On the multiplication theorems of Hadamard matrices of generalized quaternion type using M-structures[1]

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Abstract. We show that M-structures can be extended to Hadamard matrices of *generalized quaternion type* and obtain multiplication type theorems which preserve the structure.

1. Introduction

The concept of M-structures generalizes a number of concepts in Hadamard matrices, including Williamson matrices, Goethals-Seidel matrices, Wallis-Whiteman matrices and generalized quaternion matrices. We found many symmetric Williamson matrices and many Hadamard matrices using the concept of M-structures [4], [5], [6]. Furthermore, the concept of M-structures leads to the new concept of strong Kronecker products introduced by Jennifer Seberry and Xian-mo Zhang [8]. This was used by Craigen, Seberry and Zhang [1] to prove that if there exist Hadamard matrices of orders $4p$, $4q$, $4r$, and $4s$, then we have an Hadamard matrix of order $16pqrs$.

An orthogonal matrix of order $4t$ can be divided into sixteen $t \times t$ blocks M_{ij}. This partitioned matrix is said to be an *M-structure*. If the orthogonal matrix can be partitioned into sixty-four blocks M_{ij}, it will be called a 64 block M-structure.

First we give some definitions.

Definition 1: The matrices X and Y are said to be *amicable matrices* if

$$XY^t = YX^t,$$

where X^t and Y^t are the transpose matrices of X and Y respectively.

Definition 2: *Williamson matrices* of order w are four circulant symmetric matrices A, B, C, D which have entries 1 or -1 and which satisfy

$$AA^t + BB^t + CC^t + DD^t = 4wI_w$$

where I_w is a unit matrix of order w.

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Definition 3: *Williamson-type matrices* of order w are four pairwise amicable matrices A, B, C, D which have entries 1 or -1 and which satisfy

$$AA^t + BB^t + CC^t + DD^t = 4wI_w.$$

A generalized quaternion group Q_s, of order 2^{s+2} is a group generated by the two elements ρ, j such that

$$\rho^{2^{s+1}} = 1,\ j^2 = \rho^{2^s},\ j\rho j^{-1} = \rho^{-1}.$$

Let G be a semi-direct product of a cyclic group of an odd order n by the generalized quaternion group Q_s of order 2^{s+2}. That is, G is generated by ρ, ξ and j with the relations

$$\rho^{2^s} = -1,\ j^2 = -1,\ j\rho j^{-1} = \rho^{-1},\ \rho\zeta\rho^{-1} = \zeta,\ j\zeta j^{-1} = \zeta^{-1},\ \zeta^n = 1.$$

We consider the ring $\mathcal{R}$ obtained from the group ring $\mathbf{Z}G$ by identifying the elements ± 1 in the center of Q_s with ± 1 of the rational integer ring $\mathbf{Z}$. Put $\mathcal{H} = \{p^k\zeta^l : 0 \le k \le 2^s - 1, 0 \le l \le n-1\}$ and choose the basis $\mathcal{L} = \mathcal{H} \cup \mathcal{H}j$ of $\mathcal{R}$. An element ξ in $\mathcal{R}$ takes the following form.

$$\xi = \sum_{k=0}^{2N-1}\sum_{l=0}^{n-1} a_{k,l}\zeta^l\rho^k + \sum_{k=0}^{2N-1}\sum_{l=0}^{n-1} b_{k,l}\zeta^l\rho^k j = \alpha + \beta j,\ N = 2^{s-1}, \tag{1}$$

where

$$\alpha = \sum_{k=0}^{2N-1}\sum_{l=0}^{n-1} \alpha_{k,l}\zeta^l\rho^k \text{ and } \beta = \sum_{k=0}^{2N-1}\sum_{l=0}^{n-1} b_{k,l}\zeta^l\rho^k.$$

We define the conjugate $\xi = \overline{\alpha} - \beta j$ of $\xi = \alpha + \beta j$ based on the automorphism $\tau : \rho \to \rho^{-1}, \zeta \to \zeta^{-1}$ of G. Furthermore, we define the norm $\mathcal{N}(\xi) = \xi\overline{\xi}$ so that:

$$\mathcal{N}(\xi) = \alpha\overline{\alpha} + \beta\overline{\beta}$$
$$\mathcal{N}(\xi\eta) = \mathcal{N}(\xi)\mathcal{N}(\eta) \text{ for } \xi, \eta \in \mathcal{R}.$$

For an arbitrary element $\xi \in \mathcal{R}$ we construct the right regular representation matrix $R(\xi)$, defined by

$$(\rho^k\zeta^l\xi) = R(\xi)(\rho^k\zeta^l).$$

More precisely, for an element ξ of $\mathcal{R}$ with the form (1) the right regular representation matrix $R(\xi)$ is given by

$$R(\xi) = \begin{pmatrix} A & B \\ -B^t & A^t \end{pmatrix}$$

$$A = \begin{pmatrix} A_0 & A_1 & \cdots & A_{2N-1} \\ -A_{2N-1} & A_0 & \cdots & A_{2N-2} \\ \vdots & \vdots & & \vdots \\ -A_1 & -A_2 & \cdots & A_0 \end{pmatrix}$$

$$B = \begin{pmatrix} B_0 & B_1 & \cdots & B_{2N-1} \\ -B_{2N-1} & B_0 & \cdots & B_{2N-2} \\ -B_{2N-2} & B_{2N-1} & \cdots & B_{2N-3} \\ \vdots & \vdots & & \vdots \\ -B_1 & -B_2 & \cdots & B_0 \end{pmatrix}$$

where $A_k = \sum_{l=0}^{n-1} a_{k,l}T^l$ and $B_k = \sum_{l=0}^{n-1} a_{k,l}T^l$ are the circulant matrices of order n where T denotes the basic circulant matrix of order n

$$T = \begin{pmatrix} 0 & 1 & 0 & & & & 0 \\ 0 & 0 & 1 & & & & 0 \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $R(\xi) = R(\xi)^t$, we have

$$R(\xi)R(\xi) = R(\xi)R(\bar{\xi}) = R(\xi\bar{\xi}) = \begin{pmatrix} AA^t + BB^t & 0 \\ 0 & AA^t + BB^t \end{pmatrix}$$

Definition 4: If an element in $\mathcal{R}$ which is given by the equation (1) above satisfies

(i) all the coefficients $a_{k,l}$, $b_{k,l}$ are from $\{1, -1\}$ and

(ii) $\mathcal{N}(\xi) = 2^{s+1}n = 4nN$,

then the right regular representation matrix $R(\xi)$ becomes an Hadamard matrix of order $2^{s+1}n = 4nN$, which is called an *Hadamard matrix of generalized quaternion type*.

Similarly, if the following conditions are satisfied:

(iii) $a_{k,k} = 0$ and all other coefficients $a_{k,l}$, $b_{k,l}$ are from $\{1, -1\}$ and

(iv) $\mathcal{N}(\xi) = 2^{s+1}n - 1 = 4nN - 1$,

then $R(\xi)$ is a C-matrix of order $2^{s+1}n = 4nN$, which we call a *C-matrix of generalized quaternion type*.

We abbreviate *generalized quaternion type* as GQ *type* for convenience sake.

Let us express the conditions (i), (ii) in terms of the component matrices A_k, and B_k:

$$\sum_{k=0}^{2N-1} A_k A_k^t + \sum_{k=0}^{2N-1} B_k B_k^t = 4nNI,$$

$$\sum_{k=0}^{t-1}(A_k A_{2N-t+k}^t + B_{2N-t+k}^t) - \sum_{k=0}^{2N-t-1} (A_k^t A_{k+t} + B_k^t B_{k+t}) = 0 \text{ for } 1 \le t \le 2N-1.$$

In particular, in the case $N = 1$ the conditions will become

$$A_0 A_0^t + A_1 A_1^t + B_0 B_0^t + B_1 B_1^t = 4nI,$$
$$A_0 A_1^t - A_1 A_0^t + B_0 B_1^t + B_1 B_0^t = 0.$$

Moreover, suppose that A_0, A_1, B_0 and B_1 are symmetric, that is, Williamson matrices, or suppose they are pairwise amicable, that is, Williamson-type matrices, then the second condition is trivial.

2. M-structure Hadamard matrices

We consider Hadamard matrices of GQ type as an M-structure. Namely, an Hadamard matrix H of GQ type is partitioned into sixteen blocks,

$$H = \begin{pmatrix} A & B \\ -B^t & A^t \end{pmatrix} = \begin{pmatrix} C_0 & C_1 & D_0 & D_1 \\ -C_1 & C_0 & -D_1 & D_0 \\ -D_0^t & D_1^t & C_0^t & -C_1^t \\ -D_1^t & -D_0^t & C_1^t & C_0^t \end{pmatrix} \tag{3}$$

where

$$C_0 = \begin{pmatrix} A_0 & A_1 & \cdots & A_{N-1} \\ -A_{2N-1} & A_0 & \cdots & A_{N-2} \\ -A_{N+1} & -A_{N+2} & \cdots & A_0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} A_N & A_{N+1} & \cdots & A_{2N-1} \\ -A_{N-1} & A_N & \cdots & A_{2N-2} \\ -A_1 & -A_2 & \cdots & A_N \end{pmatrix},$$

$$D_0 = \begin{pmatrix} B_0 & B_1 & \cdots & B_{N-1} \\ -B_{2N-1} & B_0 & \cdots & B_{N-2} \\ -B_{N+1} & -B_{N+2} & \cdots & B_0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} B_N & B_{N+1} & \cdots & B_{2N-1} \\ -B_{N-1} & B_N & \cdots & B_{2N-2} \\ -B_1 & -B_2 & \cdots & B_N \end{pmatrix},$$

Since H is an Hadamard matrix, the component matrices C_0, C_1, D_0, D_1 satisfy the following equations,

$$\begin{cases} C_0 C_0^t + C_1 C_1^t + D_0 D_0^t + D_1 D_1^t = C_0^t C_0 + C_1^t C_1 + D_0^t D + 0 + D_1^t D_1 = 4nNI \\ C_0 C_1^t - C_1 C_0^t + D_1 D_1^t - D_1 D_0^t = C_0^t C_1 - C_1^t C_0 + D_0^t D_1 - D_1^t D_0 = 0 \\ C_0 D_0 - C_1 D_1 - D_0 C_0 + D_1 C_1 = 0 \\ C_0 D_1 + C_1 D_0 - D_0 C_1 - D_1 C_0 = 0 \end{cases} \tag{4}$$

An Hadamard matrix having the form (3) will be called an *M-structure Hadamard matrix of* GQ *type*.

3. Paley type 1 matrix

The Paley type 1 matrix can be changed into the form of a C-matrix of GQ type and is defined as follows (see [3]).

Definition 5: Let q be a prime power, $q \equiv 3 \pmod 4$, $F = GF(q)$ the finite field of q elements, $K = GF(q^2)$ a quadratic extension over F, and $K^\times$ and $F^\times$ the multiplicative groups of K and F respectively. Furthermore, let η be a generator of $K^\times$, $\gamma = \eta^{(q+1)/2}$ and let $N_{K/F}$ and $S_{K/F}$ denote the relative norm and relative trace from K to F respectively. Denote by ψ the quadratic character of F. Then the matrix

$$P = (\psi(N_{K/F}\alpha)\psi(S_{K/F}\gamma^{-1}\beta\alpha^{-1}))_{\alpha,\beta\in K^\times/F^\times}$$

is called the *Paley type 1 matrix.*

We recall here the definition of Seidel-equivalence of matrices.

Definition 6: If a square matrix A can be obtained from a square matrix B by a sequence of two kinds of operations:

(i) multiplying the row and the corresponding column by -1 simultaneously,
(ii) interchanging two rows and the corresponding two columns simultaneously,

then A will be said to be *Seidel-equivalent* to B.

Theorem 1. *The Paley type 1 matrix is Seidel equivalent to a C-matrix of* GQ *type with some additional properties:*

(i) *A is skew symmetric;*
(ii) *$B_{2N-m-1} = -B_m^t$ for $m = 0,\ldots,N-1$ where $q+1 = 2^{s+1}n$, $s \geq 1$, n odd, $N = 2^{s+1}$.*

Proof: See [11]. ∎

4. Infinite series of Hadamard matrices of generalized quaternion type

Yamada constructed some infinite series of Hadamard matrices of GQ type [11]. In this section we show these constructions of finite series.

Let q be a power of a prime p, $F = GF(q)$ denote a finite field of q elements, $K = GF(q^t)$ an extension of F of degree t, $t \geq 2$. Let η be a generator of $K^\times$ and let S_K and S_F denote the absolute trace in K and F. Furthermore, let $S_{K/F}$ and $N_{K/F}$ be the relative trace and relative norm from K to F respectively.

Definition 7: Let χ be a character of F and $\zeta_p = e^{2\pi i/p}$, then the Gauss sum $\tau_F(\chi)$ is defined by

$$\tau_f(\chi) = \sum_{\alpha\in F}\chi(\alpha)\zeta_p^{S_F\alpha}.$$

If χ is a nonprincipal character of K, then the ratio

$$\theta_\chi = \frac{\tau_K(\chi)}{\tau_F(\chi)}$$

of two Gauss sums is called the *relative Gauss sum associated with* χ.

The following theorem on the relative Gauss sum is very useful.

Theorem 2. *Suppose that χ is a character of K inducing in F a nonprincipal character. Then the relative Gauss sum associated with χ can be written in the following form*

$$\theta_\chi = \sum_{\alpha \in K^\times / F^\times} \chi(\alpha)\overline{\chi}(S_{K/F}\alpha),$$

and we have the norm relation

$$\theta_\chi \overline{\theta}_\chi = q^{t-1}.$$

Proof: See [11]. ∎

Using Theorem 2 for the case $t = 2$, we give infinite series of Hadamard matrices of GQ type.

Theorem 3. *Let $q + 1 = 2^s n$, $s \geq 2$, n odd, ρ a primitive 2^{s+1} th root of unity and w an arbitrary nth root of unity. Put $\chi = \chi_{2^{s+1}}\chi_n$ where $\chi_{2^{s+1}}(\eta) = \rho$, $\chi_n(\eta) = w$, so that χ induces a quadratic character ψ in F.*

Then for the relatite Gauss sum θ_χ we have

$$\theta_\chi = \alpha + \beta\rho^n \quad \alpha, \beta \in \mathbf{Z}[\rho^2, w],$$

and the right regular representation matrix of

$$\gamma = \alpha \pm i + \beta j$$

gives an Hadamard matrix of GQ type of order $2^s n$ where i is a primitive fourth root of unity.

Proof: See [11]. ∎

Corollary 1. *Let α, β be as in Theorem 3. Then the right regular representation matrix of*

$$\gamma = (\alpha - i + \beta\rho^n j)(1 - j) = (1 - j)(\theta_\chi + ij)$$

is an Hadamard matrix of GQ type of order $2^{s+1}n$. In particular, if $s = 1$ then we get an Hadamard matrix of Turyn's type [9], [10].

Proof: See [11]. ∎

Theorem 4. *Let $q + 1 = 2n$, n odd and ρ a primitive octic root of unity. Let η and w, be as in Theorem 3. Put $\chi = \chi_8\chi_n$, $\chi_8(\eta) = \rho$. So that χ induces a biquadratic character in F.*

The right regular representation matrix of

$$\tau = (\theta_\chi + \rho^t j)(1 + i)(1 + j),\ t = 1, 3, 5, 7,$$

gives an Hadamard matrix of GQ type of order $8n$. We may change the order of factors $\theta_\chi + \rho^t j$, $1 + i$ and $1 + j$ arbitrarily.

Proof: See [11]. ∎

On the other hand, if there exists an Hadamard matrix of GQ type of order $2^s n$, we can double its order.

Theorem 5. *Assume that the right regular representation matrix of $\xi = \alpha + \beta j$ in $\mathcal{R}$ is an Hadamard matrix of* GQ *type of order* $2^s n$. *Let p be a primitive* 2^{s+1} *th root of unity. Then*

$$\gamma = (\alpha + \beta j)(1 + \rho^t j) \text{ for } t = 1, 3, 5, \ldots, 2^s - 1,$$

generates an Hadamard matrix of GQ *type of order* $2^{s+1} n$. *We can exchange the order of two factors* $\alpha + \beta j$ *and* $1 + \rho^t j$.

Proof: See [11]. ∎

5. Main theorems

Theorem 6. *Let* H *be an* M*-structure Hadamard matrix of* GQ *type of order* $4n$,

$$H = \begin{pmatrix} C_0 & C_1 & D_0 & D_1 \\ -C_1 & C_0 & -D_1 & D_0 \\ -D_0^t & D_1^t & C_0^t & -C_1^t \\ -D_1^t & -D_0^t & C_1^t & C_0^t \end{pmatrix}.$$

Furthermore, let T_0, T_1, T_2 *and* T_3 *be matrices of order* m *which have entries* $0, 1$ *or* -1 *which satisfy*

(i) $T_i \wedge T_j, i \neq j$ *($\wedge$ the Hadamard product);*
(ii) $\sum_{i=0}^{3} T_i$ *is a matrix whose entries are* ± 1 *or* -1*;*
(iii) $\sum_{i=0}^{3} T_i T_i^t = \sum_{i=0}^{3} T_i^t T_i = m I_m$*;*
(iv) $T_0 T_1^t - T_1 T_0^t + T_2 T_3^t - T_3 T_2^t = T_0^t T_1 - T_1^t T_0 + T_2^t T_3 - T_3^t T_2 = 0$,
$T_0 T_2 - T_2 T_0 - T_1 T_3 + T_3 T_1 = T_0 T_3 - T_3 T_0 + T_1 T_2 - T_2 T_1 = 0$.

Then we have an M*-structure Hadamard matrices of* GQ *type of* $4nm$.

Proof: We define the matrices α, β, γ and δ as follow.

$$\alpha = T_0 \times C_0 - T_1 \times C_1 - T_2 \times D_0^t - T_3 \times D_1^t,$$
$$\beta = T_0 \times C_1 + T_1 \times C_0 + T_2 \times D_1^t - T_3 \times D_0^t,$$
$$\gamma = T_0 \times D_0 - T_1 \times D_1 + T_2 \times C_0^t + T_3 \times C_1^t,$$
$$\delta = T_0 \times D_1 + T_1 \times D_0 - T_2 \times C_1^t + T_3 \times C_0^t,$$

It is easily verified that α, β, γ and δ satisfy the equation (4). Hence

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\beta & \alpha & -\delta & \gamma \\ -\gamma^t & \delta^t & \alpha^t & -\beta^t \\ \delta^t & -\gamma^t & \beta^t & \alpha^t \end{pmatrix}$$

is an M-structure Hadamard matrix of GQ type of order $4nm$. ∎

Corollary 2. *Let q be a prime power and $q+1 = 2^s n$, n odd. Let $m_1, \ldots, m_r$, be the orders of Williamson matrices or type 1 Williamson type matrices. Then*

(i) *when $s \geq 2$, there exists an M-structure Hadamard matrix of* GQ *type of order $2^r m_1 \ldots m_r(q+1) = 2^{r+s} r_1 \ldots m_r n$;*

(ii) *when $s = 1$, there exists an M-structure Hadamard matrix of* GQ *type of order $2^{r+1} m_1 \ldots m_r(q+1) = 2^{r+2} m_1 \ldots m_r n$.*

Proof: From Theorems 1 and 3 there exists an M-structure Hadamard matrix of GQ type of order $2^s n$. From Corollary 1 an M-structure of Hadamard matrix of GQ type of order $4n$ exists. Let W_1, W_2, W_3 and W_4 be Williamson matrices of order m or type 1 Williamson-type matrices of order m. Put

$$X_1 = 2(W_1+W_2), X_2 = \frac{1}{2}(W_1-W_2), Y_1 = \frac{1}{2}(W_3+W_4), Y_2 = \frac{1}{2}(W_3-W_4).$$

Further, put

$$T_0 = \begin{pmatrix} X_1 & \\ & X_1 \end{pmatrix}, T_1 = \begin{pmatrix} X_2 & \\ & X_2 \end{pmatrix}, T_2 = \begin{pmatrix} & Y_1 \\ Y_1 & \end{pmatrix}, T_3 = \begin{pmatrix} & Y_2 \\ Y_2 & \end{pmatrix},$$

then T_0, T_1, T_2, T_3 satisfy the conditions of Theorem 6. ∎

Theorem 7. *Let H be an M-structure Hadamard matrix of* GQ *type of order $4n$,*

$$H = \begin{pmatrix} C_0 & C_1 & D_0 & D_1 \\ -C_1 & C_0 & -D_1 & D_0 \\ -D_0^t & D_1^t & C_0^t & -C_1^t \\ -D_1^t & -D_0^t & C_1^t & C_0^t \end{pmatrix}.$$

Furthermore, let T_0 and T_1 be the matrices of order m which have entries $0, 1$ or -1 and satisfy

(i) $T_0 \wedge T_1 = 0$, *($\wedge$ the Hadamard product);*

(ii) $T_0 + T_1$ *is a matrix which has entries* 1 *or* -1;

(iii) $T_0 T_0^t + T_1 T_1^t = T_0^t T_0 + T_1^t T_1 = mI$;

(iv) $T_0 T_1^t - T_1 T_0^t = T_0^t T_1 - T_1^t T_0 = 0$.

Then we have an M-structure Hadamard matrix of GQ *type of order $4nm$.*

Proof: We define the matrices $\alpha, \beta \gamma$ and δ as follows.

$$\begin{aligned} \alpha &= T_0 \times C_0 - T_1 \times C_1, \\ \beta &= T_0 \times C_1 + T_1 \times C_0 \\ \gamma &= T_0 \times D_0 - T_1 \times D_1 \\ \delta &= T_0 \times D_1 + T_1 \times D_0 \end{aligned}$$

Then, α, β, γ and δ satisfy the equation (4). ∎

Corollary 3. *Let q be a prime power and $q + 1 = 2^s n$, n odd. Let p_i be a prime power and $p_i \equiv 1 \pmod 4$ for $1 \le i \le r$. Then*

(i) *when $s \ge 2$ there exists an M-structure Hadamard matrix of* GQ *type of order $(p_1 + 1)(p_2 + 1)\ldots(p_r + 1)(q + 1)$;*

(ii) *when $s = 1$ there exists an M-structure Hadamard matrix of* GQ *type of order $2(p_1 + 1)(p_2 + 1)\ldots(q + 1)$.*

Proof: An Hadamard matrix of order $2(p_i + 1)$ obtained from the Paley type 2 matrix has a form

$$\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}.$$

Then $T_0 = \frac{1}{2}(X + Y), T_1 = \frac{1}{2}(X - Y)$ satisfy the conditions of Theorem 7. ▮

Corollary 4. *Let q be a prime power and $q + 1 = 2^s n$, n odd. Let $m_1, \ldots, m_r$ be the orders of Williamson type (not necessarily circulant or type 1) matrices. Then*

(i) *when $s \ge 2$, there exists an M-structure Hadamard matrix of* GQ *type of order $2^r m_1 \ldots m_r(q + 1) = 2^{r+s} r_1 \ldots m_r n$;*

(ii) *when $s = 1$, there exists an M-structue Hadamard matrix of* GQ *type of order $2^{r+1} m_1 \ldots m_r(q + 1) = 2^{r+2} m_1 \ldots m_r n$.*

Proof: Let W_1, W_2, W_3 and W_4 be Williamson type matrices of order m. Then

$$\begin{pmatrix} W_1 & W_2 & W_3 & W_4 \\ -W_2 & W_1 & -W_4 & -W_3 \\ -W_3 & W_4 & W_1 & W_2 \\ W_4 & W_3 & -W_2 & W_1 \end{pmatrix}$$

is an Hadamard matrix which has an M-structure

$$\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}.$$

where

$$X = \begin{pmatrix} W_1 & W_2 \\ -W_2 & W_1 \end{pmatrix}, \quad Y = \begin{pmatrix} W_3 & W_4 \\ -W_4 & W_3 \end{pmatrix}.$$

Then $T_0 = \frac{1}{2}(X + Y), T_1 = \frac{1}{2}(X - Y)$ satisfy the conditions of Theorem 7. ▮

Corollary 5. *Let q be a prime power and $q + 1 = 2^s n$, n odd. Suppose there exists a symmetric C-matrix of order $p_i + 1$ and there exists a symmetric Hadamard matrix of order $p_i - 1$ for $1 \le i \le r$. Then*

(i) *when $s \ge 2$ we have an M-structure Hadamard matrix of* GQ *type of order $2^r p_1 \ldots p_r(q + 1) = 2^{r+s} p_1 \ldots p_r n$;*

(ii) *when $s = 1$ we have an M=structure Hadamard matrix of* GQ *type of order $2^{r+1} p_1 \ldots p_r(q + 1) = 2^{r+2} p_1 \ldots p_r n$.*

Proof: If there exists a symmetric C-matrix of order $p_i + 1$ and there exists a symmetric Hadamard matrix of $p_i - 1$, then there exists a symmetric Hadamard matrix of order $4p_i$ having a form

$$\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}.$$

for $1 \le i \le l$. $T_0 = \frac{1}{2}(X + Y)$ and $T_1 = \frac{1}{2}(X - Y)$ satisfy the conditions of Theorem 7. ∎

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