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Regular Sets of Matrices and Applications

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Abstract. Suppose $A_1, \ldots, A_s$ are $(1, -1)$ matrices of order m satisfying

$$A_i A_j = J, \qquad i, j \in \{1, \ldots s\} \tag{1}$$

$$A_i^T A_j = A_j^T A_i = J, \qquad i \neq j, \qquad i, j \in \{1, \ldots, s\} \tag{2}$$

$$\sum_{i=1}^{s} (A_i A_i^T + A_i^T A_i) = 2smI_m \tag{3}$$

$$JA_i = A_i J = aJ, \qquad i \in \{1, \ldots, s\}, a \text{ constant} \tag{4}$$

Call $A_1, \ldots, A_s$ a *regular s-set of matrices of order m* if Eq. 1–3 are satisfied and a regular s-set of regular matrices if Eq. 4 is also satisfied, these matrices were first discovered by J. Seberry and A.L. Whiteman in "New Hadamard matrices and conference matrices obtained via Mathon's construction", Graphs and Combinatorics, 4(1988), 355–377. In this paper, we prove that

(i) if there exist a regular s-set of order m and a regular t-set of order n there exists a regular s-set of order mn when $t = sm$

(ii) if there exist a regular s-set of order m and a regular t-set of order n there exists a regular s-set of order mn when $2t = sm$ (m is odd)

(iii) if there exist a regular s-set of order m and a regular t-set of order n there exists a regular $2s$-set of order mn when $t = 2sm$

As applications, we prove that if there exist a regular s-set of order m there exists

(iv) an Hadamard matrices of order $4hm$ whenever there exists an Hadamard matrix of order $4h$ and $s = 2h$

(v) Williamson type matrices of order nm whenever there exists Williamson type matrices of order n and $s = 2n$

(vi) an $OD(4mp; ms_1, \ldots, ms_u)$ whenever an $OD(4p; s_1, \ldots, s_u)$ exists and $s = 2p$

(vii) a complex Hadamard matrix of order $2cm$ whenever there exists a complex Hadamard matrix of order $2c$ and $s = 2c$

This paper extends and improves results of Seberry and Whiteman giving new classes of Hadamard matrices, Williamson type matrices, orthogonal designs and complex Hadamard matrices.

1. Introduction and Basic Definitions

This paper uses sets of matrices first introduced by Seberry and Whiteman [1] to find new classes of Hadamard matrices, Williamson type matrices, orthogonal designs and complex Hadamard matrices. We write J for the matrix of ones, I for the identity matrix and A^T for the transpose of the matrix A.

Definition 1. Suppose $A_1, \ldots, A_s$ are $(1, -1)$ matrices of order m satisfying

$$A_i A_j = J, \qquad i, j \in \{1, \ldots, s\} \tag{5}$$

$$A_i^T A_j = A_j^T A_i = J, \qquad i \neq j, \qquad i, j \in \{1, \ldots, s\} \tag{6}$$

$$\sum_{i=1}^{s} (A_i A_i^T + A_i^T A_i) = 2smI_m \tag{7}$$

$$JA_i = A_i J = aJ, \qquad i \in \{1, \ldots, s\}, a \text{ constant} \tag{8}$$

Call $A_1, \ldots, A_s$ a *regular s-set of matrices* of order m if Eq. 5–7 are satisfied [2, 1] and a *regular s-set of regular matrices* if Eq. 8 is also satisfied.

J. Seberry and A.L. Whiteman [1] proved that if $q \equiv 3 \pmod 4$ is a prime power there exists a regular $\frac{1}{2}(q+1)$-set of regular matrices of order q^2, say A_i, $i = 1, \ldots, \frac{1}{2}(q+1)$ satisfying $A_i J = JA_i = qJ$.

Definition 2. Four $(1, -1)$ matrices X_1, X_2, X_3, X_4 of order n satisfying

$$X_1 X_1^T + X_2 X_2^T + X_3 X_3^T + X_4 X_4^T = 4nI_n$$

and

$$UV^T = VU^T$$

where $U, V \in \{X_1, X_2, X_3, X_4\}$ will be called *Williamson type matrices.*

Williamson and Williamson type matrices are discussed extensively by Baumert, Miyamoto, Seberry, Whiteman, Yamada and Yamamoto [2–13].

Definition 3. The matrix $M = (m_{ij})$ of order m satisfying $m_{1,j+1} = m_{i,j+i}$, where the subscripts are the residues of m, is called a *circulant matrix.* If $m_{1,j} = m_{i,j-i+1}$, M is called a *back-circulant matrix.*

Definition 4. An *orthogonal design A*, of order p and type $(s_1, \ldots, s_u)$, denoted by $OD(p; s_1, \ldots, s_u)$, on the commuting variables $\pm x_1, \ldots, \pm x_u, 0$ is a matrix of order p with entries $\pm x_1, \ldots, \pm x_u, 0$ satisfying

$$AA^T = (s_1 x_1^2 + \cdots + s_u x_u^2) I_p$$

Definition 5. Let C be a $(1, -1, i, -i)$ matrix of order c satisfying $CC^* = cI_c$, where C^* is the Hermitian conjugate of C. We call C a *complex Hadamard matrix* of order c.

From [14], any complex Hadamard matrix has order 1 or order divisible by 2. Let $C = X + iY$, where X, Y consist of $1, -1, 0$ and $X \wedge Y = 0$ where $\wedge$ is the Hadamard product. Clearly, if C is a complex Hadamard matrix then $XX^T + YY^T = cI_c$, $XY^T = YX^T$

2. Product of Two Sets of Matrices

Theorem 1. *If there exist a regular s-set of matrices of order m and a regular $t(=sm)$-set of matrices of order n then there exists a regular s-set of matrices of order mn.*

Proof. Let $\{A_1 = (a^1_{ij}), A_2 = (a^2_{ij}), \ldots, A_s = (a^s_{ij})\}$ be the regular s-set of matrices of order m and $\{B_1, B_2, \ldots, B_t\}$ be the regular t-set of matrices of order of n.

Define $C_i = (c^i_{kj}) = (a^i_{kj} B_{(i-1)m+j+k-1})$, $i = 1, \ldots, s$ a block back circulant matrix, so that

$$C_i = \begin{bmatrix} a^i_{11} B_{(i-1)m+1} & a^i_{12} B_{(i-1)m+2} & \cdots & a^i_{1m} B_{im} \\ a^i_{21} B_{(i-1)m+2} & a^i_{22} B_{(i-1)m+3} & \cdots & a^i_{2m} B_{(i-1)m+1} \\ & & \vdots & \\ a^i_{m1} B_{im} & a^i_{m2} B_{(i-1)m+1} & \cdots & a^i_{mm} B_{im-1} \end{bmatrix}$$

Since both $\{A_1, A_2, \ldots, A_s\}$ and $\{B_1, B_2, \ldots, B_t\}$ are regular r-set of matrices, $r = s$, t respectively, we have

$$C_i C_j = J_m \times J_n = J_{mn}, \qquad i, j \in \{1, \ldots, s\}$$

$$C_i C_j^T = C_j^T C_i = J_m \times J_n = J_{mn}, \qquad i \neq j,\ i, j \in \{1, \ldots, s\}$$

To show

$$\sum_{i=1}^{s} (C_i C_i^T + C_i^T C_i) = 2smnI_{mn} \tag{9}$$

note that $(a^i_{kj})^2 = 1$ so the diagonal element of $C_i C_i^T + C_i^T C_i$ is

$$\sum_{j=1}^{m} (B_{(i-1)m+j} B^T_{(i-1)m+j} + B^T_{(i-1)m+j} B_{(i-1)m+j})$$

and hence the diagonal element of $\sum_{i=1}^{s} (C_i C_i^T + C_i^T C_i)$ is

$$\sum_{j=1}^{sm} (B_j B_j^T + B_j^T B_j) = 2tnI_n = 2smnI_n$$

The off-diagonal elements of $C_i C_i^T$ are given by

$$\sum_{j=1}^{m} (a^i_{hj} a^i_{kj} B_{(i-1)m+j+h-1} B^T_{(i-1)m+j+k-1}),\ h \neq k$$

$$= \sum_{j=1}^{m} a^i_{hj} a^i_{kj} J$$

So the off-diagonal element of $\sum_{j=1}^{s} (C_i C_i^T + C_i^T C_i)$, is, taking into account diagonal elements of Eq. 9 for $A_1, \ldots, A_s$ is zero,

$$\sum_{i=1}^{s} \sum_{j=1}^{m} (a^i_{hj} a^i_{kj} + a^i_{jh} a^i_{jk}) J = 0 \qquad \square$$

We also note that if $B_j J_n = bJ_n$ and $A_i J_m = aJ_m$, (part Eq. 8 of the Definition 1), then

$$\left(\sum_{j=1}^{m} a^i_{kj} B_l\right) J_n = \left(\sum_{j=1}^{m} a^i_{kj}\right) bJ_n = abJ_n$$

and $C_i J_{mn} = abJ_{mn}$. Similarly $J_{mn} C_i = abJ_{mn}$. Thus we have

Corollary 1. *If there exist a regular s-set of regular matrices of order m and a regular t(=sm)-set of regular matrices of order n then there exists a regular s-set of regular matrices of order mn.*

We now use a result of Seberry and Whiteman [1] who showed that if $q \equiv 3$ (*mod* 4) is a prime power there exists a regular $\frac{1}{2}(q+1)$-set of regular matrices of order q^2.

Corollary 2. *If both $q \equiv 3$ (mod 4) and $(q+1)q^2 - 1$ are prime powers there exists a regular $\frac{1}{2}(q+1)$-set of regular matrices of order $q^2((q+1)q^2-1)^2$.*

Proof. Note $(q+1)q^2 - 1 \equiv 3$ (*mod* 4). By [1], there exist both a regular $\frac{1}{2}(q+1)$-set of regular matrices of order q^2 and a regular $\frac{1}{2}(q+1)q^2$-set of regular matrices of order $((q+1)q^2-1)^2$. Using Theorem 1, we have a regular $\frac{1}{2}(q+1)$-set of matrices of order $q^2((q+1)q^2-1)^2$. □

A result of Seberry and Whiteman (see Theorem 12 of [1]) would now give the next Corollary which is new. We shall give another proof of their results in the section entitled "Williamson Type Matrices."

Corollary 3. *If both $q \equiv 3$ (mod 4) and $(q+1)q^2 - 1$ are prime powers there exists an Hadamard matrix of order $q^2(q+1)((q+1)q^2-1)^2$.*

Proof. By Theorem 1, there exists a regular $\frac{1}{2}(q+1)$-set of matrices of order $q^2(q^2(q+1)-1)^2$. On the other hand, from the Index, [2], there exists an Hadamard matrix of order $q+1$. Finally, by Theorem 12, [1], we have an Hadamard matrix of order $q^2(q+1)(q^2(q+1)-1)^2$. □

Theorem 2. *If there exist a regular s-set of matrices of order m and a regular t-set of matrices of order n then there exists a regular s-set of matrices of order mn, when $2t = sm$ (m is odd).*

Proof. Let $\{A_1 = (a_{ij}^1), A_2 = (a_{ij}^2), \ldots, A_s = (a_{ij}^s)\}$ be the regular s-set of matrices of order m and $\{B_1, B_2, \ldots, B_t\}$ be the regular t-set of matrices of order n. Note $t = \frac{1}{2}sm$, $\frac{1}{2}s$ is an integer as $2t = sm$ and m is odd. Set $\frac{1}{2}s = r$. For $i = 1, \ldots, r$, define $C_i = (c_{kj}^i) = (a_{kj}^i B_{(i-1)m+j+k-1})$, note C_i is a back circulant matrix of blocks i.e.

$$C_i = \begin{bmatrix} a_{11}^i B_{(i-1)m+1} & a_{12}^i B_{(i-1)m+2} & \cdots & a_{1m}^i B_{im} \\ a_{21}^i B_{(i-1)m+2} & a_{22}^i B_{(i-1)m+3} & \cdots & a_{2m}^i B_{(i-1)m+1} \\ & & \vdots & \\ a_{m1}^i B_{im} & a_{m2}^i B_{(i-1)m+1} & \cdots & a_{mm}^i B_{im-1} \end{bmatrix}$$

and for $i = r+1, \ldots, 2r = s$, $C_i = (c_{kj}^i) = (a_{kj}^i B_{(i-1)m+j+k-1}^T)$, i.e.

$$C_i = \begin{bmatrix} a_{11}^i B_{(i-1)m+1}^T & a_{12}^i B_{(i-1)m+2}^T & \cdots & a_{1m}^i B_{im}^T \\ a_{21}^i B_{(i-1)m+2}^T & a_{22}^i B_{(i-1)m+3}^T & \cdots & a_{2m}^i B_{(i-1)m+1}^T \\ & & \vdots & \\ a_{m1}^i B_{im}^T & a_{m2}^i B_{(i-1)m+1}^T & \cdots & a_{mm}^i B_{im-1}^T \end{bmatrix}.$$

Since both $\{A_1, A_2, \ldots, A_s\}$ and $\{B_1, B_2, \ldots, B_t\}$ are regular l-set of matrices, $l = s$, t respectively, we have

$$C_i C_j = J_m \times J_n = J_{mn}, \qquad i, j \in \{1, \ldots, s\}$$
$$C_i C_j^T = C_j^T C_i = J_m \times J_n = J_{mn}, \qquad i \neq j,\ i, j \in \{1, \ldots, s\}$$

We now prove $\sum_{i=1}^{s}(C_iC_i^T + C_i^TC_i) = 2smnI_{mn}$. Note that $(a_{kj}^i)^2 = 1$ so the diagonal element of $C_iC_i^T + C_i^TC_i$ is

$$\sum_{j=1}^{m}(B_{(i-1)m+j}B_{(i-1)m+j}^T + B_{(i-1)m+j}^TB_{(i-1)m+j})$$

for $i = 1, \ldots, r$ and

$$\sum_{j=1}^{m}(B_{(i-1)m+j}^TB_{(i-1)m+j} + B_{(i-1)m+j}B_{(i-1)m+j}^T)$$

for $i = r+1, \ldots, s$. So the diagonal element of $\sum_{i=1}^{s}(C_iC_i^T + C_i^TC_i)$ is

$$2\sum_{j=1}^{rm}(B_jB_j^T + B_j^TB_j) = 2\sum_{j=1}^{t}(B_jB_j^T + B_j^TB_j) = 2\cdot 2tnI_n = 2smnI_n$$

The off-diagonal elements of $C_iC_i^T$ are given by

$$\sum_{j=1}^{m}(a_{hj}^ia_{kj}^iB_{(i-1)m+j+h-1}B_{(i-1)m+j+k-1}^T), \qquad h \neq k$$

$$= \sum_{j=1}^{m}a_{hj}^ia_{kj}^iJ$$

for $i = 1, \ldots, r$. By the same reasoning, the off-diagonal elements of $C_iC_i^T$ are also

$$\sum_{j=1}^{m}a_{hj}^ia_{kj}^iJ$$

$h \neq k, i = r+1, \ldots, 2r = s$. Hence the off-diagonal element of $\sum_{i=1}^{2r}(C_iC_i^T + C_i^TC_i)$ is zero, using

$$\sum_{i=1}^{m}\sum_{j=1}^{m}(a_{hj}^ia_{kj}^i + a_{jh}^ia_{jk}^i)J = 0 \qquad \square$$

By the same reason as in the proof for Corollary 1, we have

Corollary 4. *If there exist a regular s-set of regular matrices of order m and a regular t-set of regular matrices of order n then there exists a regular s-set of regular matrices of order mn, when $2t = sm$ (m is odd).*

Corollary 5. *If both $q \equiv 7$ (mod 8) and $\frac{1}{2}(q+1)q^2 - 1$ are prime powers there exists a regular $\frac{1}{2}(q+1)$-set regular of matrices of order $q^2(\frac{1}{2}(q+1)q^2 - 1)^2$.*

Proof. Note $\frac{1}{2}(q+1)q^2 - 1 \equiv 3$ (*mod* 4). By [1], there exist both a regular $\frac{1}{2}(q+1)$-set of regular matrices of order q^2 and a regular $\frac{1}{4}(q+1)q^2$-set of regular matrices of order $(\frac{1}{2}(q+1)q^2 - 1)^2$. Using Theorem 2, we have a regular $\frac{1}{2}(q+1)$-set of regular matrices of order $q^2(\frac{1}{2}(q+1)q^2 - 1)^2$. $\square$

By the same reasoning as in the proof for Corollary 3, we have

Corollary 6. *If both $q \equiv 7$ (mod 8) and $\frac{1}{2}(q+1)q^2 - 1$ are prime powers there exists an Hadamard matrix of order of $q^2(q+1)(\frac{1}{2}(q+1)q^2 - 1)^2$.*

Theorem 3. *If there exist a regular s-set of matrices of order m and a regular t-set of matrices of order n then there exists a regular 2s-set of matrices of order mn, when* $t = 2sm$.

Proof. Let $\{A_1 = (a_{ij}^1), A_2 = (a_{ij}^2), \ldots, A_s = (a_{ij}^s)\}$ be the regular s-set of matrices of order m and $\{B_1, B_2, \ldots, B_t\}$ be the regular t-set of matrices of order of n.

Define

$$C_i = \begin{bmatrix} a_{11}^i B_{(i-1)m+1} & a_{12}^i B_{(i-1)m+2} & \cdots & a_{1m}^i B_{im} \\ a_{21}^i B_{(i-1)m+2} & a_{22}^i B_{(i-1)m+3} & \cdots & a_{2m}^i B_{(i-1)m+1} \\ & & \vdots & \\ a_{m1}^i B_{im} & a_{m2}^i B_{(i-1)m+1} & \cdots & a_{mm}^i B_{im-1} \end{bmatrix}$$

and

$$C_{s+i} = \begin{bmatrix} a_{11}^i B_{(i-1)m+1} & a_{21}^i B_{(i-1)m+2} & \cdots & a_{m1}^i B_{im} \\ a_{12}^i B_{(i-1)m+2} & a_{22}^i B_{(i-1)m+3} & \cdots & a_{m2}^i B_{(i-1)m+1} \\ & & \vdots & \\ a_{1m}^i B_{im} & a_{2m}^i B_{(i-1)m+1} & \cdots & a_{mm}^i B_{im-1} \end{bmatrix}$$

$i = 1, \ldots, s$. By the same reasoning as in the proofs for Theorem 1 and Theorem 2, we have

$$C_i C_j = J_m \times J_n = J_{mn}, \qquad i, j \in \{1, \ldots, s\}$$
$$C_i C_j^T = C_j^T C_i = J_m \times J_n = J_{mn}, \qquad i \neq j, \quad i, j \in \{1, \ldots, s\}$$

and

$$\sum_{i=1}^{s} (C_i C_i^T + C_i^T C_i) = 2smnI_{mn}$$

Corollary 7. *If there exist a regular s-set of regular matrices of order m and a regular t-set of regular matrices of order n then there exists a regular 2s-set of regular matrices of order mn, when* $t = 2sm$.

Corollary 8. *If both* $q \equiv 3$ (*mod* 4) *and* $2(q+1)q^2 - 1$ *are prime powers there exists a regular* $(q+1)$*-set of regular matrices of order* $q^2(2(q+1)q^2 - 1)^2$.

Proof. Note $2(q+1)q^2 - 1 \equiv 3$ (*mod* 4). By [1], there exist both a regular $\frac{1}{2}(q+1)$-set of regular matrices of order q^2 and a regular $(q+1)q^2$-set of regular matrices of order $(2(q+1)q^2 - 1)^2$. Using Theorem 3, we have a regular $(q+1)$-set of regular matrices of order $q^2(2(q+1)q^2 - 1)^2$. □

By the same reasoning as in the proof for Corollary 3, we have

Corollary 9. *If both* $q \equiv 3$ (*mod* 4) *and* $2(q+1)q^2 - 1$ *are prime powers there exists an Hadamard matrix of order of* $2q^2(q+1)(2(q+1)q^2 - 1)^2$.

We note that if $t = 2$ in Theorems 1, 2, 3 the conditions $t = sm$, $2t = sm$, $t = 2sm$ can be removed and a completely different proof obtained.

Lemma 1. *If there exist a regular 2s-set of regular matrices of order m and a regular 2-set of regular matrices of order n then there exist a regular 2s-set of regular matrices of order mn.*

Proof. Let $\{A_1, \ldots, A_{2s}\}$ be the regular $2s$-set of regular matrices of order m and $\{B, C\}$ be the regular 2-set of regular matrices of order n. Set

$$D_{2i-1} = A_{2i-1} \times \tfrac{1}{2}(B + B^T) + A_{2i} \times \tfrac{1}{2}(B - B^T)$$
$$D_{2i} = A_{2i-1} \times \tfrac{1}{2}(C - C^T) + A_{2i} \times \tfrac{1}{2}(C + C^T), \qquad i = 1, \ldots, s$$

By long verification, we prove that $\{D_1, \ldots, D_{2s}\}$ is a regular $2s$-set of regular matrices of order mn. □

Corollary 10. *If there exist a regular s-set of regular matrices of order m there exists a regular s-set of regular matrices of order* $9^i m$, *where* $i = 0, 1, \ldots$.

Proof. From Seberry-Whiteman (Lemma 2 and Corollary 3 in [1]), there exists a regular 2-set of matrices of order 9^i, $i = 1, 2, \ldots$. Using Lemma 1, we have a regular s-set of regular matrices of order $9^i m$, where $i = 0, 1, \ldots$. □

Using Corollary 10, we extend Corollary 3, 6, 9 to give

Corollary 11. (i) *If both* $q \equiv 3 \pmod 4$ *and* $(q+1)q^2 - 1$ *are prime powers there exists an Hadamard matrix of order* $q^2(q+1)((q+1)q^2-1)^2 9^i$, *where* $i = 0, 1, \ldots$.

(ii) *If both* $q \equiv 7 \pmod 8$ *and* $\frac{1}{2}(q+1)q^2 - 1$ *are prime powers there exists an Hadamard matrix of order of* $q^2(q+1)(\frac{1}{2}(q+1)q^2 - 1)^2 9^i$, *where* $i = 0, 1, \ldots$.

(iii) *If both* $q \equiv 3 \pmod 4$ *and* $2(q+1)q^2 - 1$ *are prime powers there exists an Hadamard matrix of order of* $2q^2(q+1)(2(q+1)q^2 - 1)^2 9^i$, *where* $i = 0, 1, \ldots$.

3. Hadamard Matrices

We give another proof of Seberry and Whiteman's Theorem [1].

Theorem 4. *If there exists an Hadamard matrices of order* $4h$ *and a regular* $s(=2h)$*-set of matrices of order m there exists an Hadamard matrix of order* $4hm$.

Proof. Let $\{A_1, \ldots, A_s\}$ be the regular s-set of matrices of order m and $H = (h_{ij})$ be the Hadamard matrix of order $4h$. Set $L_1 = (h_{ij}A_{j+i-1})$, $L_2 = (h_{i,2h+j}A^T_{2h+j+i-1})$, $L_3 = (h_{2h+i,j}A_{2h+j+i-1})$, $L_4 = (h_{2h+i,2h+j}A^T_{j+i-1})$, where $i, j = 1, \ldots, 2h$ and all the subscripts $j + i - 1$ are the residues of $2h$. Set

$$E = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}$$

We now prove E is an Hadamard matrix of order $4hm$. Let

$$E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_{4h} \end{bmatrix}$$

where $E_1, E_2, \ldots, E_{4h}$ are of order $m \times 4hm$. It is easy to verify $E_iE_j^T = 0$, if $i \neq j$ and $E_iE_i^i = \sum_{k=1}^{2h}(A_kA_k^T + A_k^TA_k) = \sum_{k=1}^{s}(A_kA_k^T + A_k^TA_k) = 2smI_m = 4hmIm$. Thus $EE^T = 4hmI_{4hm}$. □

4. Williamson Type Matrices

We find new constructions for Williamson matrices not given by Miyamoto [11] or Seberry and Yamada [2, 13]. This theorem differs from that of Seberry [6] as it does not need regular sets of regular matrices.

Theorem 5. *If there exist Williamson type matrices of order n and a regular s(=2n)-set of matrices of order m then there exist Williamson type matrices of order m.*

Proof. Let $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $D = (d_{ij})$ be the Williamson type matrices of order n and $\{R_1, \ldots, R_s\}$ be the regular s-set of matrices of order m. Set $E = (a_{ij}R_{j+i-1})$, $F = (b_{ij}R_{n+j+i-1})$, $G = (c_{ij}R_{j+i-1}^T)$, $H = (d_{ij}R_{n+j+i-1}^T)$, where $i, j = 1, \ldots, n$ and the subscripts $j + i - 1$ are the residue of m. It is easy to show $UV^T = VU^T$, for $U, V \in E, F, G, H$. we now prove

$$EE^T + FF^T + GG^T + HH^T = 4mnI_{mn}$$

$$\text{Let } E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{bmatrix}, \quad \text{Let } F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}, \quad \text{Let } G = \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{bmatrix}, \quad \text{Let } H = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_n \end{bmatrix}$$

where E_i, F_i, G_i, H_i are of order $m \times mn$. By the conditions of Williamson type matrices and Eq. 1–3, it is easy to verify that if $i \neq j$, $E_iE_j^T + F_iF_j^T + G_iG_j^T + H_iH_j^T = 0$. On the other hand, $E_iE_i^T + F_iF_i^T + G_iG_i^T + H_iH_i^T = \sum_{k=1}^{2n}(R_kR_k^T + R_k^TR_k) = \sum_{k=1}^{s}(R_kR_k^T + R_k^TR_k) = 2smI_m = 4mnI_m$. Thus $EE^T + FF^T + GG^T + HH^T = 4mnI_{mn}$.

Corollary 12. *If n is the order of Williamson type matrices and* $4n - 1$ *is a prime power then there exist Williamson type matrices of order* $n(4n-1)^2 9^t$, $t = 0, 1, \ldots$.

Proof. Clearly, $4n - 1 \equiv 3 \pmod 4$. By [1], there exists a regular $2n$-set of regular matrices of order $(4n-1)^2$. Using Corollary 10, we have a regular $2n$-set of regular matrices of order $(4n-1)^2 9^i$. From Theorem 4, we have Williamson type matrices of order $n(4n-1)^2 9^i$, $i = 0, 1, \ldots$. □

Let $n = 5$, $t = 1$ in Corollary 12, then we obtain new Williamson type matrices of order $5 \cdot 19^2 \cdot 9 = 16245$. Let $n = 13$, $t = 0$, then we have new Williamson type matrices of order $13 \cdot 15^2 = 33813$. Also special cases of Corollary 12 give more Williamson type matrices.

Corollary 13. *If both* $p \equiv 1 \pmod 4$ *and* $2p + 1$ *are prime powers there exist Williamson type matrices of order* $\frac{1}{2}(p+1)(2p+1)^2 9^i$, *where* $i = 0, 1, \ldots$.

Proof. From the Index of [2], there exist Williamson matrices of order $\frac{1}{2}(p+1)$. Using Corollary 12, we have Williamson type matrices of order $\frac{1}{2}(p+1)(2p+1)^2 9^i$, where $i = 0, 1, \ldots$. □

Corollary 14. *If* $28 \cdot 3^i - 1$ *is a prime power there exist Williamson type matrices of order* $7 \cdot (28 \cdot 3^i - 1)^2 3^{i+2j}$, *where* $i, j = 0, 1, \ldots$.

Proof. From the Index of [2], there exist Williamson type matrices of order of $7 \cdot 3^i$, where $i = 0, 1, \ldots$. By corollary 12, we have Williamson type matrices of order $7 \cdot (28 \cdot 3^i - 1)^2 3^{i+2j}$. □

Let $i = j = 1$ in Corollary 14, then we have new Williamson type matrices of order $7(28 \cdot 3 - 1) \cdot 27 = 15687$.

5. Orthogonal Designs

Theorem 6. *If there exist an* $OD(4h; s_1, \ldots, s_u)$, *where* $4h = \sum_{j=1}^{u} s_j$ *and a regular* $t(=2h)$*-set of matrices of order* n *then there exists an* $OD(4nh; ns_1, \ldots, ns_u)$.

Proof. The proof is the same as the proof for Theorem 4, except the Hadamard matrix is replaced by an orthogonal design. □

Corollary 15. *If there exists an* $OD(4h; s_1, \ldots, s_u)$, *where* $4h = \sum_{j=1}^{u} s_j$, *then there exists an* $OD(4h(4h-1)^2 9^i; (4h-1)^2 9^i s_1, \ldots, (4h-1)^2 9^i s_u)$, *where* $i = 0, 1, \ldots$, *when* $4h - 1$ *is a prime power.*

Proof. Since $4h - 1 \equiv 3 \ (mod\ 4)$, a prime power, by [1], there exists a regular $2h$-set of regular matrices of order $(4h-1)^2$. Note Corollary 10, then we have a regular $2h$-set of regular matrices of order $(4h-1)^2 9^i$, where $i = 0, 1, \ldots$. Using Theorem 6, we have an $OD(4h(4h-1)^2 9^i; (4h-1)^2 9^i s_1, \ldots, (4h-1)^2 9^i s_u)$, where $i = 0, 1, \ldots$. □

By corollary 15, for example, we have an $OD(20 \cdot 19^2 \cdot 9^i; 5 \cdot 19^2 \cdot 9^i, 5 \cdot 19^2 \cdot 9^i, 5 \cdot 19^2 \cdot 9^i, 5 \cdot 19^2 \cdot 9^i)$, an $OD(60 \cdot 59^2 \cdot 9^i; 15 \cdot 59^2 \cdot 9^i, 15 \cdot 59^2 \cdot 9^i, 15 \cdot 59^2 \cdot 9^i, 15 \cdot 59^2 \cdot 9^i)$, an $OD(108 \cdot 107^2 \cdot 9^i; 27 \cdot 107^2 \cdot 9^i, 27 \cdot 107^2 \cdot 9^i, 27 \cdot 107^2 \cdot 9^i, 27 \cdot 107^2 \cdot 9^i)$, an $OD(140 \cdot 139^2 \cdot 9^i; 35 \cdot 139^2 \cdot 9^i, 35 \cdot 139^2 \cdot 9^i, 35 \cdot 139^2 \cdot 9^i, 35 \cdot 139^2 \cdot 9^i)$.

6. Complex Hadamard Matrices

Theorem 7. *If there exist a complex Hadamard matrix of order* $2c$ *and a regular* $s(=2c)$*-set of matrices of order* m *then there exists a complex Hadamard matrix of order* $2cm$.

Proof. Let $\{A_1, \ldots, A_s\}$ be the regular $s(=2c)$-set of matrices of order m and $C = X + iY$ be the complex Hadamard matrix of order $2c$, where both X and Y are $(1, -1)$ matrices satisfying $X \wedge Y = 0$, $XX^T + YY^T = 2cI_{2c}$, $XY^T = YX^T$. Let $P = X + Y$ and $Q = X - Y$. Then both P and Q are $(1, -1)$ matrices of order $2c$

and $PP^T + QQ^T = 4cI_{2c}$, $PQ^T = QP^T$. Let $P = (p_{ij})$ and $Q = (q_{ij})$, $i, j = 1, \ldots, 2c$. Set $E = (p_{ij}A_{i+j-1})$ and $F = (q_{ij}A_{i+j-1})^T$, where $i, j = 1, \ldots, s$ and the subscripts $i+j-1$ are the residues of m. Clearly, both E and F are $(1, -1)$ matrices of order $2cm$, since both P and Q are $(1, -1)$ matrices of order $2c$. We now prove $EE^T + FF^T = 4cmI_{2cm}$. Rewrite $E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{bmatrix}$ and $F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}$, where E_i and F_i are matrices of order $m \times sm$. Note

$$E_iE_i^T + F_iF_i^T = \sum_{j=1}^{s} (p_{ij}p_{ij}A_{i+j-1}A_{i+j-1}^T + q_{ij}q_{ij}A_{i+j-1}^T A_{i+j-1})$$

$$= \sum_{j=1}^{s} (A_jA_j^T + A_j^T A_j) = 2smI_m$$

On the other hand, if $i \neq k$,

$$E_iE_k^T + F_iF_k^T = \sum_{j=1}^{s} (p_{ij}p_{kj}A_{i+j-1}A_{k+j-1}^T + q_{ij}q_{kj}A_{i+j-1}^T A_{k+j-1})$$

$$= \sum_{j=1}^{s} (p_{ij}p_{kj} + q_{ij}q_{kj})J_m = 0$$

Thus

$$EE^T + FF^T = 2smI_{sm} = 4cmI_{2cm}$$

Finally, Set $U = \frac{1}{2}(E + F)$ and $V = \frac{1}{2}(E - F)$. Note both E and F are $(1, -1)$ matrices of order $2cm$ then both U and V are $(1, -1, 0)$ matrices of order $2cm$ satisfying $U \wedge V = 0, UU^T + VV^T = \frac{1}{2}(EE^T + FF^T) = 2cmI_{2cm}$. Since $PQ^T = QP^T$, $EF^T = FE^T$ and $UV^T = VU^T$. Thus $U + iV$ is a complex Hadamard matrix of order $2cm$. □

Corollary 16. *If both* $p \equiv 1$ *(mod 4) and* $2p^j(p + 1) - 1$ *are prime powers then there exists a complex Hadamard matrix of order* $p^j(p + 1)(2p^j(p + 1) - 1)^2$, *where* $j = 1, 2, \ldots$.

Proof. Obviously, $2p^j(p + 1) - 1 \equiv 3$ (*mod* 4). By [1], there exists a regular $p^j(p + 1)$-set of matrices of order $(2p^j(p + 1) - 1)^2$. From Corollary 18, [15], there exists a complex Hadamard matrix of order $p^j(p + 1)$. Using Theorem 7, we have a $p^j(p + 1)(2p^j(p + 1) - 1)^2$.

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