $G = (00 + 0 + +0 + 0 - -00 + +0 - 0 + +0 + 00)$ $H = (+ - 0 + 00 + 0 - 00 + -00 + 0 - 00 - 0 - +)$
49	$5^2 + 2^2 + 4^2 + 2^2$	$E = (+0 + 0 + 0 + 0 + 0 + 0 - 0 - 0 + 0 - 0 - 0 + +)$ $F = (+0 + 0 + 0 - 0 - 0 - 0 + 0 + 0 - 0 + 0 - 0 + 0)$ $G = (+ + 0 - 00 + 0 - 00 - -00 - 0 + 00 - 0 - -)$ $H = (00 + 0 + -0 - 0 + -00 - +0 - 0 + -0 - 00)$
49	$1^2 + 4^2 + 4^2 + 4^2$	$E = (-0 - 0 + 0 - 0 + 0 + 0 + 0 + 0 - 0 - 0 - 0 + +)$ $F = (+0 + 0 + 0 + 0 - 0 + 0 - 0 + 0 + 0 - 0 - 0 + 0)$ $G = (+0 + +00 + + - + + 00 - - + - + 00 - -0 +)$ $H = (0 + 00 - +00000 + +00000 - +00 + 0)$
53	$4^2 + 1^2 + 6^2 + 0^2$	$E = (+0 - 0 - 0 + 0 - 0 - 0 - 0 + 0 + 0 - 0 - 0 - 0 - +)$ $F = (+0 + 0 + 0 + 0 - 0 + 0 - 0 - 0 + 0 - 0 - 0 - 0 + 0)$ $G = (-0 + 00 + 0 + + + 00 - +00 - + + 0 + 00 - 0 +)$ $H = (0 - 0 + -0 + 000 + +00 - +000 - 0 - -0 + 0)$
57	$1^2 + 2^2 + 6^2 + 4^2$	$E = (-0 + 0 + 0 + 0 + 0 + 0 - 0 - 0 - 0 + 0 - 0 - 0 + 0 - +)$ $F = (+0 - 0 + 0 + 0 - 0 + 0 - 0 + 0 + 0 - 0 - 0 + 0 + 0 - 0)$ $G = (+ + 0 + 0 + 0 + + + - - 0 + -0 + + + - + 0 - 0 - 0 - +)$ $H = (00 - 0 - 0 + 00000 + 00 - 00000 - 0 - 0 - 00)$
61	$2^2 + 5^2 + 4^2 + 4^2$	$E = (-0 - 0 - 0 - 0 - 0 + 0 - 0 - 0 + 0 + 0 + 0 - 0 + 0 - 0 + +)$ $F = (+0 + 0 + 0 + 0 - 0 - 0 + 0 - 0 + 0 + 0 + 0 + 0 - 0 + 0)$ $G = (- - 00 - + + 0 + 0 - + + 0 + -0 + + + 0 + 0 - - + 00 - +)$ $H = (00 + -000 + 0 + 000 - 00 + 000 - 0 + 000 + +00)$

Table 1: Near normal sequences $NN(n)$: $(E, F; G, H)$, $n = 4m + 1$

Length n	Sums of squares $a^2 + b^2 + c^2 + d^2$	Sequences
39	$0^2 + 2^2 + 5^2 + 7^2$	$A = (- - + + + + + - - + + - - + - + -)$ $B = (- - + - - - + - - + - + + + + - - + +)$ $C = (+ + + + + - - - + + + - - + + - - +)$ $D = (+ + - + + + - - + - - + + + + + - +)$
39	$8^2 + 2^2 + 3^2 + 1^2$	$A = (+ - + + + + + - + + - + - + + + -)$ $B = (+ + + + + - - - + - - + - - + + - +)$ $C = (- + + + + - - + + + + - - + - - +)$ $D = (- - - + + + - - + - - - + + - + + +)$
41	$7^2 + 5^2 + 2^2 + 2^2$	$A = (+ + + + + + - + - + - - + - - + + - +)$ $B = (- + - + - + - - - - - + + + - - - +)$ $C = (+ + + + - + - - + + + + - - + - - -)$ $D = (+ - - - + + - - + + + + - - + - + + -)$
41	$9^2 + 1^2 + 0^2 + 0^2$	$A = (+ - + + + - - + - + + - + + + - + + +)$ $B = (- - - + - - + + + + - - - + + + - + +)$ $C = (- + + + + + - - + - + - - - + + -)$ $D = (- - + + - - + - + + + + - + + - - -)$
41	$5^2 + 5^2 + 4^2 + 4^2$	$A = (+ - - - + + - + - + + + + + - - - + +)$ $B = (- - - - + + + + - + - + - + - + + +)$ $C = (+ + - - + + - + - + + - + - - + + +)$ $D = (+ + - - - + - + - + - - + + - + + +)$
41	$1^2 + 7^2 + 4^2 + 4^2$	$A = (+ - - + + + - + - + - - - + - + + - +)$ $B = (- - + + - + - - - - - + - - - - + - +)$ $C = (+ + + + - + - + - + + - + + - - - +)$ $D = (+ + - - - - + + - + + + - + + + - +)$
43	$2^2 + 0^2 + 9^2 + 1^2$	$A = (- + + - - + - + - - + + + - - + + + + -)$ $B = (- + - + - - - + + + + - + - - + - - + +)$ $C = (- + + + - + - + - + + + + + + - + + -)$ $D = (- - + - + - - + - + + + + + - - + + - -)$
43	$8^2 + 2^2 + 3^2 + 3^2$	$A = (+ - - - - - + - + - - - + + - - - +)$ $B = (+ - - - + + + + + - + - - - - + - - -)$ $C = (+ - + + - - + + - + + - + - - - + + +)$ $D = (- - + - + + - - + + + - - + + + + + -)$
43	$6^2 + 0^2 + 7^2 + 1^2$	$A = (+ + - - + + - - + - - + + + + + - + + +)$ $B = (+ + - - + + + + + + - - - - + + - + -)$ $C = (- + - + - + + - + + - + + + - - + + +)$ $D = (- - + + - - - + + + + + - + - + - - +)$
45	$9^2 + 3^2 + 0^2 + 0^2$	$A = (- + + + + - + + - + - + + + + - + + + -)$ $B = (+ + + + + - + - - - + + - - - + + + - -)$ $C = (+ + + + - + + - - - - + - + + - - - +)$ $D = (+ + + + - - - + + - - - + + - - - + - +)$
45	$3^2 + 1^2 + 8^2 + 4^2$	$A = (+ + + + - + - - - + + - + - - - + + - + +)$ $B = (- - + + + - - + - - - + + - + + - + - +)$ $C = (+ + + + + + + + - + - + - - - + + + -)$ $D = (- - - + + + + + - - + + - + - - - + + +)$
45	$9^2 + 1^2 + 2^2 + 2^2$	$A = (+ + + + - + + + - + - + + + + - - - + - +)$ $B = (+ - + - + + + + - + + - + + + - - - -)$ $C = (+ - + - - - + + - + - + + + + - - - -)$ $D = (+ - + - - + - - + + - + + - - + + + - -)$

Table 2: Base sequences $BS(n)$: $(A, B; C, D)$ with lengths $m + 1, m + 1, m, m$, where $n = 2m + 1$ and $2n = a^2 + b^2 + c^2 + d^2$.

Length	Sums of squares	Sequences
$n = 2m + 1$	$a^2 + b^2 + c^2 + d^2$	
45	$1^2 + 7^2 + 6^2 + 2^2$	$A = (+ - + - - - - + - - + + + - + - - - + - +)$ $B = (- + + - + + - - - + + + - - - + + + + + +)$ $C = (+ - + + - - + + + + + + - + - - - - +)$ $D = (- + + - + - + - + + - - + + + - - + - + -)$
47	$0^2 + 2^2 + 9^2 + 3^2$	$A = (+ + + - - - - + - - + + - - + + - - + - +)$ $B = (+ + + + - - - + + - + - - - + - + - - - + -)$ $C = (+ + + + - - + - - - + + + + - + + + + + -)$ $D = (- - - + + - + + - + + - + - + + + + - - + -)$
47	$0^2 + 6^2 + 7^2 + 3^2$	$A = (+ - + + - + - + + - - + - - - + - + - - -)$ $B = (+ - + - - + - - + + + - - - - - + - - - +)$ $C = (+ + + - + + + - - + - + + + + - - - + - +)$ $D = (+ - + + - + - + - + - - - - + - - + + + +)$
49	$7^2 + 7^2 + 0^2 + 0^2$	$A = (+ + - - - + - + - + + + - + + - - + + + + +)$ $B = (+ + + + - + + - - + + - + + - + + + + - - -)$ $C = (+ + + + + + - - - + - + - - - - - + - + + -)$ $D = (+ - + - - + - + + + - + - - + + + - - + + - -)$
49	$1^2 + 5^2 + 6^2 + 6^2$	$A = (+ - + - + - - - - - + + + + - + + - + + - +)$ $B = (- - - - - + - + - + - - - + + + - - + + + - -)$ $C = (+ - + + + + + - - - + - + + + - + + - + - +)$ $D = (- + + - + + - + + - - - + + + - - + + + + +)$
49	$7^2 + 3^2 + 6^2 + 2^2$	$A = (+ + + + + - + - + - + + - + - - + + - - + +)$ $B = (- + - + - + + + + + + - - - + + - - + - - +)$ $C = (+ + + - + - + - - + - - - - + - - + + - - -)$ $D = (+ + - - - + + + - + - - - + - - + + + - + - -)$
49	$5^2 + 3^2 + 8^2 + 0^2$	$A = (+ - + - + + - - + + + - + + + - - - - + + +)$ $B = (- - - - - + - - + + - + + + - - - + - + - - +)$ $C = (+ + + + - + + - + + + + - - + - + - - - + +)$ $D = (+ - + + + - + + - + + - - - - + - + - - - +)$
51	$8^2 + 6^2 + 1^2 + 1^2$	$A = (+ + + + + + - - + + + - + - - + + - + - - +)$ $B = (- + + + + + + - - + + + - + - - + + - + - - +)$ $C = (+ + + + + + - - + + + - + - - + + - + - - +)$ $D = (+ - + - + - - - + + + - - + + - - - + + + + -)$
53	$5^2 + 3^2 + 6^2 + 6^2$	$A = (+ - - - + + - - - + - + + + + - + - - + + + +)$ $B = (- - - - + + - - - + - + + + + - + - - + + + +)$ $C = (+ + + - - + + + - + - - + - + - - + + - + + +)$ $D = (+ + + - - + + + - + - - + - + - - + + - + + +)$
53	$3^2 + 5^2 + 6^2 + 6^2$	$A = (- - - + - + - - + + - + + + - - - + + + + + - +)$ $B = (+ - + + + + - - + + + - - - - + - - + - + + + -)$ $C = (- - + + - + + + + + - + - + - + + - - - + +)$ $D = (- + + - + + - + + + - - + + - - + + + + + - -)$
55	$0^2 + 10^2 + 1^2 + 3^2$	$A = (+ - + - - - - + + + + + - + + - - + - - - + -)$ $B = (- - + + + + - + + + - - + + - - + + + + + - +)$ $C = (+ + + - + + - - - + + + + - - - - + - + - - +)$ $D = (- + - + + + - + + + + + - - - + + + - - - + -)$
55	$8^2 + 6^2 + 3^2 + 1^2$	can be obtained by changing the signs of even elements of each sequence

Table 2 (cont.): Base sequences $BS(n)$: $(A, B; C, D)$ with lengths $m + 1, m + 1, m, m$, where $n = 2m + 1$ and $2n = a^2 + b^2 + c^2 + d^2$.

Length n	Sums of squares $a^2 + b^2 + c^2 + d^2$	Sequences
55	$4^2 + 2^2 + 3^2 + 9^2$	$A = (- + + - + + - - - + + + - - - + - + - + + +)$ $B = (+ + + - - + - + + - - + - - + - - + - - + + +)$ $C = (+ + - + - + + + + - - + - + - - + + - - - -)$ $D = (+ + - + + - + + + - + + - - + + + + + + - -)$
57	$3^2 + 1^2 + 2^2 + 10^2$	$A = (+ - - + + + + - + + + - - + - - + - - + - - +)$ $B = (+ + - - + - + - - - + - - + + + + - - + + + - - -)$ $C = (+ + - + - + + + + - - + - - + + + - - - - - +)$ $D = (+ + + + + - + + + - - - + - + + + - + - + - +)$
59	$8^2 + 6^2 + 3^2 + 3^2$	$A = (+ + + + + - + + + + - + - - + + - - + + + - - - +)$ $B = (- + + + + - + + + + - - + + - - + + + + - - - + - +)$ $C = (+ + + - + - - - + - + - - - + - - + + + - + + + - -)$ $D = (+ - + + + + - + + + + - + - - - + + + - - - + + -)$
61	$5^2 + 4^2 + 4^2 + 2^2$	$A = (+ - + - + - + - + - - + - - + - - + + + + + - + - - + +)$ $B = (+ - - - - - - - - + + + - - + + - - + - - + + + - -)$ $C = (+ + - + + - - - - + - + - - - + - - + - - + + - + -)$ $D = (+ + + - + - - + - + + - - - + + - - - + + + - + + -)$

Table 2 (cont.): Base sequences BS(n): (A, B, C, D) with lengths
 $m + 1, m + 1, m, m$,
where $n = 2m + 1$ and $2n = a^2 + b^2 + c^2 + d^2$.

4 Multiplication of sequences and the sum of squares

Koukouvinos and Seberry [11] reformulated some results of Yang [14] and explicitly gave a method to multiply certain sequences by y to get four (disjoint) T-sequences of lengths $y(2m + 1)$. We refer to a positive (preferably odd) integer, y , as a Yang number if it can be used to multiply certain sequences with desired properties to get new, longer, sequences with the required desirable properties. So base sequences of lengths $m + 1, m + 1, m, m$ can be multiplied by y to get sequences of length $y(2m + 1)$ which are T-sequences. Koukouvinos [6] shows $y = 61$ is a Yang number. Combining all known results we have:

- (i) Yang numbers exist for $y \in \{n : n \leq 33, n = 37, 39, 41, 45, 49, 51, 53, 57, 59, 61, 65, 81, \dots, \text{and } n = 2g + 1, g = 2^a 10^b 26^c, a, b, c \text{ are non-negative integers}\}$;
- (ii) Base sequences of lengths $m + 1, m + 1, m, m$ ie. BS($2m + 1$) exist for $m \in \{1, 2, \dots, 30\} \cup G$, where $G = \{g : g = 2^a 10^b 26^c, a, b, c \text{ non-negative integers}\}$;
- (iii) Base sequences of lengths $2n - 1, 2n - 1, n, n$ exist for $n \in \{2, 4, 6, \dots, 20, 22, 24\}$.

Hence using the result that, if there are base sequences of lengths $m + p, m + p, m, m$ and y is a Yang number there are T-sequences of lengths $(2m + p)y$, we have that T-sequences exist for

$$\begin{aligned}
O_1 &= \{t : t \text{ odd } \leq 65\} \cup \{71\} \cup \{s : 75 \leq s \leq 199, s \text{ not prime}\} \\
O_2 &= \{y(2m + 1) : y \text{ a Yang number, } m \text{ a base sequence}\}
\end{aligned}$$

$$O_3 = \{g + g' : g, g' \in G \text{ (eg. we may take } g' = 1)\}$$

$$E_1 = \{2yp : y \text{ a Yang number and } p \in O_1 \cup O_2 \cup O_3 \cup E_1\}.$$

The results that we now give arise from composition theorems due to C.H. Yang [14] which are reformulated by Koukouvinos and Seberry [11] and summarized in Theorem 3.

Definition 2 $A = \{a_1, \dots, a_{m+1}\}$, $B = \{b_1, \dots, b_{m+1}\}$ and $C = \{c_1, \dots, c_m\}$, $D = \{d_1, \dots, d_m\}$ are said to be *suitable sequences* of length $m+1, m+1, m, m$ if they have zero periodic or non-periodic autocorrelation function (as appropriate) and $a_j \neq 0 \Rightarrow b_j = 0$, $b_j \neq 0 \Rightarrow a_j = 0$, $c_j \neq 0 \Rightarrow d_j = 0$, and $d_j \neq 0 \Rightarrow c_j = 0$.

Theorem 3 *There exist four (disjoint) T-sequences of lengths $y(2m+1)$ corresponding to decompositions indicated in Table 3, for $y = 13, 31, 37, 39, 41, 49, 53, 57, 61$.*

Proof. Let a, b, c, d be the row sums of suitable sequences of length $m+1, m+1, m, m$ so that $2m+1 = a^2 + b^2 + c^2 + d^2$. Then using Yang's method to multiply by y we get four (disjoint) T-sequences of lengths $y(2m+1)$ corresponding to a decomposition indicated in Table 3, for $y = 13, 31, 37, 39, 41, 49, 53, 57, 61$.

These kinds of results which specify decompositions into squares have been extensively used [10, 7, 12], to calculate the excess of Hadamard matrices and find regular symmetric Hadamard matrices of order $4k^2$ and SBIBD($4k^2, 2k^2 \pm k, k^2 \pm k$) for many values of k (see Seberry and Yamada [13]).

Remark 2 We note that by permuting the variables (or making them positive or negative) may give different sums of squares as

$$\begin{aligned} & (x^2 + y^2 + z^2 + w^2)(2s+1) = (x^2 + y^2 + z^2 + w^2)(a^2 + b^2 + c^2 + d^2) \\ & = (ax + by + cz + dw)^2 + (bx - ay + dz - cw)^2 + (cx - dy - az + bw)^2 + (dx + cy - bz - aw)^2 \\ & = (bx + ay + dz + cw)^2 + (ax - by + cz - dw)^2 + (dx - cy - bz + aw)^2 + (cx + dy - az - bw)^2 \\ & = (cx + dy + az + bw)^2 + (dx - cy + bz - aw)^2 + (ax - by - cz + dw)^2 + (bx + ay - dz - cw)^2 \\ & = (dx + cy + bz + aw)^2 + (cx - dy + az - bw)^2 + (bx - ay - dz + cw)^2 + (ax + by - cz - dw)^2. \end{aligned}$$

Remark 3 We note that choosing a, b, c and d in Table 3 for the decomposition $39 = 5^2 + 3^2 + 2^2 + 1^2$ we get for different permutations different sums of squares:

(a, b, c, d)	$13 \cdot 39$	$(3a + 2b)^2 + (2a - 3b)^2 + (3c + 2d)^2 + (2c - 3d)^2$
5, 3, 2, 1	$13 \cdot 39$	$= 21^2 + 1^2 + 8^2 + 1^2$
5, 3, 1, 2	$13 \cdot 39$	$= 21^2 + 1^2 + 7^2 + 4^2$
3, 5, 2, 1	$13 \cdot 39$	$= 19^2 + 9^2 + 8^2 + 1^2$
3, 5, 1, 2	$13 \cdot 39$	$= 19^2 + 9^2 + 7^2 + 4^2$
5, 2, 3, 1	$13 \cdot 39$	$= 19^2 + 4^2 + 11^2 + 3^2$
2, 5, 3, 1	$13 \cdot 39$	$= 16^2 + 11^2 + 11^2 + 3^2$
2, 5, 3, 1	$13 \cdot 39$	$= 16^2 + 11^2 + 9^2 + 7^2$
5, 1, 3, 2	$13 \cdot 39$	$= 17^2 + 7^2 + 13^2 + 0^2$
5, 1, 2, 3	$13 \cdot 39$	$= 17^2 + 7^2 + 12^2 + 5^2$
1, 5, 3, 2	$13 \cdot 39$	$= 13^2 + 13^2 + 13^2 + 0^2$
1, 5, 2, 3	$13 \cdot 39$	$= 13^2 + 13^2 + 12^2 + 5^2$

$\frac{y = 13}{13(2s + 1)}$	$= (3d - 2a)^2 + (3c + 2b)^2 + (3b - 2c)^2 + (3a + 2d)^2$
$\frac{y = 31}{31(2s + 1)}$	$= (a - b + 5c + 2d)^2 + (a + b - 2c + 5d)^2 + (5a - 2b - c - d)^2$
$\frac{y = 37}{37(2s + 1)}$	$= (4a - b + 4c + 2d)^2 + (a + 4b - 2c + 4d)^2 + (4a - 2b - 4c - d)^2$
$\frac{y = 39}{39(2s + 1)}$	$= (5a - b + 3c + 2d)^2 + (a + 5b - 2c + 3d)^2 + (3a - 2b - 5c - d)^2$
$\frac{y = 41}{41(2s + 1)}$	$= (2a - b + 6c)^2 + (a + 2b + 6d)^2 + (6a - 2c - d)^2 + (6b + c - 2d)^2$
$\frac{y = 49}{49(2s + 1)}$	$= (-4a + 2b - 2c - 5d)^2 + (-2a - 4b + 5c - 2d)^2 + (2a - 5b - 4c - 2d)^2$ $+ (5a + 2b + 2c - 4d)^2$
$\frac{y = 53}{53(2s + 1)}$	$= (6a - c + 4d)^2 + (6b - 4c - d)^2 + (a + 4b + 6c)^2 + (-4a + b + 6d)^2$
$\frac{y = 57}{57(2s + 1)}$	$= (6a + 4b - 2c - d)^2 + (-4a + 6b + c - 2d)^2 + (2a - b + 6c - 4d)^2$ $+ (a + 2b + 4c + 6d)^2$
$\frac{y = 61}{61(2s + 1)}$	$= (4a + 4b - 2c - 5d)^2 + (4a - 4b - 5c + 2d)^2 + (-2a + 5b - 4c + 4d)^2$ $+ (5a + 2b + 4c + 4d)^2$

Table 3: Decompositions arising from Yang's composition $y(2s + 1)$ where $2s + 1 = a^2 + b^2 + c^2 + d^2$.

References

- [1] G. Cohen, D. Rubie, J. Seberry, C. Koukouvinos, S. Kounias, and M. Yamada. A survey of base sequences, disjoint complementary sequences and $OD(4t; t, t, t, t)$. *JCMCC*, 5:69-104, 1989.
- [2] Genet M. Edmondson, Jennifer Seberry and Malcolm R. Anderson. On the existence of Turyn sequences of length less than 43, *Math. Comput.*, 62:351-362, 1994.
- [3] S. Eliahou, M. Kervaire, and B. Saffari. A new restriction on the lengths of Golay complementary sequences. *J. Combin. Th. (Ser A)*, 55:49-59, 1990.

- [4] Marc Gysin, *On Normal and Near-Yang Sequences*, M.Sc. Thesis, University of Wollongong, 1993.
- [5] Marc Gysin and Jennifer Seberry, New results with near-Yang sequences, *Bull. Inst. Comb. Appl.*, to appear.
- [6] C. Koukouvinos, Construction of some sequences with zero autocorrelation function, *JCMCC*, to appear.
- [7] Christos Koukouvinos, Stratis Kounias, and Jennifer Seberry, Further Hadamard matrices with maximal excess and new SBIBD($4k^2$, $2k^2 + k$, $k^2 + k$), *Utilitas Math.*, 36:135-150, 1989.
- [8] C. Koukouvinos, S. Kounias, and K. Sotirakoglou. On base and Turyn sequences. *Math. Comput.*, 55:825-837, 1990.
- [9] C. Koukouvinos, S. Kounias, J. Seberry, C.H. Yang and J. Yang. On sequences with zero autocorrelation, *Codes, Designs and Crypt*, to appear.
- [10] Christos Koukouvinos and Jennifer Seberry, Construction of new Hadamard matrices with maximal excess and infinitely many new SBIBD($4k^2$, $2k^2 + k$, $k^2 + k$), *Graphs, Matrices and Designs: A Festschrift for Norman J. Pullman*, by Rolf Rees, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, 1992.
- [11] C. Koukouvinos and J. Seberry. Addendum to Further results on base sequences, disjoint complementary sequences, OD($4t$; t, t, t, t) and the excess of Hadamard matrices. *Congr. Num.*, 82:97-103, 1991.
- [12] Jennifer Seberry, Existence of SBIBD($4k^2$, $2k^2 + k$, $k^2 + k$) and Hadamard matrices with maximal excess, *Austral. J. Combinatorics*, 4:87-92, 1991.
- [13] Jennifer Seberry and Mieko Yamada, Hadamard matrices, Sequences, and Block Designs, *Contemporary Design Theory - A Collection of Surveys*, Eds D. J. Stinson and J. Dinitz, John Wiley and Sons, 431-560, 1992.
- [14] C.H. Yang. On composition of four-symbol δ -codes and Hadamard matrices. *Proc. Amer. Math. Soc.*, 107:763-776, 1989.
- [15] C.H. Yang. On Golay, near normal and base sequences. In preparation.

(Received 20/4/93; revised 13/12/93)