Journal of Statistical Planning and Inference 34 (1993) 423-432
North-Holland

Constructions of balanced ternary designs based on generalized Bhaskar Rao designs

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Received 18 August 1989; revised manuscript received 4 March 1991
Recommended by R.G. Stanton

Abstract: New series of balanced ternary designs and partially balanced ternary designs are obtained. Some of the designs in the series are non-isomorphic solutions for design parameters which were previously known or whose solution was obtained by trial and error, rather than by a systematic method.

AMS Subject Classification: Primary 62K05, 62K10; secondary 05B05.

Key words and phrases: Block design; balanced ternary design; generalized Bhaskar Rao design; Hadamard matrix; balanced incomplete block design; partially balanced incomplete block design.

1. Introduction

For the definitions of terms like blocks, incidence matrix of a block design, group divisible design (GDD), balanced incomplete block design (BIBD) and partially balanced incomplete block design (PBIBD) the reader is referred to Street and Street (1987). A *balanced n-ary design* is a collection of B multisets, each of size K, chosen from a set of size V in such a way that each of the V elements occurs R times altogether and $0, 1, 2, \ldots,$ or $n-1$ times in each block, and each pair of distinct elements occurs Λ times. So the inner product of any two distinct rows of the $V \times B$ incidence matrix of the balanced n-ary design is Λ. These designs were introduced by Tocher (1952), but in his definition the equireplicate property was not required.

A balanced n-ary design where $n=2$ is the well known balanced incomplete block design. A balanced n-ary design where $n=3$ is called a balanced ternary design. A balanced ternary design which has V elements, B blocks of size K, each of the

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elements occurring once in precisely ϱ_1 blocks and twice in precisely ϱ_2 blocks, and with incidence matrix having inner product of any two rows Λ, is denoted by $\mathrm{BTD}(V, B; \varrho_1, \varrho_2, R; K, \Lambda)$. Notice that $R = \varrho_1 + 2\varrho_2$. A partially balanced ternary design (PBTD) can be defined similarly. A number of authors have studied these designs; for example, see Billington (1984), Donovan (1986), Patwardhan and Sharma (1988), Sarvate (1990), Soundara Pandian (1980), and the references therein. A list of partially balanced ternary designs with small parameters is given in Mirchandani and Sarvate (1992) and a classification of ternary group divisible designs and some constructions are given in Denig and Sarvate (1992). Some balanced ternary designs are related to generalized weighing matrices (for definitions see Geramita and Seberry (1979)). Most of the time in this paper when we talk about a BTD say, M, we are referring to the incidence matrix of the BTD.

The following definition is from Seberry (1982). Suppose we have a matrix W with elements from an elementary Abelian group $G = \{h_1, h_2, \ldots, h_g\}$, where $W = h_1 A_1 + h_2 A_2 + \cdots + h_g A_g$, with $A_1, \ldots, A_g$ $v \times b$ (0,1) matrices, and the Hadamard product $A_i * A_j, i \neq j$, is zero. Suppose $(a_{i1}, \ldots, a_{ib})$ and $(b_{j1}, \ldots, b_{jb})$ are the ith and jth rows of W, then we define WW^+ by: $(WW^+)_{ij} = (a_{i1}, \ldots, a_{ib}) \cdot (b_{j1}^{-1}, \ldots, b_{jb}^{-1})$, with $\cdot$ the scalar product. Then W is a *generalized Bhaskar Rao design* or *GBRD* if

(i) $WW^+ = rI + tG(J - I)$; and

(ii) $N = A_1 + A_2 + \cdots + Ag$ satisfies $NN^{\mathrm{T}} = (r\text{-}\lambda)I + \lambda J$, that is, N is the incidence matrix of a $\mathrm{BIBD}(v, b, r, k, \lambda)$.

As a convention tG stands for t copies of $h_1 + h_2 + \cdots + h_g$; that is t gives the number of times a complete copy $h_1 + h_2 + \cdots + h_g$ of the group G occurs. Such a matrix will be denoted by $\mathrm{GBRD}(v, b, r, k, \lambda = tg;\ G)$. Keeping consistency with BIBD notation we may write $\mathrm{GBRD}(v, b, r, k, \lambda = tg;\ G)$ as $\mathrm{GBRD}(v, k, \lambda = tg; G)$.

Here and elsewhere in the present paper J will stand for an appropriate size matrix with all entries one. For example here the matrix J is a square matrix of order V. A generalized Hadamard matrix $\mathrm{GH}(tg;\ G)$ can be regarded as a $\mathrm{GBRD}(tg, tg, tg; G)$.

There are several papers in the literature where GBRD's are used to construct block designs. An early application can be found in Seberry (1984), and one of the recent papers where such application is used to construct group divisible designs is Palmer and Seberry (1988). In the present note we apply these designs in the construction of new series of n-ary designs. As mentioned in the abstract we observed that in some cases these constructions give non-isomorphic solutions for design parameters which were previously known or whose solution was obtained by trial and error, rather than by a systematic method. This suggests that the methods will produce previously unknown designs.

2. Constructions based on generalized Hadamard matrices

Theorem 1. *If a* $\mathrm{BTD}(V, B;\ \varrho_1, \varrho_2, R, K; \Lambda) = M$ *and a* $\mathrm{GH}(n, G) = N$ *with* $|G| = V$ *exist, then a* $\mathrm{PBTD}(nV, nB;\ n\varrho_1, n\varrho_2, n\varrho_1 + 2n\varrho_2;\ nK, \Lambda_1 = n\Lambda, \Lambda_2 = nRK/V)$ *exists.*

Proof. Construct a matrix P by replacing each element g in N by $T_g M$ where T_g is the right regular matrix representation of the element g of G. Observe that PP^{T} is a block matrix with diagonal entries $n((RK-\Lambda V)I+\Lambda J)$ and the off diagonal entries are $(n/V)[(RK-\Lambda V)I+\Lambda J]J$. Hence P is the incidence matrix of the required PBTD. $\square$

Now when $n=V=|G|$, P gives the incidence matrix of a PBTD(V^2, VB; $V\varrho_1, V\varrho_2, V(\varrho_1+2\varrho_2)$; $VK, \Lambda_1=V\Lambda, \Lambda_2=RK$). But we know that $\Lambda(V-1)=R(K-1)-2\varrho_2$ i.e. $\Lambda V+(\varrho_1+2\varrho_2)+2\varrho_2=RK+\Lambda$ so we augment the matrix P by a column of rows of the BTD(V,B; ϱ_1,ϱ_2,R; K,Λ) as follows:

$$[P:M\times J].$$

Here J is a column matrix of size V. The augmented matrix is a BTD($V_2,(V+1)B$; $V\varrho_1+\varrho_1, V\varrho_2+\varrho_2, (V+1)\varrho_1+2(V+1)\varrho_2$; $VK, V\Lambda+R+2\varrho_2$). Hence we have

Theorem 2. *If a* BTD(V,B; ϱ_1,ϱ_2,R; K,Λ)$=M$ *and a* GH(V,G)$=N$ *exist, then a* BTD($V^2,(V+1)B$; $V\varrho_1+\varrho_1$, $V\varrho_2+\varrho_2$, $(V+1)\varrho_1+2(V+1)\varrho_2$; VK, $V\Lambda+R+2\varrho_2$) *exists.*

Corollary 3. *If q is an odd prime power, then a* BTD($q^2,(q+1)q$; $(q+1)$, $(q-1)(q+1)/2$, $q(q+1)$; q^2, q^2+q-1) *exists.* $\square$

Proof. Saha and Dey (1973) showed that BTD($q,q,1,(q-1)/2,q;q,q-1$) exist, where q is an odd prime power, and it is well known that GH(q,G) exist. $\square$

Example 1. Consider BTD(3, 3; 1, 1, 3; 3, 2)$=M$ and GH($3,Z_3$)$=N$, that is,

$$M=\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad N=\begin{bmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{bmatrix}.$$

Then the above construction gives a BTD(9, 12; 4, 4, 12; 9, 11):

$$\begin{bmatrix}
1\,2\,0 & 1\,2\,0 & 1\,2\,0 & 1\,2\,0 \\
2\,0\,1 & 2\,0\,1 & 2\,0\,1 & 1\,2\,0 \\
0\,1\,2 & 0\,1\,2 & 0\,1\,2 & 1\,2\,0 \\
 & & & \\
1\,2\,0 & 2\,0\,1 & 0\,1\,2 & 2\,0\,1 \\
2\,0\,1 & 0\,1\,2 & 1\,2\,0 & 2\,0\,1 \\
0\,1\,2 & 1\,2\,0 & 2\,0\,1 & 2\,0\,1 \\
 & & & \\
1\,2\,0 & 0\,1\,2 & 2\,0\,1 & 0\,1\,2 \\
2\,0\,1 & 1\,2\,0 & 0\,1\,2 & 0\,1\,2 \\
0\,1\,2 & 2\,0\,1 & 1\,2\,0 & 0\,1\,2
\end{bmatrix}.$$

Now a BTD(9, 12; 4, 4, 12; 9, 11) was previously known and it is listed (number 163) in Billington and Robinson (1983). However, it is interesting to note that the above design does not contain any complete blocks and therefore it is non-isomorphic to the one given in Billington and Robinson (1983) which contains complete blocks.

Remarks. (1) As generalized Hadamard matrices GH(n, G) exist for infinitely many values of n other than the one given above, for each BTD which exists, we can construct infinitely many PBTDs.

(2) In the Appendix, we give a list of BTDs obtained via Theorem 2 and the list of BTDs with prime power V, given in Billington and Robinson (1983). It may be interesting to know that there are 34 BTDs in our list with $R \leqslant 50$.

(3) The above Theorems can be generalized to n-ary designs.

3. Constructions based on Bhaskar Rao Designs

Generalized Bhaskar Rao Designs over the group $\{1, -1\}$ are called Bhaskar Rao Designs (BRD).

Theorem 4. *Suppose there exist a* BRD$(v, b, r, k, 2\lambda; Z_2)$ *and square matrices B and C of order u, with entries from* $\{0, 1, 2, \ldots, n-1\}$, *which satisfy the following properties*:

(i) $BB^{\mathrm{T}} = CC^{\mathrm{T}} = cI + d(J-I)$,

(ii) $BC^{\mathrm{T}} = CB^{\mathrm{T}} = eI + f(J-I)$,

where c, d, e *and* f *are integers. Then there exists a matrix with entries from* $\{0, 1, 2, \ldots, n-1\}$ *such that the inner product of any two distinct rows is in the set* $\{\Lambda_1 = rd, \Lambda_2 = \lambda(c+e), \Lambda_3 = \lambda(d+f)\}$ *and the inner product of a row with itself is* rc.

Proof. Construct a matrix P by replacing 1's in the BRD by B and -1's in the BRD by C. Then a block diagonal entry of PP^{T} is rBB^{T} (using (i) $BB^{\mathrm{T}} = CC^{\mathrm{T}}$) which is equal to $rcI + rd(J-I)$. Now as the inner product of any two rows of a BRD is zero, it is clear that the product of the entries in the pairs $(1, 1)$ and $(-1, -1)$ and in the pairs $(1, -1)$ and $(-1, 1)$ occur equal number of times in the inner product. Therefore the off diagonal block entry of PP^{T} is equal to $\lambda(BB^{\mathrm{T}} + BC^{\mathrm{T}})$, which is equal to $\lambda((c+e)I + (d+f)(J-I))$ (using (ii) $BC^{\mathrm{T}} = CB^{\mathrm{T}}$). Hence the result follows. □

Let B be a symmetric balanced ternary design. Then

$$BB^{\mathrm{T}} = (R^2 - \Lambda V)I + \Lambda J.$$

Now we use Theorem 4 with $B = C$, to get

Corollary 5. *A symmetric* $\mathrm{BTD}(V,B;\varrho_1,\varrho_2,R;K,\Lambda)$ *and a* $\mathrm{BRD}(v,b,r,k,2\lambda;Z_2)$ *give a* $\mathrm{PBTD}(vV,bV;r\varrho_1,r\varrho_2,rR;kR,\Lambda_1=r\Lambda,\Lambda_2=2\lambda(R^2-\Lambda V+\Lambda),\ \Lambda_3=2\lambda\Lambda)$.

Similarly,

Corollary 6. *A symmetric* $\mathrm{BTD}(V,B;\varrho_1,\varrho_2,R;K,\Lambda)$ *and a Hadamard matrix of order* $4t$ *give a* $\mathrm{PBTD}(4tV,4tV;4t\varrho_1,4t\varrho_2,4tR;4tR,\Lambda_1=4t\Lambda,\Lambda_2=4t(R^2-\Lambda V+\Lambda))$.

Remark. Let B be a symmetric balanced ternary design. Then as above

$$BB^{\mathrm{T}}=(R^2-\Lambda V)I+\Lambda J,$$

and furthermore

$$(2J-B)(2J-B)^{\mathrm{T}}=4JJ^{\mathrm{T}}-2JB^{\mathrm{T}}-2BJ^{\mathrm{T}}+BB^{\mathrm{T}}$$
$$=4VJ-4RJ+BB^{\mathrm{T}};$$

$$B(2J-B)^{\mathrm{T}}=2BJ-BB^{\mathrm{T}}=2RJ-BB^{\mathrm{T}},$$

and

$$(2J-B)B^{\mathrm{T}}=2JB-BB^{\mathrm{T}}=2RJ-BB^{\mathrm{T}}.$$

Theorem 7. *Let B be a symmetric balanced ternary design with $R=V$ and suppose a* $\mathrm{BRD}(v,b,r,k,2\lambda;Z_2)$ *exists. Then a* $\mathrm{PBTD}(vV,bV;r\varrho_1,r\varrho_2,rR;kR,\Lambda_1=r\Lambda,\Lambda_2=2\lambda R)$ *exists.*

Proof. Suppose that in the statement of Theorem 4, B is a symmetric BTD and $C=2J-B$. Also note that $V=R$ and therefore $V-\varrho_1-\varrho_2=\varrho_2$ and therefore the values of ϱ_1 and ϱ_2 do not change in $2J-B$. □

Example 2. A $\mathrm{BTD}(11,11;1,5,11;11,10)$ exists [2, no. 113] and a $\mathrm{BRD}(4,3,2;Z_2)$ exists, therefore a $\mathrm{PBTD}(44,44;3,15,33;33,\Lambda_1=30,\Lambda_2=22)$ exists.

Corollary 8. *Let B be a symmetric* BTD *with $R=V$. Suppose a* $\mathrm{BRD}(v,b,r,k,2\lambda;Z_2)$ *exists such that $r\Lambda=2\lambda R$; then a* $\mathrm{BTD}(vV,bV;r\varrho_1,r\varrho_2,rR;kR,r\Lambda)$ *exists.*

Example 3. $\mathrm{BRD}(4,3,2;Z_2)$ and $\mathrm{BTD}(3,3;1,1,3;3,2)$ give a $\mathrm{BTD}(12,12,3,3,9;9,6)$. Now this design also exists [Billington and Robinson, 1983, no. 57], but the solution is given by listing all the blocks.

We observe that if there exist a $\mathrm{BTD}(V,V;\varrho_1,\varrho_2,R=V,V,\Lambda)$ and a $\mathrm{BRD}(v,b,r,k,2\lambda;Z_2)$ for which $r\Lambda=2\lambda R$, then the $\mathrm{BTD}(vV,bV;r\varrho_1,r\varrho_2,rR;kR,2\Lambda R)$ constructed by using Theorem 7 and the $\mathrm{BRD}(v,b,r,k,2\lambda;Z_2)$ can also be used to give a BTD. In other words:

Theorem 9. *If there exists a* $\mathrm{BTD}(V, V; \varrho_1, \varrho_2, V; V, \Lambda)$ *and a* $\mathrm{BRD}(v, b, r, k, 2\lambda; Z_2)$ *for which* $r\Lambda = 2\lambda R$, *then the* $\mathrm{BTD}(v^t V, b^t V; r^t\varrho_1, r^t\varrho_2, r^t R; k^t R, (2\lambda)^t R)$ *exists for all integers* $t \geqslant 0$.

Proof. Suppose that the new BTD constructed by Theorem 7 has the replication number $R' = rR$ and the index $\Lambda' = 2\lambda R$. We wish to show that R' and Λ' satisfy Corollary 8, i.e., $r(\Lambda') = 2\lambda R'$, but $r(2\lambda R) = 2\lambda(rR)$. □

Example 3 now gives

Corollary 10. *A* $\mathrm{BTD}(4^t \cdot 3, 4^t \cdot 3; 3^t, 3^t, 3 \cdot 3^t; 3 \cdot 3^t, 2^t \cdot 3)$ *exists for all* $t \geqslant 0$.

Now we will construct some balanced ternary designs via Theorem 7 using a particular BTD

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} \quad \text{and} \quad C = 2J - B.$$

Corollary 11. *If* $v(v-1) \equiv 0 \pmod{12}$, *then there exists a* $\mathrm{BRD}(v, b, r, 3, 2\lambda; Z_2)$ *(see Seberry, 1982) and therefore a* $\mathrm{PBTD}(3v, 3b; r, r, 3r; 9, \Lambda_1 = 2r, \Lambda_2 = 6\lambda)$ *exists. Furthermore if* $2r = 6\lambda$ *then we get a* BTD.

Example 4. $\mathrm{BRD}(4, 4\lambda, 3\lambda, 3, 2\lambda; Z_2)$ *gives a* $\mathrm{BTD}(12, 12\lambda; 3\lambda, 3\lambda, 9\lambda; 9, 6\lambda)$.

Remark. Several families of $\mathrm{PBTD}(3v, 3b; r, r, 3r; 12, \Lambda_1 = 2r, \Lambda_2 = 6\lambda)$ can be constructed by using the existence results of $\mathrm{BRD}(v, b, r, 4, 2t\lambda; Z_2)$, which are given by de Launey and Seberry (1984).

A result similar to Theorem 4 can be given as follows:

Theorem 12. *Let a* $\mathrm{BRD}(v, b, r, k, 2\lambda; Z_2)$ *with the following properties exist*:

(a) *In the inner product of any two distinct rows the number of occurrences of the pairs* $(1,1)$, $(-1,-1)$, $(1,-1)$ *and* $(-1,1)$ *are constants, say* c_1, c_2, c_3 *and* c_4 *respectively, (with* $c_1 + c_2 = c_3 + c_4 = \lambda$).

(b) *Each row of the* BRD *contains constant number of* 1*'s and* -1*'s, say* d_1 *and* d_2 *(with* $d_1 + d_2 = r$).

Assume further that there exist square matrices B and C of the same order with entries from $\{0, 1, 2, \ldots, n-1\}$, *and which satisfy*:

(i) $BB^{\mathrm{T}} = sI + q(J - I)$;
(ii) $CC^{\mathrm{T}} = uI + w(J - I)$;
(iii) $BC^{\mathrm{T}} = xI + y(J - I)$;
(iv) $CB^{\mathrm{T}} = zI + a(J - I)$;

where s, q, u, w, x, y, z, *and* a *are integers. Then there exists a matrix with entries from* $\{0, 1, 2, \dots, n-1\}$ *such that the inner product of any two distinct rows is in the set* $\{\Lambda_1 = d_1 q + d_2 w,\ \Lambda_2 = c_1 q + c_2 w + c_3 y + c_4 a,\ \Lambda_3 = c_1 s + c_2 u + c_3 x + c_4 z\}$ *and the inner product of a row with itself is* $d_1 s + d_2 u$.

Theorem 13. *Suppose a* SBIBD$(4t^2, 2t^2+t, t^2+t)$ *and a symmetric* BTD$(V, V; \varrho_1, \varrho_2, R; R, \Lambda)$ *exist. Further suppose that the Hadamard matrix corresponding to the* SBIBD *satisfies the properties required in Theorem* 12 *for the* BRD. (*Note that the Hadamard matrix corresponding to the* SBIBD *is a* BRD$(4t^2, 4t^2, 4t^2, 4t^2, 4t^2, Z_2)$.) *Then there exists a* PBTD$(4t^2V, 4t^2V, 4t^2\varrho_1, (2t^2-t)(V-\varrho_1) + 2t\varrho_2, 2V(2t^2-t) + 2tR; 2V(2t^2\text{-}t) + 2tR, \Lambda_1 = 4t^2\Lambda + 4(2t^2-t)(V-R), \Lambda_2 = 4t^2R + 4(t^2-t)(V-R))$.

Proof. SBIBD$(4t^2, 2t^2+t, t^2+t)$ gives a regular (constant row and column sum $2t$) Hadamard matrix, H, when its zeros are replaced by -1's. That means in the proof of Theorem 12, $c_1 = t^2+t, c_2 = t^2-t, c_3 = c_4 = t^2$. Now replace the ones by the symmetric BTD B and -1's by $C = 2J - B$. Then the block matrix so constructed has $2t^2+t$ copies of B and $2t^2-t$ copies of C in each of its row. Now using the remark after Corollary 6, and the notation of Theorem 12, we have

$$s = R^2 - \Lambda(V-1), \quad q = \Lambda; u = R^2 - \Lambda(V-1) + 4(V-R), \quad w = \Lambda + 4(V-R);$$

$$x = z = 2R - (R^2 - \Lambda(V-1)); \quad y = a = 2R - \Lambda; \quad d_1 = 2t^2+t, \quad d_2 = 2t^2-t.$$

Therefore we get the required PBTD$(4t^2V, 4t^2V; 4t^2\varrho_1, (2t^2-t)(V-\varrho_1) + 2t\varrho_2, 2V(2t^2-t) + 2tR; 2V(2t^2-t) + 2tR, \Lambda_1 = 4t^2\Lambda + 4(2t^2-t)(V-R), \Lambda_2 = 4t^2R + 4(t^2-t)(V-R))$. □

Corollary 14. *If* $V = 2R - \Lambda$ *then we get a* BTD$(4t^2V, 4t^2V; 4t^2\varrho_1, (2t^2-t)(V-\varrho_1) + 2t\varrho_2, 2V(2t^2-t) + 2tR; 2V(2t^2-t) + 2tR, 4t^2R + 4(t^2-t)(V-R))$.

Example 5. BTD$(6, 6; 2, 1, 4; 4, 2)$ exists [Billington and Robison, 1983 no. 3] and SBIBD$(4, 3, 2)$ $(t = 1)$ exists (first row: 1 1 1 0). Here $c_2 = 0, 4t^2R + 4c_2(V-R) = 16$ and $4t^2\Lambda + 4(2t^2-t)(V-R) = 8 + 8 = 16$ and therefore a BTD$(24, 24; 8, 6, 20; 20, 16)$ exists.

Remark. The SBIBD$(4t^2, 2t^2+t, t^2+t)$ used in Theorem 13 have been extensively studied in Koukouvinos, Kounias and Seberry (1989) and Seberry (1992).

Appendix

A list of BTDs obtained via Theorem 2 is given. The last column gives the number of the BTD used with prime power V in Billington and Robinson (1983).

No.	V	B	ϱ_1	ϱ_2	R	K	Λ	No. in B&R
1	9	12	4	4	12	9	11	1
2	9	16	8	4	16	9	15	2
3	9	20	12	4	20	9	19	4
4	49	56	24	8	40	35	28	6
5	25	30	6	12	30	25	29	7
6	9	24	16	4	24	9	23	9
7	64	72	36	9	54	48	40	11
8	25	60	12	12	36	15	20	12
9	9	24	8	8	24	9	22	13
10	25	36	12	12	36	25	35	14
11	9	28	20	4	28	9	27	15
12	121	132	60	12	84	77	53	18
13	9	28	12	8	28	9	26	21
14	25	42	18	12	42	25	41	22
15	16	35	5	15	35	16	33	23
16	49	56	8	24	56	49	55	25
17	9	32	24	4	32	9	31	26
18	361	380	120	20	160	152	67	29
19	9	32	16	8	32	9	30	31
20	121	264	48	24	96	44	34	32
21	25	60	24	12	48	20	37	34
22	25	48	24	12	48	25	47	35
23	16	40	10	15	40	16	38	37
24	49	64	16	24	64	49	63	38
25	81	270	70	10	90	27	29	40
26	25	90	42	6	54	15	31	41
27	9	36	28	4	36	9	35	42
28	121	32	84	12	108	99	88	46
29	64	216	45	18	81	24	29	47
30	9	36	20	8	36	9	34	48
31	25	54	30	12	54	25	53	49
32	49	168	24	24	72	21	29	51
33	16	45	15	15	45	16	43	53
34	9	36	12	12	36	9	33	54
35	16	45	15	15	45	16	43	55
36	49	72	24	24	72	49	71	56
37	81	90	10	40	90	81	89	59
38	9	40	32	4	40	9	39	60
39	64	180	72	9	90	32	44	61
40	81	150	80	10	100	54	66	63
41	529	552	192	24	240	230	104	66
42	81	300	60	20	100	27	32	68
43	9	40	24	8	40	9	38	69
44	361	760	120	40	200	95	52	71
45	169	364	84	28	140	65	53	72
46	49	112	48	16	80	35	56	74
47	25	60	36	12	60	25	59	75
48	9	40	16	12	40	9	37	77

No.	V	B	ϱ_1	ϱ_2	R	K	Λ	No. in B&R
83	49	112	48	24	96	42	81	148
84	49	96	48	24	96	49	95	149
85	361	380	120	60	240	228	151	152
86	81	360	40	40	120	27	38	154
87	25	120	24	24	72	15	40	155
88	9	48	16	16	48	9	44	156
89	64	216	36	36	108	32	52	157
90	25	90	24	24	72	20	55	158
91	25	72	24	24	72	25	70	159
92	81	120	40	40	120	81	119	163
93	121	144	24	60	144	121	143	168
94	81	390	110	10	130	27	42	169
95	9	52	44	4	52	9	51	170
96	529	552	264	24	312	299	176	175
97	9	52	36	8	52	9	50	178
98	64	234	81	18	117	32	57	179
99	25	78	54	12	78	25	77	181
100	9	52	28	12	52	9	49	186
101	16	65	35	15	65	16	63	188
102	49	104	56	24	104	49	103	189
103	9	52	20	16	52	9	48	193
104	25	78	30	24	78	25	76	194
105	81	130	50	40	130	81	129	195
106	121	156	36	60	156	121	155	199
107	16	65	5	30	65	16	61	200
108	169	182	14	84	182	169	181	205
109	9	56	48	4	56	9	55	206
110	961	992	384	32	448	434	202	215
111	256	272	204	17	238	224	208	218
112	81	420	100	20	140	27	45	219
113	9	56	40	8	56	9	54	220
114	25	84	60	12	84	25	83	222
115	289	612	180	36	252	119	103	226
116	121	384	120	24	168	77	106	227
117	81	180	100	20	140	63	108	228
118	9	56	32	12	56	9	53	230
119	16	70	40	15	70	16	68	232
120	49	112	64	24	112	49	111	235
121	9	56	24	16	56	9	52	242
122	25	84	36	24	84	25	82	243
123	81	140	60	40	140	81	139	247
124	121	168	48	60	168	121	167	253
125	16	70	10	30	70	16	66	256
126	169	364	28	84	196	91	104	258
127	49	112	16	48	112	49	110	261
128	169	196	28	84	196	169	195	264
129	64	360	117	9	135	24	49	268
130	25	150	78	6	90	15	52	269

No.	V	B	ϱ_1	ϱ_2	R	K	Λ	No. in B&R
49	16	50	20	15	50	16	48	78
50	49	80	32	24	80	49	79	80
51	169	182	56	42	140	130	107	83
52	81	180	20	40	100	45	54	84
53	25	60	12	24	60	25	58	85
54	81	100	20	40	100	81	99	87
55	9	44	36	4	44	9	43	89
56	361	380	180	20	220	209	127	95
57	169	182	126	14	154	143	130	96
58	9	44	28	8	44	9	42	97
59	25	66	42	12	66	25	65	98
60	81	330	50	30	110	27	35	100
61	9	44	20	12	44	9	41	101
62	16	55	25	15	55	16	53	102
63	49	88	40	24	88	49	87	103
64	25	66	18	24	66	25	64	107
65	81	110	30	40	110	81	109	108
66	121	132	12	60	132	121	131	113
67	9	48	40	4	48	9	47	115
68	121	528	96	24	144	33	38	123
69	25	120	48	12	72	15	41	125
70	9	48	32	8	48	9	46	126
71	289	918	144	36	216	68	50	127
72	81	270	80	20	120	36	52	128
73	25	90	48	12	72	20	56	129
74	25	72	48	12	72	25	71	130
75	81	180	80	20	120	54	79	133
76	64	144	72	18	108	48	80	134
77	49	224	48	24	96	21	39	139
78	16	80	30	15	60	12	42	140
79	9	48	24	12	48	9	45	141
80	256	816	102	51	204	64	50	142
81	49	168	48	24	96	28	53	144
82	16	60	30	15	60	16	58	146

No.	V	B	ϱ_1	ϱ_2	R	K	Λ	No. in B&R
131	9	60	52	4	60	9	59	270
132	2809	2862	702	54	810	795	229	275
133	289	306	234	18	270	255	238	277
134	9	60	44	8	60	9	58	279
135	225	720	176	32	240	75	79	281
136	81	270	110	20	150	45	82	282
137	64	216	99	18	135	40	83	283
138	25	90	66	12	90	25	89	284
139	169	910	126	42	210	39	47	287
140	81	450	90	30	150	27	48	288
141	49	280	72	24	120	21	49	289
142	25	150	54	18	90	15	51	290
143	16	100	45	15	75	12	53	291
144	9	60	36	12	60	9	57	292
145	16	75	45	15	75	16	73	293
146	361	1140	180	60	300	95	78	295
147	49	168	72	24	120	35	84	297
148	49	120	72	24	120	49	119	298
149	9	60	28	16	60	9	56	304
150	25	90	42	24	90	25	88	306
151	81	150	70	40	150	81	149	308
152	121	660	60	60	180	33	47	309
153	9	60	20	20	60	9	55	312
154	64	270	45	45	135	32	65	313
155	121	180	60	60	180	121	179	321
156	16	75	15	30	75	16	71	327
157	169	546	42	84	210	65	79	328
158	81	270	30	60	150	45	81	329
159	49	168	24	48	120	35	83	330
160	25	90	18	36	90	25	87	331
161	49	120	24	48	120	49	118	335
162	169	210	42	84	210	169	209	336
163	64	135	9	63	135	64	133	340
164	225	240	16	112	240	225	239	344

Acknowledgement

We are grateful to the referees for useful comments.

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