New Results with Near-Yang Sequences

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Abstract. We construct new TW-sequences, weighing matrices and orthogonal designs using near-Yang sequences. In particular we construct new $OD(60(2m+1)+4t; 13(2m+1), 13(2m+1), 13(2m+1), 13(2m+1))$ and new $W(60(2m+1)+4t; 13s(2m+1))$ for all $t \geq 0$, $m \leq 30$, $s = 1,2,3,4$.

1. Introduction

For definitions we refer the reader to [9, Introduction] and [11, Section 2]. We give one new definition.

Definition 1. (near-Yang sequences) *A triple (F, G, H) of sequences is said to be a set of near-Yang sequences for length n (abbreviated as $NY(n)$) if the following conditions are satisfied.*

(i) $F = (f_k)$ *is a* $(0, 1, -1)$ *sequence of length* n.
(ii) $G = (g_k)$ *and* $H = (h_k)$ *are sequences of length* n *with entries* $0, 1, -1$, *such that* $G + H = (g_k + h_k)$ *and* $G - H = (g_k - h_k)$ *are both* $(0, 1, -1)$ *sequences of length* n.
(iii)

$$\begin{aligned} g_s + g_{n-s+1} &\equiv 0 \pmod 2 \\ h_s + h_{n-s+1} &\equiv 0 \pmod 2 \end{aligned} \qquad s = 1, \ldots, [\tfrac{n}{2}]$$

(iv) $N_F(s) + N_G(s) + N_H(s) = 0, \quad s = 1, \ldots, n-1.$

where

$$N_X(s) = \sum_{i=1}^{n-s} x_i x_{i+s}.$$

2. Computational Results

In Koukouvinos, Kounias, Seberry, Yang and Yang [6] it is shown that if in (ii) of the definition $G \pm H$ are both $(1, -1)$ sequences then conditions (i), (ii) and (iv) imply condition (iii) but this is not true for near-Yang sequences. These sequences are normal sequences $NS(\ell)$.

We searched for normal sequences $NS(\ell)$. $NS(\ell)$ do exist for the following lengths $\ell \in \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 15, 16, 18, 19, 20, 25, 26, 29, 32, \ldots\}$ and they do not exist for $\ell \in \{6, 14, 17, 21, 22, 23, 30, 46, 56, 62, 78,$

Utilitas Mathematica 45(1994), pp. 85-89

Table 1: Normal sequences via Simulated Annealing.

Length ℓ	Sequences	Weight
20	$F = --++++++-++-+-+++-+-$	20
	$G = 0+-000+000000+000-+0$	6
	$H = -00+++0-++--+0---00+$	14
25	$F--+-+-+++---++--+--+$	25
	$G = 0-0-0+0+0+0-0+0-0-0-0+0+0$	12
	$H = -0-0-0+0+0-0+0-0+0+0-0-0+$	13

$94, \ldots\}$ [3,6,16]. We note here that we found normal sequences of length 25 and 20 (Table 1) using simulated annealing; this is described in Gysin [3]. Sequences of length 25 can be obtained from Turyn sequences of lengths 13 and 12 and a complete search for these was carried out over 20 years ago. It is known that there are eight inequivalent sets of Turyn sequences of lengths 13 and 12 [2] and hence by the construction discussed in [6] probably at least sixteen inequivalent sets of normal sequences of length 25. It would be interesting to know if there are $NS(25)$ which cannot be made from Turyn sequences. There exist $NS(20)$ which cannot be made from Turyn sequences, an example is given in Table 1. In those small cases where $NS(\ell)$ do not exist we searched for $NY(n)$, which contain more zeros in appropiate positions. We obtained the following new results:

$NY(n)$ with weight $u = 12$ exist for the following lengths: $n \in \{7, 11, 13, 15\}$.

In Table 2 two conditions were imposed in counting the number of inequivalent triples of sequences: two triples of sequences were considered equivalent if one triple of sequences can be changed into the other triple of sequences by reversing and/or negating one or more sequences of the triple; if the three sequences F, G and H all started or ended with '0' they were considered to be of smaller length and not counted for this length n.

This allows us to find new 4-complementary sequences of lengths $15(2m+1)$, $23(2m+1), 27(2m+1), 31(2m+1)$ and weights $13(2m+1), m \leq 30$.

3. Construction

Definition 2. (suitable sequences) [5,8,11] *A, B, C, D are suitable sequences $SS(m+p, m; w)$ with elements 0, 1, -1 of lengths $m+p$, $m+p$, m, m and total total weight w if A and B are disjoint, C and D are disjoint and A, B, C, and D have zero non-periodic autocorrelation function.*

We use a modified version of Yang's [5,8,16] theorem

Table 2: New near-Yang sequences.

Length n	No of Seq.	Sequence Examples	Weight
7	2	$F = +++0-+-$	6
		$G = 0++0-+0$	4
		$H = +00000+$	2
		$F = +++0-+-$	6
		$G = 0+000+0$	2
		$H = +0+0-0+$	4
11	12	$F = -0+0+00000+$	4
		$G = 0+0+000+0-0$	4
		H = +0 + 00000-0+	4
13	24	$F = +0+0+0-0+000-$	6
		$G = 0+000000000+0$	2
		$H = +00+00000-00+$	4
15	26	$F = +0+00000+00000-$	4
		$G = 0+0000+0-0000+0$	4
		$H = +00+0000000-00+$	4

Theorem 1. *Let A, B, C, D be $SS(m+p, m; w)$ and F, G, H be $NY(n)$ with total weight u and $0', 0$ be sequences of zeros of length $m+p$ and m respectively and X^* be the reverse sequence of X then*

$$Q = \{Af_n, Cg_1 - Dh_1; 0', 0; Af_{n-1}, C_{g_2} - D_{h_2}; 0', 0; \ldots; A_{f_1}, C_{g_n} - D_{h_n}; 0', 0; B^*, 0\}$$

$$R = \{Bf_n, Dg_n + Ch_n; 0', 0; Bf_{n-1}, Dg_{n-1} + Ch_{n-1}; 0', 0; \ldots; B_{f_1}, D_{g_1} + C_{h_1}; 0', 0; -A^*; 0\}$$

$$S = \{0', 0; Ag_n + Bh_1, -Cf_1; 0', 0; Ag_{n-1} + Bh_2, -Cf_2; \ldots; 0', 0; A_{g_1} + B_{h_n}, -C_{f_n}; 0', D^*\}$$

$$T = \{0', 0; -Bg_1 + Ah_n, Df_1; 0', 0; -Bg_2 + Ah_{n-1}, Df_2; \ldots; 0', 0; -Bg_n + Ah_1, Df_n; 0', C^*\}$$

are TW-sequences of length $(2m+p)(2n+1)$ and total weight $(u+1)w$.

This gives many new TW-sequences, weighing matrices and orthogonal designs. Many other corollaries are also possible.

Example 1. Let $F = \{+++0-+-\}$, $G = \{0+000+0\}$, $H = \{+0+0-0+\}$ and A, B, C, D be suitable sequences of length $m+p$ and m and total weight w. Then with $0'$ and 0 zero vectors of length $m+p$ and m respectively we have

$$Q = \{-A,-D;0',0;A,C;0',0;-A,-D;0',0;0',0;0',0;$$
$$A,D;0',0;A,C;0',0;A,-D;0',0;B^*,0\}$$
$$R = \{-B,C;0',0;B,D;0',0;-B,-C;0',0;0',0;0',0;$$
$$B,C;0',0;B,D;0',0;B,C;0',0;-A^*,0\}$$
$$S = \{0',0;B,-C;0',0;A,-C;0',0;B,-C;0',0;0',0;$$
$$0',0;-B,C;0',0;A,-C;0',0;B,C;0',D^*\}$$
$$T = \{0',0;A,D;0',0;-B,D;0',0;-A,D;0',0;0',0;$$
$$0',0;A,-D;0',0;-B,D;0',0;A,-D;0',C^*\}$$

are TW-sequences of of length $15(2m+p)$ and total weight $13w$.

Corollary 1. *Suppose there are suitable sequences of length $m+p$, $m+p$, m, m and total weight w, $SS(m+p,m;w)$ and near-Yang sequences of length n and total weight u. Then there are TW-sequences of length $(2n+1)(2m+p)$ and total weight $(u+1)w$.*

Corollary 2. *Suppose there exist $SS(m+p,m;w)$. Then since there are near-Yang sequences of length 7 and total weight 12 there are TW-sequences of length $15(2m+p)$ and total weight $13w$.*

From [11] we see $SS(m+1,m;2m+1)$ exist for all $m \le 30$ hence there exist TW-sequences of length $15(2m+1)$ and weight $13(2m+1)$ for all $m \le 30$. Using theorems 3.6, 3.7, 3.8 of [11] we have $OD(60(2m+1)+4t; 13(2m+1), 13(2m+1), 13(2m+1), 13(2m+1))$ for all $t \ge 0$ and $m \le 30$. Furthermore Q, R, S, T can be used in the Goethals-Seidel array to form $OD(4t; 13w, 13w, 13w, 13w)$ for every $t > 15(2m+1)$.

Recalling that variables in an OD can be set equal or set zero to give weighing matrices, we obtain $W(60(2m+1)+4t; 13s(2m+1))$ and $W(4t; 13sw)$, $s = 1,2,3,4$. Since $NS(n)$ are $NY(n)$ with total weight $2n$.

Corollary 3. *Suppose there exist $SS(m+p,m;w)$. Then since there are normal sequences of length 25 and total weight 50 there are TW-sequences of length $51(2m+p)$ and total weight $51w$. If $w = 2m+p$ then we have T-sequences.*

Again using theorems 3.6, 3.7, 3.8 of [11] we have $OD(104(2m+1)+4t; 51(2m+1), 51(2m+1), 51(2m+1), 51(2m+1))$ for all $t \ge 0$ and $m \le 30$. Furthermore Q, R, S, T can be used in the Goethals-Seidel array to form $OD(4t; 51w, 51w, 51w, 51w)$ for every $t > (2n+1)(2m+1)$.

Acknowledgement

We wish to thank the University of Trondheim, Norway for letting us use hundreds of CPU-hours on their machines "fiolilla" and "lise".

References

[1] Joan Cooper and Jennifer Wallis, *A construction for Hadamard arrays*, Bull. of the Austr. Math. Soc. **7, No 2** (1972), 269–277.

[2] G. M. Edmondson, J. Seberry, M. R. Anderson, *On the existence of Turyn sequences of length less than 43*, Math. Comp. to appear.

[3] M. Gysin, On Normal and Near-Yang Sequences. M.Sc. Thesis, University of Wollongong, to be published.

[4] A.V. Geramita and J. Seberry, "Orthogonal Designs: Quadratic Forms and Hadamard Matrices", Marcel Dekker, New York - Basel, 1979.

[5] C. Koukouvinos, S. Kounias, and J. Seberry, *Further results on base sequences, disjoint complementary sequences, $OD(4t; t, t, t, t)$ and the excess of Hadamard matrices*, Ars Combinatoria **30** (1990), 241–256.

[6] C. Koukouvinos, S. Kounias, J. Seberry, C.H. Yang and J. Yang. Multiplication of sequences with zero autocorrelation, to appear.

[7] C. Koukouvinos, S. Kounias, J. Seberry, C.H. Yang and J. Yang. On sequences with zero autocorrelation, to appear.

[8] C. Koukouvinos and J. Seberry, *Addendum to further results on base sequences, disjoint complementary sequences, $OD(4t; t, t, t, t)$ and the excess of Hadamard matrices*, Congr. Num. **82** (1991), 97–103.

[9] C.Koukouvinos and J.Seberry. On weighing matrices, Utilitas Math., to appear.

[10] C.Koukouvinos and J.Seberry. On sequences with zero autocorrelation function, submitted.

[11] Jennifer Seberry and Mieko Yamada, *Hadamard matrices, sequences and block designs*, in "Contemporary Design Theory - a Collection of Surveys, eds J. Dinitz and D.J. Stinson", John Wiley and Sons, New York, 1992, pp. 431–560.

[12] C.H. Yang, *Hadamard matrices and δ-codes of length $3n$*, Proc. Amer. Math. Soc. **85** (1982), 480–482.

[13] C.H. Yang, *A composition theorem for δ-codes*, Proc. Amer. Math. Soc. **89** (1983), 375–378.

[14] C.H. Yang, *Lagrange identity for polynomials and δ-codes of lengths $7t$ and $13t$*, Proc. Amer. Math. Soc. **88** (1983), 746–750.

[15] C.H. Yang, *On composition of four-symbol δ-codes and Hadamard matrices*, Proc. Amer. Math. Soc. **107** (1989), 763–776.

[16] C.H. Yang. On Golay, near normal and base sequences, in preparation.