

## The Excess of Complex Hadamard Matrices

H. Kharaghani<sup>1\*</sup> and Jennifer Seberry<sup>2\*\*</sup>

<sup>1</sup> Department of Mathematical Sciences, The University of Lethbridge, Lethbridge, Alberta, T1K 3M4, Canada

<sup>2</sup> Department of Computer Science, University of Wollongong, Wollongong, NSW 2500, Australia

**Abstract.** A complex Hadamard matrix,  $C$ , of order  $n$  has elements  $1, -1, i, -i$  and satisfies  $CC^* = nI_n$  where  $C^*$  denotes the conjugate transpose of  $C$ . Let  $C = [c_{ij}]$  be a complex Hadamard matrix of order  $n$ .  $S(C) = \sum_{ij} c_{ij}$  is called the sum of  $C$ .  $\sigma(C) = |S(C)|$  is called the excess of  $C$ . We study the excess of complex Hadamard matrices. As an application many real Hadamard matrices of large and maximal excess are obtained.

### 1. Introduction

A complex Hadamard matrix,  $C$ , has elements  $1, -1, i, -i$  where  $i = \sqrt{-1}$ , is necessarily of order  $1$  or  $2c$  and satisfies  $CC^* = 2cI_{2c}$  where  $C^*$  denotes the Hermitian conjugate (transpose, complex conjugate) of  $C$ . These were first introduced by Turyn [24], and further discussed by Seberry Wallis [23]. Their generalization to *complex orthogonal designs* is discussed by Geramita and Geramita [4].

Let  $C = [a_{ij}]$  be a complex Hadamard matrix of order  $n$  and denote by  $S(C)$  the sum of all entries of  $C$ , so  $S(C) = \sum_{ij} a_{ij}$ .  $\sigma(C) = |S(C)|$  is called the *excess* of  $C$ . Let  $\sigma_C(n) = \max\{\sigma(C) : C \text{ a complex Hadamard matrix of order } n\}$ . The maximum excess problem has been extensively studied for real Hadamard matrices [1–3, 7–19, 25, 26]. In this paper we will study the excess of complex Hadamard matrices. As an application we get many real Hadamard matrices of large excess and new classes of Hadamard matrices of maximum excess.

### 2. Preliminaries

First we note some basic properties of complex Hadamard matrices of large excess:

\* Supported by an NSERC grant.

\*\* Supported by Telecom grant 7027, an ATERB and ARC grant # A48830241.

**Lemma 1.** Let  $H = [h_{ij}]$  be a complex Hadamard matrix of order  $n$ . Let  $C_j$  denote the sum of the elements of the  $j$ th column of  $C$ , then

$$(i) \sum_j |C_j|^2 = n^2$$

$$(ii) \sum_j |C_j| \leq n\sqrt{n} \text{ with equality iff } |C_j| = \sqrt{n} \text{ for each } j$$

*Proof.*

(i) Let  $e$  be the  $1 \times n$  matrix of ones, then

$$eHH^*e^t = [C_1 C_2 \cdots C_n][C_1 C_2 \cdots C_n]^* = \sum_j |C_j|^2$$

also, since  $HH^* = nI$

$$eHH^*e^t = enIe^t = n^2$$

so

$$\sum_j |C_j|^2 = n^2$$

$$(ii) \sum_j (|C_j| - \sqrt{n})^2 = \sum_j |C_j|^2 + n^2 - 2\sqrt{n} \sum_j |C_j|$$

$$= 2n^2 - 2\sqrt{n} \sum_j |C_j| \geq 0 \text{ with equality iff } |C_j| = \sqrt{n} \text{ for each } j. \quad \square$$

A complex Hadamard matrix  $H = [h_{ij}]$  of order  $n$  is called *regular* if  $C_j = \sum_j h_{ij}$  remains the same for each  $j$ . In this case  $|C_j| = \sqrt{n}$  for each  $j$  and the maximum excess of  $n\sqrt{n}$  is attained. Since  $C_j = a + bi$  for some integers  $a, b$ , it follows that for a regular complex Hadamard matrix of order  $n$ ,  $n$  must be a sum of two squares. It is easy to see that an odd integer  $n$  is a sum of two squares iff  $2^{n_0}n$  is a sum of two squares for some  $n_0$ .

**Conjecture 2.** Suppose the odd integer  $m$  is a sum of two squares, then there is a regular complex Hadamard matrix of order  $2m$ .

If the odd integer  $m$  is not a sum of two squares, then there can be no regular complex Hadamard matrix of order  $2^{n_0}m$  for any  $n_0 \geq 1$ . Consequently, if there exists a complex Hadamard matrix of order  $n = 2^{n_0}m$ , then the maximum excess of such a matrix is strictly less than  $n\sqrt{n}$ . The maximum excess problem in this case is hard and for many large values of  $n$  seems hopeless.

### 3. Basic Constructions

We begin with a lemma which implies that the construction of certain complex Hadamard matrices is equivalent to the Williamson construction of real Hadamard matrices.

**Lemma 3.** *There is an Hadamard matrix of order  $4m$  of the form*

$$\begin{bmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{bmatrix}$$

*if and only if there is a complex Hadamard matrix of order  $2m$  of the form  $\begin{bmatrix} S & T \\ -\bar{T} & \bar{S} \end{bmatrix}$  where  $\bar{T}$  denotes the complex conjugate of  $T$ .*

*Proof.* To form the complex Hadamard matrix, let

$$X = \frac{1}{2}(A + B), \quad Y = \frac{1}{2}(A - B), \quad V = \frac{1}{2}(C - D), \quad W = \frac{1}{2}(C + D)$$

and the matrix is

$$\begin{bmatrix} X + iY & V + iW \\ -V + iW & X - iY \end{bmatrix}.$$

Conversely, if there is a complex Hadamard matrix of the form  $\begin{bmatrix} S & T \\ -\bar{T} & \bar{S} \end{bmatrix}$ , let

$$S = X + iY, \quad T = V + iW$$

Then

$$A = X + Y, \quad B = X - Y, \quad C = V + W, \quad D = W - V$$

can be used to form the required Hadamard matrix.  $\square$

Starting with four Williamson matrices  $A, B, C, D$  of order  $m$  in Lemma 3, the derived complex Hadamard matrix will have a sum of  $\sigma(A) + \sigma(B) + i(\sigma(C) + \sigma(D))$  and an excess of  $\sqrt{[\sigma(A) + \sigma(B)]^2 + [\sigma(C) + \sigma(D)]^2}$ . We use this method and known Williamson matrices [21, 22] to form Table 1.

**Lemma 4.** *Let  $C = H + iK$  be a complex Hadamard matrix of order  $m$ . Then the matrix  $H_{2m} = \begin{bmatrix} H + K & H - K \\ -H + K & H + K \end{bmatrix}$  is an Hadamard matrix of excess  $2(\sigma(H) + \sigma(K))$  and the matrix*

$$H_{4m} = \begin{bmatrix} -(H + K) & H - K & H + K & H - K \\ H - K & H + K & H - K & -(H + K) \\ H + K & -H + K & H + K & H - K \\ H - K & H + K & -H + K & H + K \end{bmatrix}$$

*is an Hadamard matrix of excess  $8\sigma(H)$ .*

Table 1. <sup>a</sup> indicates the maximum excess

$m$	Complex order	Williamson decomposition	Sum	Excess
3	6	$3^2 + 1^2 + 1^2 + 1^2$	$6(2 + i)$	$6\sqrt{5}$
5	10	$3^2 + 3^2 + 1^2 + 1^2$	$10(3 + i)$	$10\sqrt{10}^a$
7	14	$3^2 + 3^2 + 3^2 + 1^2$	$14(3 + 2i)$	$14\sqrt{13}$
9	18	$3^2 + 3^2 + 3^2 + 3^2$	$18(3 + 3i)$	$18\sqrt{18}^a$
11	22	$5^2 + 3^2 + 3^2 + 1^2$	$22(4 + 2i)$	$22\sqrt{20}$
13	26	$5^2 + 3^2 + 3^2 + 3^2$	$26(4 + 3i)$	$26\sqrt{25}$
		$5^2 + 5^2 + 1^2 + 1^2$	$26(5 + i)$	$26\sqrt{26}^a$
15	30	$5^2 + 5^2 + 3^2 + 1^2$	$30(5 + 2i)$	$30\sqrt{29}$
17	34	$5^2 + 5^2 + 3^2 + 3^2$	$34(5 + 3i)$	$34\sqrt{34}^a$
19	38	$7^2 + 5^2 + 1^2 + 1^2$	$38(6 + i)$	$38\sqrt{37}$
21	42	$5^2 + 5^2 + 5^2 + 3^2$	$42(5 + 4i)$	$42\sqrt{41}$
23	46	$7^2 + 5^2 + 3^2 + 3^2$	$46(6 + 3i)$	$46\sqrt{45}$
25	50	$5^2 + 5^2 + 5^2 + 5^2$	$50(5 + 5i)$	$50\sqrt{50}^a$
		$7^2 + 7^2 + 1^2 + 1^2$	$50(7 + i)$	$50\sqrt{50}^a$

Table 2. <sup>a</sup> indicates maximum possible excess achieved; <sup>b</sup> indicates maximum real excess known

Complex $m$	Sum	Excess $\sigma(C_m)$	Excess $\sigma(H_{2m})$	Excess $\sigma(H_{4m})$
6	$6(2 + i)$	$6\sqrt{5}$	$36^a$	
10	$10(3 + i)$	$10\sqrt{10}^a$	$80^a$	$240^b$
14	$14(3 + 2i)$	$14\sqrt{13}$	$140^a$	
18	$18(3 + 3i)$	$18\sqrt{18}^a$	$216^a$	
22	$22(4 + 2i)$	$22\sqrt{20}$	$264$	
26	$26(4 + 3i)$	$26\sqrt{25}$	$364^a$	
	$26(5 + i)$	$26\sqrt{26}^a$		$1040^b$
30	$30(5 + 2i)$	$30\sqrt{29}$	$420$	
34	$34(5 + 3i)$	$34\sqrt{34}^a$	$544$	
38	$38(6 + i)$	$38\sqrt{37}$		$1824^b$
42	$42(5 + 4i)$	$42\sqrt{41}$	$756^a$	
46	$46(6 + 3i)$	$46\sqrt{45}$	$828$	$2204^b$
50	$50(5 + 5i)$	$50\sqrt{50}^a$	$1000^a$	
	$50(7 + i)$	$50\sqrt{50}^a$		$2800^b$

*Proof.* Note that  $C$  being a complex Hadamard matrix implies that  $HH^t + KK^t = mI$  and  $HK^t = KH^t$ . Consequently,  $H_{2m}$  and  $H_{4m}$  are Hadamard. The excess part is trivial.  $\square$

Suppose now that  $\sigma(C) = a + bi$  and  $|\sigma(c)| = \sqrt{a^2 + b^2}$  is close to  $n\sqrt{n}$ , then if  $a$  is close to  $b$ ,  $H_{2m}$  will have a large excess, whereas, if  $a > b$  and  $a$  is far from  $b$ , then  $H_{4m}$  will have a large excess. Table 2 shows how this property can be effectively used to find real Hadamard matrices of large excess.

**Theorem 5.** *Let  $p \equiv 1 \pmod{4}$  be a prime power, then there is a complex Hadamard matrix of order  $p + 1$  with excess  $(p + 1)\sqrt{p}$ .*

*Proof.* It is known that if  $p \equiv 1 \pmod{4}$  is a prime power, then there are two circulant symmetric matrices,  $A$  and  $B$ ,  $A$  with zero diagonal and all other elements  $\pm 1$ ,  $B$  with elements  $\pm 1$ , of order  $\frac{1}{2}(p + 1)$  which satisfy  $AA^T + BB^T = pI$  [6, 23].

Let  $a, b$  be the row sums of  $A, B$  respectively, then  $p = a^2 + b^2$ , where  $b$  is odd.

Now, the matrix  $C_{p+1} = \begin{bmatrix} I + iA & B \\ B & -I + iA \end{bmatrix}$  is a complex Hadamard matrix with sum  $(p + 1)(b + ia)$  and consequently it has an excess of  $(p + 1)\sqrt{a^2 + b^2} = (p + 1)\sqrt{p}$ .

**Corollary 6.** *Let  $p = k^2 + (k + 1)^2 \equiv 1 \pmod{4}$  be a prime power. Furthermore, suppose  $p$  is a sum of two squares only as above. Then there is an Hadamard matrix of order  $2(p + 1)$  with maximal excess of  $2(p + 1)(2k + 1)$ .*

*Proof.* Using the matrix  $C_{p+1}$  of the preceding theorem in Lemma 4, we get a Hadamard matrix of order  $2(p + 1)$  with excess  $2(p + 1)(2k + 1)$ . It is shown in [17] that this is the maximum excess possible. Table 3 is drawn from this corollary.  $\square$

**Lemma 7.** *Let  $q^2 = p^{2\alpha} \equiv 1 \pmod{4}$  be a prime power which is decomposable as sum of two squares in only one way, namely,  $q^2 = p^{2\alpha} = (p^\alpha)^2 + 0 = q^2 + 0$ . Then there is a regular complex Hadamard matrix of order  $q^2 + 1$  and an Hadamard matrix of order  $4(q + 1)$  with excess of  $8q(q^2 + 1)$ .*

**Table 3.** \* indicates maximum possible excess

Prime $p \equiv (\pmod{4})$	Hadamard matrix order	Excess
$41 = 4^2 + 5^2$	84	756*
$61 = 5^2 + 6^2$	124	1364*
$113 = 7^2 + 8^2$	228	3420*
$181 = 9^2 + 10^2$	364	6916*
$313 = 12^2 + 13^2$	628	15700*
$421 = 14^2 + 15^2$	844	24476*

**Table 4.** \* indicates maximum excess known

Prime $q^2 = p^{2\alpha} \equiv 1 \pmod{4}$	Hadamard matrix order	Excess
$7^2$	200	2800*
$3^4$	328	5904*
$11^2$	488	10736*

*Proof.* Consider the matrices  $A, B$  in the proof of Theorem 5. Let  $D_{q^2+1} = \begin{bmatrix} iI + A & B \\ B & iI - A \end{bmatrix}$ , then  $D_{q^2+1}$  is regular. Using  $D_{q^2+1}$  in Lemma 4, we get an Hadamard matrix  $H_{4(q^2+1)}$  with excess  $8q(q^2 + 1)$ .  $\square$

Table 4 is drawn from this lemma.

#### 4. The Main Theorem

Kounias and Farmakis [17] showed that the maximal excess of a Hadamard matrix of order  $4m(m - 1)$  is given by

$$\sigma(4m(m - 1)) \leq 4(m - 1)^2(2m + 1).$$

Kharaghani [9] showed that this maximal excess can be attained if  $m$  is the order of a skew-Hadamard matrix. Koukouvinos and Seberry [14] subsequently proved that this maximal excess is attained if  $m = 2 \pmod{4}$  is the order of a conference matrix.

In this section we will show that some special kinds of complex Hadamard matrices can be used to construct Hadamard matrices which attain the above upperbound. This covers both values of  $m$  which are the order of a skew-Hadamard matrix and  $m \equiv 2 \pmod{4}$  which is the order of a conference matrix.

**Theorem 8.** *If there is a skew-complex Hadamard matrix of order  $m$  and a core of order  $m - 1$ , then there is a Hadamard matrix of order  $4m(m - 1)$  with maximum excess of  $4(m - 1)^2(2m + 1)$ .*

*Proof.* Note that  $m$ , being the order of a complex Hadamard matrix, is an even integer. We will divide the proof in two parts:

- (i)  $m - 1 \equiv 1 \pmod{4}$ . We may assume that the skew-complex Hadamard matrix is of the form

$$\begin{bmatrix} + & i & i & \cdots & i \\ i & & & & \\ i & & B + I & & \\ \vdots & & & & \\ i & & & & \end{bmatrix},$$

where  $B^* = -B$  and  $I$  is the identity matrix.

Insert the matrix  $J_{m-1}$ , the matrix of ones of order  $m - 1$ , on the main diagonal and elsewhere  $I + i$  core. By Corollary 6.8 of [23] the resulting matrix is a complex Hadamard matrix of order  $m(m - 1)$ . The sum of each of the top  $m - 1$  rows of this matrix is  $m - 1 + (m - 1)i$  and the remaining rows have a sum of  $m - 1 + i$  each. Thus, the total sum of this matrix is

$$(m - 1)[(m - 1) + (m - 1)i] + (m - 1)^2[m - 1 + i]$$

Using this Hadamard matrix and the two special Hadamard matrices,  $H$  and  $K$ .

$$H = \begin{bmatrix} + & + & + & - \\ + & + & - & + \\ + & - & + & + \\ - & + & + & + \end{bmatrix}, \quad K = \begin{bmatrix} - & + & + & + \\ + & - & + & + \\ - & - & + & - \\ - & - & - & + \end{bmatrix}$$

we can construct a Hadamard matrix  $H'$  of order  $4m(m - 1)$  following Turyn's method [24].

The sum of the matrix  $H'$  is  $8[(m - 1)^2 + (m - 1)^3]$ . Furthermore, the matrix  $H'$  will have a block  $J \times H$  on the upper left corner. Half of the rows and columns corresponding to this block have zero sum. Pick all such rows and columns corresponding to the  $(3, 1)$  elements of  $H$  and multiply them by a minus. As a result a total of  $4(m - 1)^2$  will be added to the previous sum [9]. The resulting Hadamard matrix will have the maximum excess of  $8[(m - 1)^2 + (m - 1)^3] + 4(m - 1)^2 = 4(m - 1)^2[2m + 1]$ .

- (ii) Let  $m - 1 \equiv 3 \pmod{4}$ . We may assume that the skew-complex Hadamard matrix is of the form

$$\begin{bmatrix} - & + & + & \cdots & + \\ + & & & & \\ + & & & & \\ \vdots & & & B + I & \\ + & & & & \end{bmatrix}$$

where  $B^* = -B$ .

Insert the matrix  $J_{m-1}$  on the main diagonal and elsewhere  $I$  + core. Follow a similar procedure as part (i) to construct the required Hadamard matrix.

*Example.* Consider the skew-complex Hadamard matrix

$$i \begin{bmatrix} -i & + & + & + & + & + \\ + & -i & + & - & - & + \\ + & + & -i & + & - & - \\ + & - & + & -i & + & - \\ + & - & - & + & -i & + \\ + & + & - & - & + & -i \end{bmatrix}$$

Let

$$C = \begin{bmatrix} 0 & + & - & - & + \\ + & 0 & - & + & - \\ - & - & 0 & + & + \\ - & + & + & 0 & - \\ + & - & + & - & 0 \end{bmatrix}$$

be a core of order 5. Then

$$H' = \begin{bmatrix} J \times H & I \times K - C \times H & I \times K - C \times H & I \times K - C \times H & I \times K - C \times H & I \times K - C \times H \\ I \times K - C \times H & \cdot & \cdot & \cdot & \cdot & \cdot \\ I \times K - C \times H & \cdot & \cdot & \cdot & \cdot & \cdot \\ I \times K - C \times H & \cdot & \cdot & \cdot & \cdot & \cdot \\ I \times K - C \times H & \cdot & \cdot & \cdot & \cdot & \cdot \\ I \times K - C \times H & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

where

$$J \times H = \begin{bmatrix} H & H & H & H & H \\ H & H & H & H & H \\ H & H & H & H & H \\ H & H & H & H & H \\ H & H & H & H & H \end{bmatrix},$$

$$I \times K + C \times H = \begin{bmatrix} K & H & -H & -H & H \\ H & K & -H & H & -H \\ -H & -H & K & H & H \\ -H & H & H & K & -H \\ H & -H & H & -H & K \end{bmatrix}$$

$$H = \begin{bmatrix} + & + & + & - \\ + & + & - & + \\ + & - & + & + \\ - & + & + & + \end{bmatrix}, \quad K = \begin{bmatrix} - & + & + & + \\ + & - & + & + \\ - & - & + & - \\ - & - & - & + \end{bmatrix}$$

The total sum of  $H'$  is 1200. Following the procedure in the theorem, one should multiply the following rows and columns of  $H'$  by minus;

row number: 3, 7, 11, 15, 19

column number: 1, 5, 9, 13, 17.

As a result the new matrix will have a sum of 1300 which is the maximum possible sum for the order  $n = 120$ .

*Example.*

Start with the matrix  $\begin{bmatrix} - & + & + & + \\ + & + & - & + \\ + & + & + & - \\ + & - & + & + \end{bmatrix}$ . Let  $C = \begin{bmatrix} 0 & + & - \\ - & 0 & + \\ + & - & 0 \end{bmatrix}$  be a core of

order 3.

Then

$$H' = \begin{bmatrix} -J \times H & (I + C) \times H & (I + C) \times H & (I + C) \times H \\ (I + C) \times H & \cdot & \cdot & \cdot \\ (I + C) \times H & \cdot & \cdot & \cdot \\ (I + C) \times H & \cdot & \cdot & \cdot \end{bmatrix}$$



where

$$J \times H = \begin{bmatrix} H & H & H \\ H & H & H \\ H & H & H \end{bmatrix}, \quad (I + C) \times H = \begin{bmatrix} H & H & -H \\ -H & H & H \\ H & -H & H \end{bmatrix},$$

$$H = \begin{bmatrix} + & + & + & - \\ + & + & - & + \\ + & - & + & + \\ - & + & + & + \end{bmatrix}.$$

$H'$  has a sum of 288. Following the procedure in the theorem, one should multiply the following rows and columns of  $H'$  by minus;

$$\begin{aligned} \text{row number:} & \quad 4, 8, 12 \\ \text{column number:} & \quad 1, 5, 9 \end{aligned}$$

As a result, the new matrix will have a sum of 324 which is the maximum possible sum for the order  $n = 48$ .

Theorem 8 is more powerful than those of maximal excess found in [9] and [14] and induces and unifies those results.

## References

1. Best, M.R.: The excess of a Hadamard matrix, *Indag. Math.* **39**, 357–361 (1977)
2. Enomoto, H., Miyamoto, M.: On maximal weights of Hadamard matrices, *J. Comb. Theory Ser. A*, **29**, 94–100 (1980)
3. Farmakis, N., Kounias, S.: The excess of Hadamard matrices and optimal designs, *Discrete Math.*, **67**, 165–176 (1987)
4. Geramita, A.V., Geramita, J.M.: Complex orthogonal designs, *J. Comb. Theory Ser. A*, **25**, 211–225 (1978)
5. Geramita, A.V., Seberry, J.: *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York (1979)
6. Goethals, J.M., Seidel, J.J.: Orthogonal matrices with zero diagonal, *Canad. J. Math.*, **19**, 1001–1010 (1967)
7. Hammer, J., Levingston, R., Seberry, J.: A remark on the excess of Hadamard matrices and orthogonal designs, *Ars Combinatoria*, **5**, 237–254 (1978)
8. Jenkins, B.A., Koukouvinos, C., Kounias, S., Seberry, J., Seberry, R.: Some results on the excesses of Hadamard matrices, *JCMCC*, **4**, 155–188 (1988)
9. Kharaghani, H.: An infinite class of Hadamard matrices of maximal excess, *Discrete Math.* (to be published)
10. Kharaghani, H., Seberry, J.: Regular complex Hadamard matrices, *Congr. Numerantium*, **75**, 187–201 (1990)
11. Koukouvinos, C.: Construction of some *SBIBD*( $4k^2, 2k^2 + k, k^2 + k$ ) and Hadamard matrices with maximal excess, *J. Stat. Plann. Inference* (to be published)
12. Koukouvinos, C., Kounias, S.: Construction of some Hadamard matrices with maximum excess, *Discrete Math.*, **85**, 295–300 (1990)
13. Koukouvinos, C., Kounias, S., Seberry, J.: Further results on base sequences, disjoint complementary sequences, *OD*( $4t; t, t, t, t$ ) and the excess of Hadamard matrices, *Ars Combinatoria*, **30**, 241–256 (1990)

14. Koukouvinos, C., Seberry, J.: Hadamard matrices of order  $\equiv 8 \pmod{16}$  with maximal excess, *Discrete Math.* **92**, 173–176 (1991)
15. Koukouvinos, C., Seberry, J.: Construction of new Hadamard matrices with maximal excess and infinitely many new *SBIBD*( $4k^2, 2k^2 + k, k^2 + k$ ), *Ann. Discrete Math.*, (A special edition edited by Rolf Rees in honour of Norman J. Pullman) (to be published)
16. Koukouvinos, C., Seberry, J.: Addendum to further results on base sequences, disjoint complementary sequences, *OD*( $4t; t, t, t, t$ ) and the excess of Hadamard matrices, *Congr. Numerantium* **82**, 97–103 (1991)
17. Kounias, S., Farmakis, N.: On the excess of Hadamard matrices, *Discrete Math.*, **68**, 59–69 (1988)
18. Sathé, Y.S., Shenoy, R.G.: Construction of maximal weight Hadamard matrices of order 48 and 80, *Ars Combinatoria*, **19**, 25–35 (1985)
19. Schmidt, K.W., Wang, E.T.H.: The weights of Hadamard matrices, *J. Comb. Theory Ser. A*, **23**, 257–263 (1977)
20. Seberry, J.: *SBIBD*( $4k^2, 2k^2 + k, k^2 + k$ ) and Hadamard matrices of order  $4k^2$  with maximal excess are equivalent, *Graphs and Combinatorics*, **5**, 373–383 (1989)
21. Seberry, J., Yamada, M.: Products of Hadamard matrices, Williamson matrices and other orthogonal matrices using *M*-structures, *Proceedings of Combinatorics Conference, Kobe, Japan, November* (1989)
22. Seberry, J., Yamada, M.: On products of Hadamard matrices, Williamson matrices and other orthogonal matrices using *M*-structures, *JCMCC*, **7**, 97–137 (1990)
23. Seberry Wallis, J.: Hadamard matrices, Part IV, *Combinatorics: Room Squares, Sum free sets and Hadamard Matrices*. In: Wallis W.D., Penfold Street A, Seberry Wallis J. (eds.) *Lecture Notes in Mathematics*, vol 292, Springer-Verlag, Berlin, Heidelberg, New York 1972
24. Turyn, R.J.: *Complex Hadamard matrices, Combinatorial Structures and Their Applications*, pp. 435–437 Gordon & Breach, New York 1970
25. Wallis, W.D.: On the weights of Hadamard matrices, *Ars Combinatoria*, **3**, 287–292 (1977)
26. Yamada, M.: On a series of Hadamard matrices of order  $2^t$  and the maximal excess of Hadamard matrices of order  $2^{2t}$ , *Graphs and Combinatorics*, **4**, 297–301 (1988)

Received: August 10, 1990

Revised: December 9, 1991