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Hadamard Matrices, Sequences, and Block Designs

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1 INTRODUCTION

One hundred years ago, in 1893, Jacques Hadamard [31] found square matrices of orders 12 and 20, with entries ± 1 , which had all their rows (and columns) pairwise orthogonal. These matrices, $X = (x_{ij})$, satisfied the equality of the following inequality,

$$|\det X|^2 \leq \prod_{i=1}^n \sum_{j=1}^n |x_{ij}|^2,$$

and so had maximal determinant among matrices with entries ± 1 . Hadamard actually asked the question of finding the maximal determinant of matrices with entries on the unit disc, but his name has become associated with the question concerning real matrices.

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Hadamard was not the first to study these matrices, for J. J. Sylvester in 1857, in his seminal paper, "Thoughts on inverse orthogonal matrices, simultaneous sign-successions and tessellated pavements in two or more colors with application to Newton's rule, ornamental tile work and the theory of numbers" [97], had found such matrices for all orders that are powers of two. Nevertheless, Hadamard showed that matrices with entries ± 1 and maximal determinant could exist only for orders 1, 2, and $4t$. The Hadamard conjecture states that "there exists an *Hadamard matrix*, or square matrix with every entry ± 1 and row (column) vectors pairwise orthogonal for these orders." This survey indicates the progress that has been made in the past 100 years.

Hadamard's inequality applies to matrices with entries from the unit circle. Matrices with entries ± 1 , $\pm i$, and pairwise orthogonal rows (and columns) are called *complex Hadamard matrices* (note the scalar product is $a \cdot b = \sum a_i b_i^*$ for complex numbers). These matrices were first studied by R. J. Turyn [104]. We believe complex Hadamard matrices exist for every order $n \equiv 0 \pmod{2}$. The truth of this conjecture would imply the truth of the Hadamard conjecture.

We begin by mentioning a few practical applications of Hadamard matrices. We note that it was M. Hall, Jr., L. Baumert, and S. Golomb [4] working with the U.S. Jet Propulsion Laboratories (JPL) who sparked the interest in Hadamard matrices in the past 30 years. In the 1960s the JPL was working toward building the *Mariner* and *Voyager* space probes to visit Mars and the other planets of the solar system. Those of us who saw early black-and-white pictures of the back of the moon remember that whole lines were missing. The black-and-white television pictures from the first landing on the moon were extremely poor quality. How many of us remember that the recent flyby of Neptune was by a space probe launched in the seventies? We take the high-quality color pictures of Jupiter, Saturn, Uranus, Neptune, and their moons for granted.

In brief, these high-quality color pictures are made by using three black-and-white pictures taken, in turn, through red, green, and blue filters. Each picture is then considered as a 1000×1000 matrix of black-and-white pixels. Each pixel is graded on a scale of 1 to 16, according to its greyness. So white is 1, and black is 16. These grades are then used to choose a codeword in an eight error correction code based on the Hadamard matrix of order 32. The codeword is transmitted to Earth, error corrected, the three black-and-white pictures are reconstructed, and then a computer is used to obtain the colored pictures.

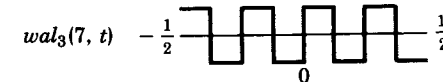
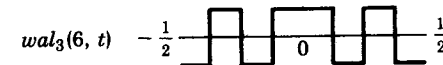
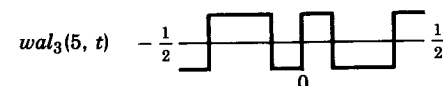
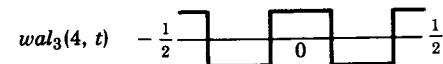
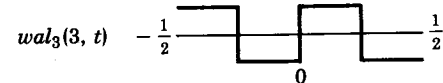
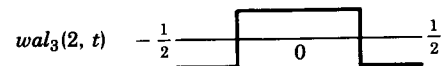
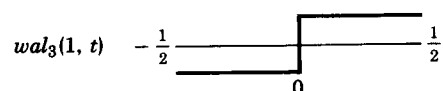
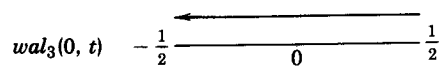
Hadamard matrices were used for these codewords for two reasons. First, error correction codes based on Hadamard matrices have maximal error correction capability for a given length of codeword. Second, the Hadamard matrices of powers of two are analogous to the Walsh functions, and thus all the computer processing can be accomplished using additions (which are very fast and easy to implement in computer hardware) rather than multiplications (which are far slower).

Sylvester's original construction for Hadamard matrices is equivalent to finding Walsh functions [48] which are the discrete analogue of Fourier Series.

Example 1.1. Let H be a Sylvester-Hadamard matrix (see Section 2) of order $8 = 2^3$.

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

The Walsh function wal_3 generated by H is the following:



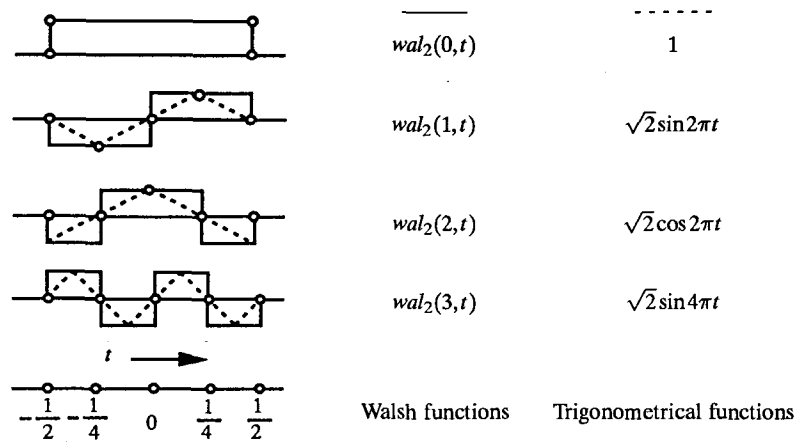


Figure 1.1. Walsh functions and trigonometrical functions.

The Walsh function wal_n is constructed in a similar way from the Sylvester-Hadamard matrix of order 2^n . The points of intersections of Walsh functions are identical with those of trigonometrical functions. See Figure 1.1.

As Figure 1.1 shows, by mapping $w(i, t) = wal_n(i, t)$ into the interval $[-\frac{1}{2}, 0]$, and then by extending the graph symmetrically into $[0, \frac{1}{2}]$, we get $w(2i, t)$, which is an even function. By operating similarly, we get $w(2i - 1, t)$, an odd function.

Just as any curve can be written as an infinite Fourier series,

$$\sum_n a_n \sin nt + b_n \cos nt,$$

the curve can be written in terms of Walsh functions,

$$\sum_n a_n sal_n(i, t) + b_n cal_n(i, t) = \sum_n c_n wal_n(i, t),$$

where $sal_n(i, t)$ and $cal_n(i, t)$ are, respectively, even and odd components of the Walsh function $wal_n(i, t)$. The hardest curve to model with Fourier series is the step function $wal_2(0, t)$, and errors lead to the Gibbes phenomenon. Similarly, the hardest curve to model with Walsh functions is the basic $\sin 2\pi t$ or $\cos 2\pi t$ curve. Still, we see that we can transform each form to the other.

Many problems require Fourier transforms to be taken, but Fourier transforms require many multiplications that are slow and expensive to execute. On the other hand, the fast Walsh-Hadamard transform uses only additions and subtractions (addition of the complement) and so is used extensively to transform power sequency spectrum density, band compression of television signals or facsimile signals or image processing.

Walsh functions are easy to extend to higher dimensions (and higher dimensional Hadamard matrices) to model surfaces in three and higher dimensions—

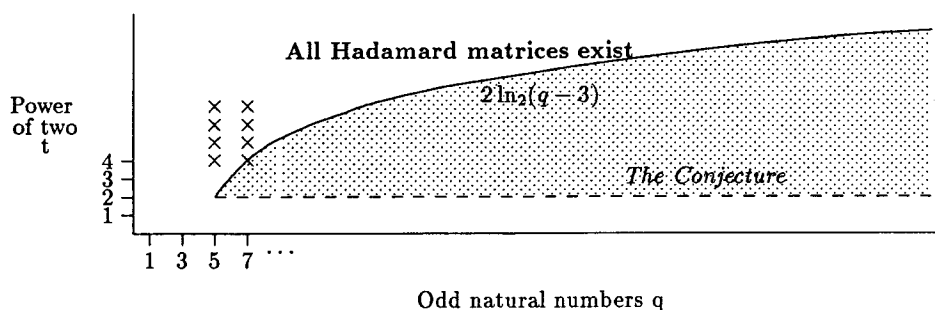


Figure 1.2. Hadamard matrices of order $2^t q$.

Fourier series are more difficult to extend. Walsh-Hadamard transforms in higher dimensions are also effected using only additions (and subtractions).

We now give an overview of construction methods for Hadamard matrices. Constructions for Hadamard matrices can be roughly classified into three types:

1. Multiplication theorems;
2. "Plug-in" methods;
3. Direct constructions.

In 1976, Jennifer Seberry Wallis, in her paper, "On the existence of Hadamard matrices" [121], showed that "given any odd natural number q , there exists a $t \approx 2 \log_2(q-3)$ so that there is an Hadamard matrix of order $2^t q$ (and hence for all orders $2^s q$, $s \geq t$)." This is represented graphically in Figure 1.2.

In fact, as we show in our Appendix, Hadamard matrices are known to exist of order $2^2 q$ for most $q < 3000$ (we have results up to 40000 that are similar). In many other cases, Hadamard matrices of order $2^3 q$ or $2^4 q$ exist. A quick look at the Appendix shows most of the very difficult cases are for q (prime) $\equiv 3 \pmod{4}$.

Hadamard's original construction for Hadamard matrices is a "multiplication theorem" as it uses the fact that the Kronecker product of Hadamard matrices of orders $2^a m$ and $2^b n$ is an Hadamard matrix of order $2^{a+b} mn$. Our graph shows that we would like to reduce this power of two. In his book, *Hadamard Matrices and Their Applications*, Agayan [1] shows how to multiply these Hadamard matrices to get an Hadamard matrix of order $2^{a+b-1} mn$ (which lowers the curve in our graph except for q prime).

Paley's 1933 "direct" construction [66], which gives Hadamard matrices of order $\prod_{i,j} (p_i + 1)(2(q_j + 1))$, p_i (prime power) $\equiv 3 \pmod{4}$, q_j (prime power) $\equiv 1 \pmod{4}$, is extremely productive of Hadamard matrices, but we note again the proliferation of powers of two as more products are taken.

Many people do not realize that in the same issue of the *Journal of Mathematics and Physics* as Paley's paper appeared, J. A. Todd showed the equivalence of Hadamard matrices of order $4t$ and $(4t-1, 2t-1, t-1)$ -SBIBD (see

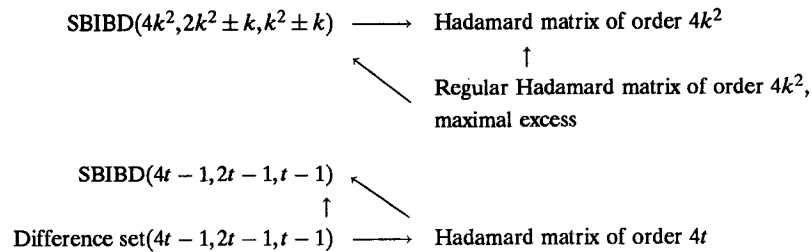


Figure 1.3. Relationship between SBIBD and Hadamard matrices.

Figure 1.3). This family of SBIBD, its complementary family $(4t - 1, 2t, t)$ -SBIBD, and the family $(4s^2, 2s^2 \pm s, s^2 \pm s)$ -SBIBD are called *Hadamard designs*. The latter family satisfies the constraint $v = 4(k - \lambda)$, for $v = 4s^2$, $k = 2s^2 \pm s$, and $\lambda = s^2 \pm s$, which appears in some constructions (e.g., Shrikhande [91]). Hadamard designs have the maximum number of one's in their incidence matrices among all incidence matrices of (v, k, λ) -SBIBD (see Tsuzuku [103]).

In 1944, J. Williamson [128], who coined the name *Hadamard matrices*, first constructed what have come to be called *Williamson matrices*, or with a small set of conditions, *Williamson type matrices*. These matrices are used to replace the variables of a formally orthogonal matrix. We say Williamson type matrices are "plugged in" to the second matrix. Other matrices that can be "plugged in" to arrays of variables are called *suitable matrices*. Generally the arrays into which suitable matrices are plugged are *orthogonal designs*, which have formally orthogonal rows (and columns) but may have variations such as Goethals-Seidel arrays, Wallis-Whiteman arrays, Spence arrays, generalized quaternion arrays, Agayan families, Kharaghani's methods, and regular s -sets of regular matrices that give new matrices. This is an extremely prolific method of construction. We will discuss methods that give matrices to "plug in" and matrices to "plug into."

As a general rule, if we want to check if an Hadamard matrix of a specific order $4pq$ exists, we would first check if there are Williamson type matrices of order p, q, pq ; then we would check if there is an $\text{OD}(4t; t, t, t)$ for $t = q, p, pq$. This failing, we would check the "direct" constructions. Finally, we would use a "multiplication theorem." When we talk of "strength" of a construction, this reflects a personal preference.

Before we proceed to more detail, we will consider diagrammatically some of the linkages between conjectures that will arise in this survey: The conjecture implied is "the necessary conditions are sufficient for the existence of (say) Hadamard matrices" (see Figure 1.4). (A *weighing matrix* W has entries $0, \pm 1$, is square, and satisfies $WW^T = kI$.)

The hierarchy of conjectures for weighing matrices and ODs is more straightforward. Settling the OD conjecture in Table 1.1 would settle the weighing matrix conjecture to its left. This survey emphasizes those constructions, selected by us, which we believe show the most promise toward solving the Hadamard conjecture and which were found in the last 15 years.

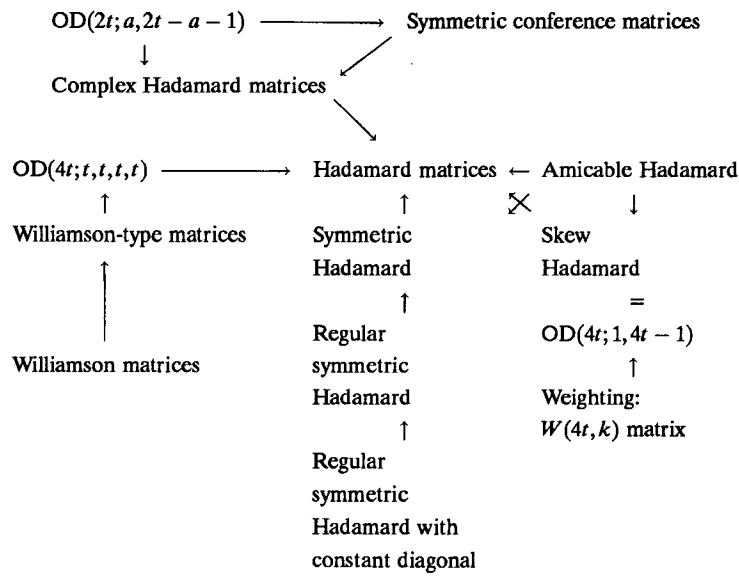


Figure 1.4. Conjecture: "The necessary conditions are sufficient for the existence of (say) Hadamard matrices."

TABLE 1.1 Weighing Matrix and OD Conjectures

| | Matrices | OD's |
|-----------|-------------------------------------|---|
| Strongest | Skew-weighing | $OD(n; 1, k)$ |
| | Weighting $W(n; k)$, n odd | |
| | Weighting $W(2n, k)$, n odd | $OD(2n; a, b)$, n odd |
| | Weighting $W(4n, k)$, n odd | $OD(4n; a, b, c, d)$, n odd |
| Weakest | $W(2^s n, k)$, n odd, $s \geq 3$ | $OD(2^s n; u_1, u_2, \dots, u_s)$, n odd |

2 HADAMARD MATRICES

A square matrix with elements ± 1 and order h , whose distinct row vectors are orthogonal is an *Hadamard matrix* of order h . The smallest examples are

$$[1], \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix},$$

where we write $-$ for -1 . These were first studied by J. J. Sylvester [97] who observed that if H is an Hadamard matrix, then

$$\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

is also an Hadamard matrix. Indeed, using the matrix of order 2, we have

Lemma 2.1 (Sylvester [97]). *There is an Hadamard matrix of order 2^t for all integers t .*

We call matrices of order 2^t constructed by Sylvester's construction *Sylvester-Hadamard matrices*. We have seen that these matrices are naturally associated with the discrete orthogonal functions called *Walsh functions*. Using Sylvester's method, the first few Hadamard matrices obtained are

$$\begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix}, \quad \begin{array}{c|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & - & - & 1 & 1 & - & - & 1 \\ \hline 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & - & 1 & - & - & 1 & - & 1 \\ 1 & 1 & - & - & - & - & 1 & 1 \\ 1 & - & - & 1 & - & 1 & 1 & - \end{array}.$$

For these matrices, we count, row by row, the number of times the sign changes; for example, $1 - -1$ changes sign twice. This gives

for the matrix of order 2 : 0, 1;

for the matrix of order 4 : 0, 3, 1, 2;

for the matrix of order 8 : 0, 7, 3, 4, 1, 6, 2, 5.

Indeed, we will see that the set of the numbers of sign changes in the rows of a Sylvester-Hadamard matrix of order n is $\{0, 1, \dots, n-1\}$, corresponding to the number times the Walsh functions cross the x -axis.

In 1893, Jacques Hadamard [31] gave examples of Hadamard matrices for a few small orders and conjectured that they exist for every order divisible by 4. Some examples for order 12 are

We have given these matrices in full because, unfortunately, an earlier survey contains errors.

Two Hadamard matrices are said to be *Hadamard equivalent* (or just *equivalent*) if one can be obtained from the other by a sequence of operations of the following two types:

1. Permute rows (or columns).
2. Multiply any row (or column) by -1 .

Although the Hadamard matrices of order 12 presented above appear to be different, it is possible to show that they are equivalent.

In fact, we know that there are 5 inequivalent matrices of order 16 [32], 3 of order 20 [33], 60 of order 24 [37, 47], 486 of order 28 [44], over 15 of order 32 (N. Ito, personal communication, 1989), and over 109 of order 36 [11].

An Hadamard matrix of order 20 is given in Figure 2.1. This figure is more easily described by calling the rows 0 to 19 and saying that the zeroth row is all ones, the first row has ones in positions

$$\{1, 2, 5, 6, 7, 8, 10, 12, 17, 18\},$$

the second row has ones in positions

$$\{2, 3, 6, 7, 8, 9, 11, 13, 18, 19\},$$

the third row has ones in positions

$$\{4, 5, 8, 9, 10, 11, 13, 15, 1, 2\},$$

and so on.

This example illustrates the use of difference sets with the parameters $(4t - 1, 2t - 1, t - 1)$ in the construction of Hadamard matrices. $\{1, 2, 5, 6, 7, 8, 10, 12, 17, 18\}$ is a difference set with parameters $(19, 9, 4)$. For more information on difference sets, see the survey by Jungnickel in this volume [40].

Hadamard matrices can also be constructed using supplementary difference sets. The existence of supplementary difference sets in the abelian group $Z_3 \times Z_3$ and can be used to construct another Hadamard matrix of order 20 given in Figure 2.2.

We now recall some basic properties of Hadamard matrices:

Lemma 2.2. *Let H be an Hadamard matrix of order h . Then the following hold:*

1. $HH^T = hI_h$.
2. $|\det H| = h^{(1/2)h}$.
3. $HH^T = H^T H$.

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| - | 1 | 1 | - | - | 1 | 1 | 1 | 1 | - | 1 | - | 1 | - | - | - | - | 1 | 1 | - |
| - | - | 1 | 1 | - | - | 1 | 1 | 1 | 1 | - | 1 | - | 1 | - | - | - | - | 1 | 1 |
| - | 1 | - | 1 | 1 | - | - | 1 | 1 | 1 | 1 | - | 1 | - | 1 | - | - | - | - | 1 |
| - | 1 | 1 | - | 1 | 1 | - | - | 1 | 1 | 1 | 1 | - | 1 | - | 1 | - | - | - | - |
| - | - | 1 | 1 | - | 1 | 1 | - | - | 1 | 1 | 1 | 1 | - | 1 | - | 1 | - | - | - |
| - | - | - | 1 | 1 | - | 1 | 1 | - | - | 1 | 1 | 1 | 1 | - | 1 | - | 1 | - | - |
| - | - | - | - | 1 | 1 | - | 1 | 1 | - | - | 1 | 1 | 1 | 1 | - | 1 | - | 1 | - |
| - | - | - | - | - | 1 | 1 | - | 1 | 1 | - | - | 1 | 1 | 1 | 1 | - | 1 | - | 1 |
| - | 1 | - | - | - | - | 1 | 1 | - | 1 | 1 | - | - | 1 | 1 | 1 | 1 | - | 1 | - |
| - | - | 1 | - | - | - | - | 1 | 1 | - | 1 | 1 | - | - | 1 | 1 | 1 | 1 | - | 1 |
| - | 1 | - | 1 | - | - | - | - | 1 | 1 | - | 1 | 1 | - | - | 1 | 1 | 1 | 1 | - |
| - | - | 1 | - | 1 | - | - | - | - | 1 | 1 | - | 1 | 1 | - | - | 1 | 1 | 1 | 1 |
| - | 1 | - | 1 | - | 1 | - | - | - | - | 1 | 1 | - | 1 | 1 | - | - | 1 | 1 | 1 |
| - | 1 | 1 | - | 1 | - | 1 | - | - | - | - | 1 | 1 | - | 1 | 1 | - | - | 1 | 1 |
| - | 1 | 1 | 1 | - | 1 | - | 1 | - | - | - | - | 1 | 1 | - | 1 | 1 | - | - | 1 |
| - | 1 | 1 | 1 | 1 | 1 | - | 1 | - | 1 | - | - | - | 1 | 1 | - | 1 | 1 | - | - |
| - | - | 1 | 1 | 1 | 1 | 1 | - | 1 | - | 1 | - | - | - | 1 | 1 | - | 1 | 1 | 1 |
| - | 1 | - | - | 1 | 1 | 1 | 1 | - | 1 | - | 1 | - | - | - | 1 | 1 | - | 1 | 1 |

Figure 2.1. An Hadamard matrix of order 20.

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | - | - | 1 | - | 1 | - | - | - | 1 | - | 1 | - | - | 1 | - | - |
| 1 | 1 | 1 | 1 | 1 | - | - | - | - | 1 | - | - | 1 | - | - | - | - | - | 1 | - |
| 1 | 1 | 1 | 1 | - | 1 | - | - | 1 | - | - | - | - | 1 | - | - | - | - | - | - |
| 1 | - | 1 | - | 1 | 1 | 1 | - | - | 1 | - | - | - | - | 1 | - | - | - | - | 1 |
| 1 | - | - | 1 | 1 | 1 | 1 | 1 | - | - | - | - | - | 1 | - | 1 | - | 1 | - | - |
| 1 | 1 | - | - | 1 | 1 | 1 | 1 | - | 1 | - | - | - | - | 1 | 1 | - | - | - | 1 |
| 1 | - | - | 1 | - | 1 | - | 1 | 1 | 1 | - | - | - | - | - | 1 | 1 | - | - | 1 |
| 1 | 1 | - | - | - | - | 1 | 1 | 1 | 1 | - | - | - | - | - | 1 | 1 | - | - | 1 |
| 1 | - | - | 1 | - | 1 | - | - | - | - | - | - | - | - | - | 1 | 1 | 1 | - | - |
| 1 | - | 1 | 1 | - | - | 1 | - | - | - | 1 | 1 | - | - | 1 | 1 | - | 1 | - | 1 |
| 1 | 1 | 1 | - | - | 1 | - | - | 1 | - | - | - | - | - | - | 1 | 1 | - | - | 1 |
| 1 | - | 1 | - | - | 1 | 1 | - | - | - | 1 | 1 | - | - | - | 1 | 1 | - | - | 1 |
| 1 | - | - | 1 | 1 | - | 1 | 1 | - | - | 1 | 1 | - | - | - | - | 1 | 1 | - | - |
| 1 | 1 | - | - | - | - | 1 | - | 1 | - | - | - | - | - | - | - | 1 | 1 | - | - |
| 1 | - | 1 | - | 1 | - | - | 1 | 1 | - | - | - | - | - | - | - | 1 | 1 | - | - |

Figure 2.2. A second Hadamard matrix of order 20.

4. Every Hadamard matrix is equivalent to an Hadamard matrix that has every element of its first row and column +1 (matrices of this latter form are called normalized).
5. $h = 1, 2,$ or $4n,$ n an integer.

6. If H is a normalized Hadamard matrix of order $4n$, then every row (column) except the first has $2n$ minus ones and $2n$ plus ones in each row (column); further, n minus ones in any row (column) overlap with n minus ones in each other row (column).

Definition 2.1. An Hadamard matrix H is said to be *regular* if the sum of all the elements in each row or column is a constant k . Hence $HJ = JH = kJ$, where J is the matrix of all ones.

Definition 2.2. If $M = (m_{ij})$ is a $m \times p$ matrix and $N = (n_{ij})$ is an $n \times q$ matrix, then the *Kronecker product* $M \times N$ is the $mn \times pq$ matrix given by

$$M \times N = \begin{bmatrix} m_{11}N & m_{12}N & \cdots & m_{1p}N \\ m_{21}N & m_{22}N & \cdots & m_{2p}N \\ \vdots & & & \vdots \\ m_{m1}N & m_{m2}N & \cdots & m_{mp}N \end{bmatrix}.$$

Example 2.1. Let

$$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

Then

$$M \times N = \begin{bmatrix} N & N \\ N & -N \end{bmatrix} = \left[\begin{array}{cccc|cccc} -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ \hline -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \end{array} \right].$$

Lemma 2.3 (Hadamard [31]). Let H_1 and H_2 be Hadamard matrices of orders h_1 and h_2 . Then $H = H_1 \times H_2$ is an Hadamard matrix of order $h_1 h_2$.

We now prove a stronger result than Hadamard's, first proved by Agayan and Sarukhanyan, and then strengthened by Seberry and Yamada [87] and

Agayan-Sarukhanyan [1]. These theorems have the advantage of reducing the power of two in the resulting Hadamard matrix.

Lemma 2.4 (The Multiplication Theorem of Agayan-Sarukhanyan [1]). *Let H_1 and H_2 be Hadamard matrices of orders $4h$ and $4k$. Then there is an Hadamard matrix of order $8hk$.*

Proof. Write the two Hadamard matrices as

$$H_1 = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \quad \text{and} \quad H_2 = \begin{bmatrix} K & L \\ M & N \end{bmatrix}.$$

We note that since $H_1 H_1^T = 4hI$ and $H_2 H_2^T = 4kI$, we have

$$PP^T + QQ^T = RR^T + SS^T = 2hI, \quad PR^T + QS^T = O = RP^T + SQ^T;$$

$$KK^T + LL^T = MM^T + NN^T = 2kI, \quad KM^T + LN^T = O = MK^T + NL^T.$$

The required Hadamard matrix of order $8hk$ is

$$\begin{bmatrix} \frac{1}{2}(P+Q) \times K + \frac{1}{2}(P-Q) \times M & \frac{1}{2}(P+Q) \times L + \frac{1}{2}(P-Q) \times N \\ \frac{1}{2}(R+S) \times K + \frac{1}{2}(R-S) \times M & \frac{1}{2}(R+S) \times L + \frac{1}{2}(R-S) \times N \end{bmatrix}$$

which can be verified by simple algebraic manipulation. \square

Example 2.2. There are Hadamard matrices of orders 12 and 20. Sylvester's lemma guarantees the existence of an Hadamard matrix of order 240, while the Agayan-Sarukhanyan guarantees the existence of one of order 120.

This can also be strengthened.

Theorem 2.5 (Craig-Seberry-Zhang [14]). *Suppose that there are Hadamard matrices of orders $4a, 4b, 4c, 4d$. Then there is an Hadamard matrix of order $16abcd$.*

So, for example, we can get an Hadamard matrix of order $16 \cdot 15 \cdot 15$ from this theorem.

3 THE STRONGEST HADAMARD CONSTRUCTION THEOREMS

For easy reference, we will now give the strongest construction theorems for Hadamard matrices. These theorems do not give all the known orders but give

the vast majority of those known. We leave the proofs until our later book as well as details of when these conditions can be satisfied.

Theorem 3.1 (Paley [66]). *Let $p \equiv 3 \pmod{4}$ be a prime power. Then there is an Hadamard matrix of order $p + 1$.*

Theorem 3.2 (Paley [66]). *Let $p \equiv 1 \pmod{4}$ be a prime power. Then there is an Hadamard matrix of order $2(p + 1)$.*

Theorem 3.3 (Goethals-Seidel [25]). *Suppose that there is an Hadamard matrix of order h . Then there is a regular symmetric Hadamard matrix with constant diagonal of order h^2 .*

Since Hadamard matrices are of order $h \equiv 0 \pmod{4}$ and Hadamard's inequality studies matrices on the unit disc, it is natural to consider matrices with complex entries.

Definition 3.1. A matrix C of order $2n$ with elements $\pm 1, \pm i$ that satisfies $CC^* = 2nI$ will be called a *complex Hadamard matrix*.

The strongest theorem using complex Hadamard matrices is the following "multiplication theorem":

Theorem 3.4 (Turyn [104]). *Suppose that there is a complex Hadamard matrix of order $2n$ and an Hadamard matrix of order $4h$. Then there is an Hadamard matrix of order $8hn$.*

This means that the complex Hadamard conjecture is intricately woven with the Hadamard conjecture.

Definition 3.2. X and Y are said to be *amicable matrices* if

$$XY^T = YX^T. \quad (1)$$

Now we look more precisely at definitions of matrices to "plug in."

Definition 3.3. Four circulant symmetric ± 1 matrices A, B, C, D of order w that satisfy

$$AA^T + BB^T + CC^T + DD^T = 4wI_w$$

will be called *Williamson matrices*. Four ± 1 matrices A, B, C, D of order w that satisfy both

$$XY^T = YX^T \quad \text{for } X, Y \in \{A, B, C, D\}$$

(that is, A, B, C, D are pairwise amicable), and

$$AA^T + BB^T + CC^T + DD^T = 4wI_w, \quad (2)$$

will be called *Williamson-type matrices*.

Analogously, eight circulant ± 1 matrices A_1, A_2, \dots, A_8 of order w which are symmetric and which satisfy

$$\sum_{i=1}^8 A_i A_i^T = 8wI_w$$

will be called *8-Williamson matrices*. Eight ± 1 amicable matrices A_1, A_2, \dots, A_8 of order w which satisfy both

$$\sum_{i=1}^8 A_i A_i^T = 8wI_w \quad \text{and} \quad A_j A_i^T = A_i A_j^T, \quad i, j = 1, \dots, 8,$$

will be called *8-Williamson-type matrices*.

The most common structure matrices are "plugged into" is the orthogonal design, defined as follows:

Definition 3.4. An orthogonal design of order n and type (s_1, \dots, s_u) , s_i positive integers, is an $n \times n$ matrix X , with entries $\{0, \pm x_1, \dots, \pm x_u\}$ (the x_i commuting indeterminates) satisfying

$$XX^T = \left(\sum_{i=1}^u s_i x_i^2 \right) I_n. \quad (3)$$

We write this as $OD(n; s_1, s_2, \dots, s_u)$.

Alternatively, each row of X has s_i entries of the type $\pm x_i$, and the distinct rows are orthogonal under the euclidean inner product. We may view X as a matrix with entries in the field of fractions of the integral domain $Z[x_1, \dots, x_u]$ (Z the rational integers), and if we let $f = (\sum_{i=1}^u s_i x_i^2)$, then X is an invertible matrix with inverse $(1/f)X^T$. Thus, $XX^T = fI_n$, and so our alternative definition that the row vectors are orthogonal applies equally well to the column vectors of X .

An orthogonal design with no zeros and in which each of the entries is replaced by $+1$ or -1 is an Hadamard matrix. A special orthogonal design, the $OD(4t; t, t, t, t)$, is especially useful in the construction of Hadamard matrices. An $OD(12; 3, 3, 3, 3)$ was first found by L. Baumert and M. Hall, Jr. [6], and an $OD(20; 5, 5, 5, 5)$ by Welch (see below). $OD(4t; t, t, t, t)$ are sometimes called *Baumert-Hall arrays*.

Another set of matrices of a very different kind can be obtained by partitioning a matrix as follows: Let M be a matrix of order tm . Then M can be expressed as a t^2 block M -structure when M is an orthogonal matrix:

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ M_{21} & M_{22} & \cdots & M_{2t} \\ \vdots & & & \vdots \\ M_{t1} & M_{t2} & \cdots & M_{tt} \end{bmatrix},$$

where M_{ij} is of order m ($i, j = 1, 2, \dots, t$).

Some orthogonal designs of special interest are the following:

1. The Williamson array—the OD(4; 1, 1, 1, 1):

$$\begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix} \quad \text{the right representation of the quaternions;}$$

$$\begin{bmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{bmatrix} \quad \text{the left representation of the quaternions.}$$

2. The OD(8; 1, 1, 1, 1, 1, 1, 1, 1):

$$\begin{array}{cccc|cccc} A & B & C & D & E & F & G & H \\ -B & A & D & -C & F & -E & -H & G \\ -C & -D & A & B & G & H & -E & -F \\ -D & C & -B & A & H & -G & F & -E \\ \hline -E & -F & -G & -H & A & B & C & D \\ -F & E & -H & G & -B & A & -D & C \\ -G & H & E & -F & -C & D & A & -B \\ -H & -G & F & E & -D & -C & B & A \end{array}$$

3. The Baumert-Hall array—the OD(12;3,3,3,3):

$$A(x,y,z,w) =$$

| | | | | | | | | | | | |
|------|------|------|------|------|------|------|------|------|------|------|------|
| y | x | x | x | $-z$ | z | w | y | $-w$ | w | z | $-y$ |
| $-x$ | y | x | $-x$ | w | $-w$ | z | $-y$ | $-z$ | z | $-w$ | $-y$ |
| $-x$ | $-x$ | y | x | w | $-y$ | $-y$ | w | z | z | w | $-z$ |
| $-x$ | x | $-x$ | y | $-w$ | $-w$ | $-z$ | w | $-z$ | $-y$ | $-y$ | $-z$ |
| $-y$ | $-y$ | $-z$ | $-w$ | z | x | x | x | $-w$ | $-w$ | z | $-y$ |
| $-w$ | $-w$ | $-z$ | y | $-x$ | z | x | $-x$ | y | y | $-z$ | $-w$ |
| w | $-w$ | w | $-y$ | $-x$ | $-x$ | z | x | y | $-z$ | $-y$ | $-z$ |
| $-w$ | $-z$ | w | $-z$ | $-x$ | x | $-x$ | z | $-y$ | y | $-y$ | w |
| $-y$ | y | $-z$ | $-w$ | $-z$ | $-z$ | w | y | w | x | x | x |
| z | $-z$ | $-y$ | $-w$ | $-y$ | $-y$ | $-w$ | $-z$ | $-x$ | w | x | $-x$ |
| $-z$ | $-z$ | y | z | $-y$ | $-w$ | y | $-w$ | $-x$ | $-x$ | w | x |
| z | $-w$ | $-w$ | z | y | $-y$ | y | z | $-x$ | x | $-x$ | w |

or alternatively (using the Cooper-J.Wallis theorem [12]), the OD(12;3,3,3,3) is

| | | | | | | | | | | | |
|------|------|------|------|------|------|------|------|------|------|------|------|
| a | b | c | $-b$ | a | d | $-c$ | $-d$ | a | $-d$ | c | $-b$ |
| c | a | b | a | d | $-b$ | $-d$ | a | $-c$ | c | $-b$ | $-d$ |
| b | c | a | d | $-b$ | a | a | $-c$ | $-d$ | $-b$ | $-d$ | c |
| b | $-a$ | $-d$ | a | b | c | $-d$ | $-b$ | c | c | $-a$ | d |
| $-a$ | $-d$ | b | c | a | b | $-b$ | c | $-d$ | $-a$ | d | c |
| $-d$ | b | $-a$ | b | c | a | c | $-d$ | $-b$ | d | c | $-a$ |
| c | d | $-a$ | d | b | $-c$ | a | b | c | $-b$ | d | a |
| d | $-a$ | c | b | $-c$ | d | c | a | b | d | a | $-b$ |
| $-a$ | c | d | $-c$ | d | b | b | c | a | a | $-b$ | d |
| d | $-c$ | b | $-c$ | a | $-d$ | b | $-d$ | $-a$ | a | b | c |
| $-c$ | b | d | a | $-d$ | $-c$ | $-d$ | $-a$ | b | c | a | b |
| b | d | $-c$ | $-d$ | $-c$ | a | $-a$ | b | $-d$ | b | c | a |

4. The Plotkin array—the OD(24;3,3,3,3,3,3,3,3):

Let $A(x,y,z,w)$ be as in array 3, and let

$$B = (x, y, z, w)$$

$$= \begin{bmatrix} y & x & x & x & -w & w & z & y & -z & z & w & -y \\ -x & y & x & -x & -z & z & -w & -y & w & -w & z & -y \\ -x & -x & y & x & -y & -w & y & -w & -z & -z & w & z \\ -x & x & -x & y & w & w & -z & -w & -y & z & y & z \\ -w & -w & -z & -y & z & x & x & x & -y & -y & z & -w \\ y & y & -z & -w & -x & z & x & -x & -w & -w & -z & y \\ -w & w & -w & -y & -x & -x & z & x & z & y & y & z \\ z & -w & -w & z & -x & x & -x & z & y & -y & y & w \\ z & -z & y & -w & y & y & w & -z & w & x & x & x \\ y & -y & -z & -w & -z & -z & -w & -y & -x & w & x & -x \\ z & z & y & -z & w & -y & -y & w & -x & -x & w & x \\ -w & -z & w & -z & -y & y & -y & z & -x & x & -x & w \end{bmatrix},$$

then $\begin{bmatrix} A(x_1, x_2, x_3, x_4) & B(x_5, x_6, x_7, x_8) \\ B(-x_5, x_6, x_7, x_8) & -A(-x_1, x_2, x_3, x_4) \end{bmatrix}$ is the required design.

5. The Welch array—the OD(20; 5, 5, 5, 5) constructed from 16-block circulant matrices is an M -structure:

| | | | | | | | | | | | | | | | | | | | |
|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| $-D$ | B | $-C$ | $-C$ | $-B$ | C | A | $-D$ | $-D$ | $-A$ | $-B$ | $-A$ | C | $-C$ | $-A$ | A | $-B$ | $-D$ | D | $-B$ |
| $-B$ | $-D$ | B | $-C$ | $-C$ | $-A$ | C | A | $-D$ | $-D$ | $-A$ | $-B$ | $-A$ | C | $-C$ | $-B$ | A | $-B$ | $-D$ | D |
| $-C$ | $-B$ | $-D$ | B | $-C$ | $-D$ | $-A$ | C | A | $-D$ | $-C$ | $-A$ | $-B$ | $-A$ | C | D | $-B$ | A | $-B$ | $-D$ |
| $-C$ | $-C$ | $-B$ | $-D$ | B | $-D$ | $-D$ | $-A$ | C | A | C | $-C$ | $-A$ | $-B$ | $-A$ | $-D$ | D | $-B$ | A | $-B$ |
| B | $-C$ | $-C$ | $-B$ | $-D$ | A | $-D$ | $-D$ | $-A$ | C | $-A$ | C | $-C$ | $-A$ | $-B$ | $-B$ | $-D$ | D | $-B$ | A |
| $-C$ | A | D | D | $-A$ | $-D$ | $-B$ | $-C$ | $-C$ | B | $-A$ | B | $-D$ | D | B | $-B$ | $-A$ | $-C$ | C | $-A$ |
| $-A$ | $-C$ | A | D | D | B | $-D$ | $-B$ | $-C$ | $-C$ | B | $-A$ | B | $-D$ | D | $-A$ | $-B$ | $-A$ | $-C$ | C |
| D | $-A$ | $-C$ | A | D | $-C$ | B | $-D$ | $-B$ | $-C$ | D | B | $-A$ | B | $-D$ | C | $-A$ | $-B$ | $-A$ | $-C$ |
| D | D | $-A$ | $-C$ | A | $-C$ | $-C$ | B | $-D$ | $-B$ | $-D$ | D | B | $-A$ | B | $-C$ | C | $-A$ | $-B$ | $-A$ |
| A | D | D | $-A$ | $-C$ | $-B$ | $-C$ | $-C$ | B | $-D$ | B | $-D$ | D | B | $-A$ | $-A$ | $-C$ | C | $-A$ | $-B$ |
| B | $-A$ | $-C$ | C | $-A$ | A | B | $-D$ | D | B | $-D$ | $-B$ | C | C | B | $-C$ | A | $-D$ | $-D$ | $-A$ |
| $-A$ | B | $-A$ | $-C$ | C | B | A | B | $-D$ | D | B | $-D$ | $-B$ | C | C | $-A$ | $-C$ | A | $-D$ | $-D$ |
| C | $-A$ | B | $-A$ | $-C$ | D | B | A | B | $-D$ | C | B | $-D$ | $-B$ | C | $-D$ | $-A$ | $-C$ | A | $-D$ |
| $-C$ | C | $-A$ | B | $-A$ | $-D$ | D | B | A | B | C | C | B | $-D$ | $-B$ | $-D$ | $-D$ | $-A$ | $-C$ | A |
| $-A$ | $-C$ | C | $-A$ | B | B | $-D$ | D | B | A | $-B$ | C | C | B | $-D$ | A | $-D$ | $-D$ | $-A$ | $-C$ |
| $-A$ | $-B$ | $-D$ | D | $-B$ | B | $-A$ | C | $-C$ | $-A$ | C | A | D | D | $-A$ | $-D$ | B | C | C | $-B$ |
| $-B$ | $-A$ | $-B$ | $-D$ | D | $-A$ | B | $-A$ | C | $-C$ | $-A$ | C | A | D | D | $-B$ | $-D$ | B | C | C |
| D | $-B$ | $-A$ | $-B$ | $-D$ | $-C$ | $-A$ | B | $-A$ | C | D | $-A$ | C | A | D | C | $-B$ | $-D$ | B | C |
| $-D$ | D | $-B$ | $-A$ | $-B$ | C | $-C$ | $-A$ | B | $-A$ | D | D | $-A$ | C | A | C | C | $-B$ | $-D$ | B |
| $-B$ | $-D$ | D | $-B$ | $-A$ | $-A$ | C | $-C$ | $-A$ | B | A | D | D | $-A$ | C | B | C | C | $-B$ | $-D$ |

6. The Ono-Sawade-Yamamoto array—the OD(36; 9, 9, 9, 9) constructed from 16 type one matrices is an M -structure and is given on the facing page.

a a a b c d b b d c b a a b c d b d c c a a b d c d b d c d a a a b c d d a a b c d b d c
a a a c d b d c b d a b a d b c c b d a c a d b c c b d a c a d b c c b d a a d c d b d c b
-b-d-c a a a b c d b d c b a a b c d b d c b a a c c d b d c b a a c c d b d c b a a d c d c b
-c-b-d a a a d b c c c b d a b a d b c c b d a c a a b c d b d c a c a a b c d c b d a d a d b c
-d-c-b a a a c d b d c b a a b c d b d c b d c b a a c c d b d c b a a b c d b a a d c d b
b c d b d c a a a b c d b d c b a a b c d b d c b a a d b c c d b d c c a a b c d b a a d a a
d b c c b d a a a d b c c b d a b a b c d b d c b d a c a a d b c c b d a c a a d b c c b d a d a
c d b d c b a a a c d b d c b a a b c d b d c b a a c c d b d c b d c b a a d c b d c b a a d
-b a a b c d b d c a a a b c d b d c d a a b c d b d c c a a b c d b d c c a a b c d b d c d b d c
-a b a d b c c b d a a a d b c c b d a a d a d b c c b d a d a d b c c b d a c a d b c c b d
a a b c d b d c b a a a c d b d c b a a a c d b d c b a a d c d b d c b a a c c d b d c b
-d-c-b a a a b c d b d c b a a b c d b d c b a a a b c d b d c b a a d c d b d c b a a d c d
-c-b-d a a a b c d b d c b a a b c d b d c b a a a b c d b d c b a a d c d b d c b a a d c d b
d c b a a b c d b d c b a a a c d b d c b a a a c d b d c b a a a c d b d c b a a c c d b
-b c d b d c b a a b c d b d c b a a b c d b d c b a a b c d b d c b a a b c d b d c b a a
-d b c c b d a a a d b c c b d a a a b c d b d c b d a a a b c d b d c b d a a b c d b d c b a a
c d b d c b a a a c d b d c b a a a c d b d c b a a a c d b d c b a a a c d b d c b a a a c
-c a a b c d b d c d a a b c d b d c a a a b c d b d c a a a b c d b d c b a a b c d b d c
-a c a d b c c b d a d a d b c c b d a a a d b c c b d a a a d b c c b d a b a d b c c b d
a a c c d b d c b a a d c d b d c b a a a c d b d c b a a a c d b d c b a a a b c d b d c b
b d c c a a b c d b d c d a a b c d b d c d b d c a a a b c d b d c a a b c d b d c b
-c b d a a c a d b c c b d a d a d b c d c b d a a a d b c d c b d a a a d b c d c b d a b c d
-d c b a a c c d b d c b a a a d c d b d c b a a a c d b d c b a a a c d b d c b a a d b c
-b c d b d c c a a b c d b d c b a a b c d b d c b a a b c d b d c b a a b c d b d c b a a
d b c c b d a c a d b c c b d a d a d b c c b d a d a d b c c b d a d a d b c c b d a b a
-c d b d c b a a a c d b d c b a a a c d b d c b a a a c d b d c b a a a c d b d c b a a a b

7. The Goethals-Seidel array [27] (see also J. Wallis-Whiteman [113]):

$$\begin{bmatrix} A & BR & CR & DR \\ -BR & A & -D^T R & C^T R \\ -CR & D^T R & A & -B^T R \\ -DR & -C^T R & B^T R & A \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^T R & -C^T R \\ -CR & -D^T R & A & B^T R \\ -DR & C^T R & -B^T R & A \end{bmatrix},$$

where A, B, C, D are circulant (type one) matrices satisfying (2) and R is the back diagonal (equivalent type two) $(0, 1)$ matrix.

Definition 3.5. Suitable matrices of order w for an $\text{OD}(n; s_1, s_2, \dots, s_u)$ are u pairwise amicable (i.e., pairwise satisfy (1)) matrices, $A_i, i = 1, \dots, u$, that have entries $+1$ or -1 and that satisfy

$$\sum_{i=1}^u s_i A_i A_i^T = (\sum s_i) w I_w. \quad (4)$$

They are used in the following theorem:

Theorem 3.5 (Geramita-Seberry). Suppose that there exists an $\text{OD}(\sum s_i; s_1, \dots, s_u)$ and u suitable matrices of order m . Then there is an Hadamard matrix of order $(\sum s_i)m$.

If we generalize the definition of suitable matrices so that entries $0, +1, -1$ are allowed, then *weighing matrices* rather than Hadamard matrices could be constructed.

An overview of matrices to “plug in” and “plug into” is given in Table 3.1.

The most prolific method for constructing matrices to “plug into” uses T -matrices or T -sequences:

Definition 3.6 (T -matrices). A set of 4 T -matrices, $T_i, i = 1, \dots, 4$ of order t are four circulant or type one matrices that have entries $0, +1$ or -1 and that satisfy

1. $T_i * T_j = 0, i \neq j$ ($*$ denotes the Hadamard product);
 2. $\sum_{i=1}^4 T_i$ is a $(1, -1)$ matrix;
 3. $\sum_{i=1}^4 T_i T_i^T = t I_t$; and for r/v
 4. $t = t_1^2 + t_2^2 + t_3^2 + t_4^2$, where t_i is the row [column] sum of T_i .
- (5)

T -matrices are known (see Cohen, Rubie, Koukouvinos, Kounias, Seberry, Yamada [10] for a recent survey) (71 occurs in [58]) for many orders including the following:

TABLE 3.1 The Relationship Between Matrices to “Plug in” and Matrices to “Plug into”

| | Matrices to “Plug in” | Matrices to “Plug into” |
|-----------------|---|--|
| Hardest to find | Williamson Williamson-type | OD(4t; t, t, t, t) |
| | 8-Williamson 8-Williamson-type | OD(8t; t, t, t, t, t, t, t, t) |
| Easiest to find | Suitable matrices | OD(2 ^t n; u ₁ , u ₂ , ..., u ₃) |
| | 4 circulant suitable matrices | Goethals-Seidel |
| | 4 type one suitable matrices | J. Wallis-Whiteman |
| | Near suitable | “Bordered arrays” |
| | Regular s-sets M-structures Kharaghani matrices | Latin squares |

1, ..., 72, 74, ..., 78, 80, ..., 82, 84, ..., 88, 90, ..., 96, 98, ..., 102, 104, ..., 106, 108, 110, ..., 112, 114, ..., 126, 128, ..., 130, 132, ..., 136, 138, 140, ..., 148, 150, 152, ..., 156, 158, ..., 162, 164, ..., 166, 168, ..., 172, 174, ..., 178, 180, 182, 184, ..., 190, 192, 194, ..., 196, 198, 200, ..., 210, ... T-matrices of order t give Hadamard matrices of order 4t.

Definition 3.7 (T-sequences). A set of four sequences $A = \{ \{a_{11}, \dots, a_{1n}\}, \{a_{21}, \dots, a_{2n}\}, \{a_{31}, \dots, a_{3n}\}, \{a_{41}, \dots, a_{4n}\} \}$ of length n, with entries 0, 1, -1 so that exactly one of $\{a_{1j}, a_{2j}, a_{3j}, a_{4j}\}$ is ± 1 (three are zero) for $j = 1, \dots, n$ and with zero nonperiodic autocorrelation function, that is, $N_A(j) = 0$ for $j = 1, \dots, n - 1$, where

$$N_A(j) = \sum_{i=1}^{n-j} (a_{1i}a_{1,i+j} + a_{2i}a_{2,i+j} + a_{3i}a_{3,i+j} + a_{4i}a_{4,i+j}),$$

are called T-sequences.

T-matrices are a slightly weaker structure than T-sequences, being defined on finite abelian groups rather than the infinite cyclic group. They are known for a few important small orders, for example, 61 and 67 [36, 75] for which no T-sequences are yet known. Sequences are discussed extensively in Section 5. They are also known for even orders t for which no T-sequences of length t are known [53].

The following result, in a slightly different form, was also discovered by R. J. Turyn. It is the single, most useful method for constructing OD(4n; n, n, n, n), that is, matrices to “plug into.”

Theorem 3.6 (Cooper–J. Wallis [12]). *Suppose there exist circulant T -matrices (T -sequences) $X_i, i = 1, \dots, 4$, of order n . Let a, b, c, d be commuting variables. Then*

$$\begin{aligned} A &= aX_1 + bX_2 + cX_3 + dX_4, \\ B &= -bX_1 + aX_2 + dX_3 - cX_4, \\ C &= -cX_1 - dX_2 + aX_3 + bX_4, \\ D &= -dX_1 + cX_2 - bX_3 + aX_4, \end{aligned}$$

can be used in the Goethal-Seidel (or J. Wallis-Whiteman) array to obtain an $\text{OD}(4n; n, n, n, n)$ and an Hadamard matrix of order $4n$.

Corollary 3.7. *If there are T -matrices of order t , then there is an $\text{OD}(4t; t, t, t, t)$.*

The results on T -matrices and T -sequences as applied to Hadamard matrices are given in Section 5.

The appropriate theorem for the construction of Hadamard matrices (it is implied by Williamson, Baumert-Hall, Welch, Cooper–J. Wallis, Turyn) is

Theorem 3.8. *Suppose that there exists an $\text{OD}(4t; t, t, t, t)$ and four suitable matrices A, B, C, D of order w that satisfy*

$$AA^T + BB^T + CC^T + DD^T = 4wI_w.$$

Then there is an Hadamard matrix of order $4wt$.

Williamson matrices (which are discussed further in a later section) are suitable matrices for $\text{OD}(4t; t, t, t, t)$, and as such, Williamson matrices are plugged into the OD.

Corollary 3.9. *If there are circulant T -matrices of order t and there are Williamson matrices of order w , there is an Hadamard matrix of order $4tw$. Alternatively, if there are an $\text{OD}(4t; t, t, t, t)$ and Williamson matrices of order w , there is an Hadamard matrix of order $4tw$.*

We modify a construction of Turyn to obtain the first theorem which capitalized on M -structures. The $\text{OD}(4s; u_1, \dots, u_n)$ of the next theorem is an M -structure of which the Welch and Ono-Sawade-Yamamoto arrays are powerful examples.

Theorem 3.10 (Seberry-Yamada-Turyn [87, 108]). *Suppose that there are T -matrices of order t . Further suppose that there is an $\text{OD}(4s; u_1, \dots, u_n)$ constructed of 16 circulant (or type one) $s \times s$ blocks on the variables x_1, \dots, x_n .*

Then there is an $OD(4st; tu_1, \dots, tu_n)$. In particular, if there is an $OD(4s; s, s, s, s)$ constructed of 16 circulant (or type one) $s \times s$ blocks, then there is an $OD(4st; st, st, st, st)$.

Proof. We write the OD as (N_{ij}) , $i, j = 1, 2, 3, 4$, where each N_{ij} is circulant (or type one). Hence, we are considering the OD purely as an M -structure. Since we have an OD,

$$N_{i1}N_{j1}^T + N_{i2}N_{j2}^T + N_{i3}N_{j3}^T + N_{i4}N_{j4}^T = \begin{cases} \sum_{k=1}^4 u_k x_k^2 I_s, & i = j; \\ 0, & i \neq j. \end{cases}$$

Suppose that the T -matrices are T_1, T_2, T_3, T_4 . Then form the matrices

$$\begin{aligned} A &= T_1 \times N_{11} + T_2 \times N_{21} + T_3 \times N_{31} + T_4 \times N_{41}, \\ B &= T_1 \times N_{12} + T_2 \times N_{22} + T_3 \times N_{32} + T_4 \times N_{42}, \\ C &= T_1 \times N_{13} + T_2 \times N_{23} + T_3 \times N_{33} + T_4 \times N_{43}, \\ D &= T_1 \times N_{14} + T_2 \times N_{24} + T_3 \times N_{34} + T_4 \times N_{44}. \end{aligned}$$

Now

$$AA^T + BB^T + CC^T + DD^T = t \sum_{k=1}^4 u_k x_k^2 I_{st},$$

and since A, B, C, D are type one, they can be used in the J. Wallis-Whiteman generalization of the Goethals-Seidel array to obtain the result. \square

Use the Welch and Ono-Sawade-Yamamoto arrays to see

Corollary 3.11. *Suppose that the T -matrices are of order t . Then there are orthogonal designs $OD(20t; 5t, 5t, 5t, 5t)$ and $OD(36t; 9t, 9t, 9t, 9t)$.*

Note that to prove the Hadamard conjecture "there is an Hadamard matrix of order $4t$ for all $t > 0$," it would be sufficient to prove:

Conjecture 3.12. *There exists an $OD(4t; t, t, t, t)$ for every positive integer t .*

The most encompassing theorem presently known, in that it gives a result for every odd q , is proved using a "plug in" technique:

Theorem 3.13 (Seberry [121]). *Let q be any odd natural number. Then there exists an integer $t \leq [2\log_2(q - 3)] + 1$ so that there is an Hadamard matrix of order $2^t q$. (The best known bounds are $t \leq [\log_2(q - 3)(q - 7) - 1]$ for q (prime) $\equiv 3 \pmod{4}$ and $t \leq [\log_2(q - 1)(q - 5)] + 1$ for p (prime) $\equiv 1 \pmod{4}$.)*

The proof of this theorem allows a number of cases of interest and stronger results in some cases where q is not prime.

Corollary 3.14 (Seberry [121]). *Let q be any odd natural number. Then there exists a regular symmetric Hadamard matrix with constant diagonal of order $2^{2t}q^2$, $t \leq [2\log_2(q-3)] + 1$.*

Corollary 3.15 (Seberry, unpublished).

1. *Let p and $p+2$ be twin prime powers. Then there exists a $t \leq [\log_2(p+3)(p-1)(p^2+2p-7)] - 2$ so that there is an Hadamard matrix of order $2^t p(p+2)$.*
2. *Let $p+1$ be the order of a symmetric Hadamard matrix. Then there exists a $t \leq [\log_2(p-3)(p-7)] - 2$ so that there is an Hadamard matrix of order $2^t p$.*

Corollary 3.16 [81]. *Let pq be an odd natural number. Suppose that all $OD(2^s p; 2^r a, 2^r b, 2^r c)$ exist, $s \geq s_0$, $2^{s-r}p = a + b + c$. Then there exists an Hadamard matrix of order $2^t \cdot p \cdot q$, $s \leq t \leq [2\log_2((q-3)/p)] + r + 1$. (The best-known bounds are $s \leq t \leq [\log_2((q-3)(q-7)/p)] - 1 + r$ for q (prime) $\equiv 3 \pmod{4}$ and $st \leq [\log_2((q-1)(q-5)/p)] + r + 1$ for q (prime) $\equiv 1 \pmod{4}$.)*

Example 3.1. Often we can find better results than indicated by Theorem 3.13. Let $q = 3 \cdot 491$. We know there is an Hadamard matrix of order 12. Now, using the proof of Theorem 3.13, rather than the enunciation, we can find an Hadamard matrix of order $2^{15} \cdot 491$. So there is an Hadamard matrix of order $2^{16} \cdot 3 \cdot 19$ using the multiplication theorem. On the other hand, the proof of the corollary gives an Hadamard matrix of order $2^{13} \cdot 3 \cdot 491$ using the $OD(2^{12} \cdot 3; 22, 3, 2^{12} \cdot 3 - 25)$.

Other similar results are known. The Appendix gives an indication of the smallest t for each odd natural number q for which an Hadamard matrix is known. A list of the construction methods used is given in Section A.3 of the Appendix.

Theorem 3.13 changes ideas for evaluating construction methods: We consider a method to be more powerful if it lowers the power of two for the resultant odd number. Thus, Agayan's theorem, which gives Hadamard matrices of order $8mn$ from Hadamard matrices of order $4m$ and $4n$, is more powerful than that of Hadamard, which gives a matrix of order $16mn$.

We now see another way to lower the power in a multiplication method. First, we introduce some notation.

Let $M = (M_{ij})$ and $N = (N_{gh})$ be orthogonal matrices or t^2 block M -structures of orders tm and tn , respectively, where M_{ij} is of order m ($i, j = 1, 2, \dots, t$) and N_{gh} is of order n ($g, h = 1, 2, \dots, t$).

We now define the operation \circ as the following:

$$M \circ N = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1t} \\ L_{21} & L_{22} & \cdots & L_{2t} \\ \vdots & & & \vdots \\ L_{t1} & L_{t2} & \cdots & L_{tt} \end{bmatrix},$$

where L_{ij} is of order of mn , and

$$L_{ij} = M_{i1} \times N_{1j} + M_{i2} \times N_{2j} + \cdots + M_{it} \times N_{tj},$$

$i, j = 1, 2, \dots, t$. We call this the *strong Kronecker* multiplication of two matrices. We note that the strong Kronecker product preserves orthogonality but not necessarily with entries in a useful form (i.e. equal to $0, \pm 1$).

Theorem 3.17. *Let A be an $OD(tm; p_1, \dots, p_u)$ with entries x_1, \dots, x_u , and let B be an $OD(tn; q_1, \dots, q_s)$ with entries y_1, \dots, y_s , then*

$$(A \circ B)(A \circ B)^T = \left(\sum_{j=1}^u p_j x_j^2 \right) \left(\sum_{j=1}^s q_j y_j^2 \right) I_{tmn}.$$

($A \circ B$ is not an orthogonal design but an orthogonal matrix.) If A is a $W(tm, p)$ and B is a weighing matrix $W(tn, q)$, then $A \circ B = C$ satisfies $CC^T = pqI_{tmn}$.

Hereafter, let $H = H_{ij}$ and $N = (N_{ij})$ of order $4h$ and $4n$, respectively, be 16 block M -structures. So

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix},$$

where

$$\sum_{j=1}^4 H_{ij} H_{ij}^T = 4hI_h = \sum_{j=1}^4 H_{ji} H_{ji}^T,$$

for $i = 1, 2, 3, 4$, and

$$\sum_{j=1}^4 H_{ij} H_{kj}^T = 0 = \sum_{j=1}^4 H_{ji}^T H_{jk},$$

for $i \neq k, i, k = 1, 2, 3, 4$, and similarly for N .

For ease of writing, we define $X_i = \frac{1}{2}(H_{i1} + H_{i2})$, $Y_i = \frac{1}{2}(H_{i1} - H_{i2})$, $Z_i = \frac{1}{2}(H_{i3} + H_{i4})$, and $W_i = \frac{1}{2}(H_{i3} - H_{i4})$, where $i = 1, 2, 3, 4$. Then both $X_i \pm Y_i$ and $Z_i \pm W_i$ are $(1, -1)$ matrices with $X_i \wedge Y_i = 0$ and $Z_i \wedge W_i = 0$, where \wedge is the Hadamard product.

Let

$$S = \begin{bmatrix} X_1 & -Y_1 & Z_1 & -W_1 \\ X_2 & -Y_2 & Z_2 & -W_2 \\ X_3 & -Y_3 & Z_3 & -W_3 \\ X_4 & -Y_4 & Z_4 & -W_4 \end{bmatrix}.$$

Obviously, S is a $(0, 1, -1)$ matrix.

Write

$$R = \begin{bmatrix} Y_1 & X_1 & W_1 & Z_1 \\ Y_2 & X_2 & W_2 & Z_2 \\ Y_3 & X_3 & W_3 & Z_3 \\ Y_4 & X_4 & W_4 & Z_4 \end{bmatrix},$$

also a $(0, 1, -1)$ matrix.

We note $S \pm R$ is a $(1, -1)$ matrix, $R \wedge S = 0$, and by the previous theorem,

$$SS^T = RR^T = 2hI_{4h}.$$

Lemma 3.18. *If there exists an Hadamard matrix of order $4h$, there exists an $OD(4h; 2h, 2h)$.*

Proof. Form S and R as above. Now $H = S + R$. Note that $HH^T = SS^T + RR^T + SR^T + RS^T = 4hI_{4h}$ and $SS^T = RR^T = 2hI_{4h}$. Hence, $SR^T + RS^T = 0$. Let x and y be commuting variables; then $E = xS + yR$ is the required orthogonal design. \square

In fact, exploiting the strong Kronecker product, Seberry and Zhang show

Lemma 3.19. *If there exist Hadamard matrices of order $4h$ and $4n$, there exists a $W(4hn, 2hn)$. If there exists an Hadamard matrix of order $4h$, there exists a $W(4h, 2h)$ ($h > 1$).*

Theorem 3.20. *Suppose that $4h$ and $4n$ are the orders of Hadamard matrices; then there exist two disjoint amicable $W(4hn, 2hn)$ whose sum and difference are $(1, -1)$ matrices. Suppose that there exists an Hadamard matrix of order $4h$; then there exists disjoint amicable $W(4h, 2h)$ whose sum and difference are $(1, -1)$ matrices.*

We now proceed to use the idea of *orthogonal pairs* or ± 1 matrices, S and P of order n , satisfying

1. $SS^T + PP^T = 2nI_n$,
2. $SP^T = PS^T = 0$,

first introduced by R. Craigen [13] who showed

Lemma 3.21 (Craigen). *If there exist Hadamard matrices of order $4p$ and $4q$, then there exist two $(1, -1)$ matrices, S and P of order $4pq$, satisfying*

1. $SS^T + PP^T = 8pqI_{4pq}$,
2. $SP^T = PS^T = 0$.

Proof. By Theorem 3.20, there exist two $W(4pq, 2pq)$, X and Y , satisfying $X \wedge Y = 0$; $X \pm Y$ is a $(1, -1)$ matrix, and $XY^T = YX^T$. Let $S = X + Y, P = X - Y$. Then both S and P are $(1, -1)$ matrices of order $4pq$. Note that

$$SS^T + PP^T = 2(XX^T + YY^T) = 8pqI_{4pq}$$

and

$$SP^T = XX^T - YY^T = 0.$$

Similarly, $PS^T = 0$. So S and P are the required matrices. □

These results can be combined to give

Theorem 3.22 (Craigen-Seberry-Zhang [14]). *If there exist Hadamard matrices of order $4m, 4n, 4p, 4q$, then there exists an Hadamard matrix of order $16mnpq$.*

Proof. Let U, V be amicable $W(4mn, 2mn)$ constructed in Theorem 3.20. By Lemma 3.21, there exist two $(1, -1)$ matrices S and P of order $4pq$ satisfying conditions 1 and 2 in Lemma 3.21.

Let $H = U \times S + V \times P$. Then H is a $(1, -1)$ matrix, and

$$\begin{aligned} HH^T &= UU^T \times SS^T + VV^T \times PP^T = 2mnI_{4mn}(SS^T + PP^T) \\ &= 2mnI_{4mn} \times 8pqI_{4pq} = 16mnpqI_{16mnpq}. \end{aligned}$$

Thus H is the required Hadamard matrix. □

The theorem gives an improvement and extension for the result of Agayan [1] that if there exist Hadamard matrices of order $4m$ and $4n$, then there exists an Hadamard matrix of order $8mn$, since using Agayan's theorem repeatedly on four Hadamard matrices of order $4m, 4n, 4p, 4q$ gives an Hadamard matrix of order $32mnpq$.

$$\begin{array}{cccc}
 \begin{bmatrix} x & y \\ y & -x \end{bmatrix} & \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} & \begin{bmatrix} a & b & b & d \\ -b & a & d & -b \\ -b & -d & a & b \\ -d & b & -b & a \end{bmatrix} & \begin{bmatrix} a & 0 & -c & 0 \\ 0 & a & 0 & c \\ c & 0 & a & 0 \\ 0 & -c & 0 & a \end{bmatrix} \\
 \text{(a)} & \text{(b)} & \text{(c)} & \text{(d)} \\
 \text{OD}(2; 1, 1); & \text{OD}(4; 1, 1, 1, 1); & \text{OD}(4; 1, 1, 2); & \text{OD}(4; 1, 1).
 \end{array}$$

Figure 4.1. Orthogonal designs.

Other similar results exist.

4 ORTHOGONAL DESIGNS AND ASYMPTOTIC EXISTENCE

The primary result regarding the asymptotic existence of Hadamard matrices is the theorem of Seberry Wallis (Theorem 4.11 of this section). In this section we outline the proof of this theorem. We begin this section with a discussion of orthogonal designs. These are key ingredients in the proof of the main theorem.

4.1. Orthogonal Designs

An orthogonal design is a generalization of an Hadamard matrix (see Definition 3.8). First we collect a few preliminary results and give some examples.

Example 4.1. Some small orthogonal designs are shown in Figure 4.1. Notice that Figure 4.1(b) is the Williamson array.

The following lemma gives some properties of orthogonal designs.

Lemma 4.1. *Let D be an orthogonal design $\text{OD}(n; u_1, u_2, \dots, u_t)$ on the commuting variables x_1, x_2, \dots, x_t . Then D can be written as*

$$D = x_1 A_1 + x_2 A_2 + \dots + x_t A_t,$$

where, for each $i, j \in \{1, \dots, t\}$,

1. A_i is an $n \times n$ matrix with entries $0, \pm 1$;
2. $A_i A_i^T = u_i I_n$;
3. $A_i A_j^T + A_j A_i^T = 0, i \neq j$.

We need one further basic result:

Lemma 4.2. *Let D be an orthogonal design $\text{OD}(n; u_1, u_2, \dots, u_t)$, on the t commuting variables x_1, x_2, \dots, x_t . Then the following orthogonal designs exist:*

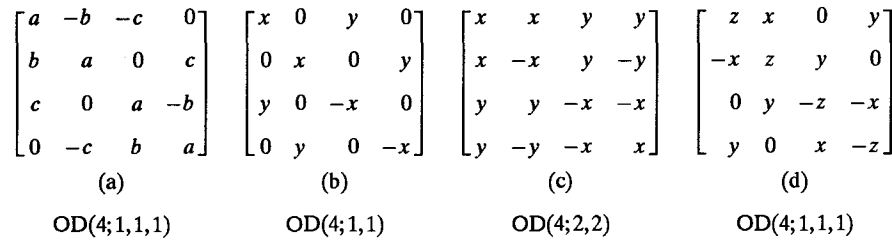


Figure 4.2. Orthogonal designs.

1. OD($n; u_1, u_2, \dots, u_i + u_j, \dots, u_t$) on $t - 1$ variables (i.e., $u_i + u_j$ replaces $u_i, u_j, i \neq j$);
2. OD($n; u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_t$) on $t - 1$ variables;
3. OD($2n; u_1, u_2, \dots, u_t$) on t variables;
4. OD($2n; 2u_1, 2u_2, \dots, 2u_t$) on t variables;
5. OD($2n; u_1, u_1, u_2, \dots, u_t$) on $t + 1$ variables;
6. OD($2n; u_1, u_1, 2u_2, \dots, 2u_t$) on $t + 1$ variables.

The techniques of this lemma are exhibited in the following example:

Example 4.2. Let D_1 and D_2 be the designs of Figure 4.2(b) and (a), respectively. Applying Lemma 4.2 to these designs gives examples as follows: D_1 is an OD(4;1,1,1,1); letting $b = c$ as in case 1 of Lemma 4.2 gives the OD(4;1,1,2) design in Figure 4.2(c); letting $d = 0$ as in case 2 gives the OD(4;1,1,1) design in Figure 4.2(a). D_2 is a (2;1,1) design; replacing variables by 2×2 matrices as in cases 3, 4, and 5 gives the designs OD(4;1,1), OD(4;2,2), OD(4;1,1,1), in Figure 4.2(b), (c), and (d), respectively.

Lemma 4.2 now lets us show

Lemma 4.3. Suppose that for all choices of nonnegative integers a, b, c with $a + b + c = n$, an orthogonal design OD($n; a, b, c$) exists. Then for all choices of nonnegative integers x, y, z with $x + y + z = 2n$, an orthogonal design OD($2n; x, y, z$) exists.

Proof. Notice first that we make the convention that an OD($n; a, b$) may also be considered as an OD($n; a, b, 0$), and so on.

Let x, y, z be nonnegative integers such that $x + y + z = 2n$, and assume that $0 \leq x \leq y \leq z$, so that $y \leq n$. Four cases arise:

1. Both x and y are even, so we may write $x = 2a, y = 2b$, and $a + b < n$. By hypothesis, an OD($n; a, b, c$) exists, where $c = n - a - b$. Hence, by case 6 of Lemma 4.2, an OD($2n; a, a, 2b, 2c$) exists and, by case 1, an OD($2n; 2a, 2b, 2c$) also exists. This is the design we want.

2. Next, let x be even and y odd, so we may take $x = 2a, y = 2a + 1$. Now $a + y = 3a + 1$, and $z = 2n - 4a - 1$. Since $y \leq z$, we have $3a + 1 \leq n$. Thus, an $\text{OD}(n; y, a, n - a - y)$ exists, and as before, this means that an $\text{OD}(2n; y, y, 2a, 2n - 2a - 2y)$ also exists. Setting $x_1 = x_4$, we get an $\text{OD}(2n; y, 2a, 2n - 2a - y)$. Since $2a = x$ and $2n - 2a - y = z$, the last design is the required one.
3. If x is odd and y is even, we can take $x = 2a + 1, y = 2b$ and $z = 2t + 1$. Since $x + y + z = 2n$, we have $a + b + t + 1 = n$. Now, by assumption, $a < t$, so $x + b = 2a + b + 1 < n$. Hence, we have the following orthogonal designs: $\text{OD}(n; x, b, n - x - b)$, $\text{OD}(2n; x, x, 2b, 2n - 2x - 2b)$, and $\text{OD}(2n; x, 2b, 2n - x - 2b)$. Since $y = 2b$ and $z = 2n - x - y$, we have the required design.
4. Finally, if x and y are both odd, we let $y = x + 2b$, where $b \geq 0$. Since $x + b \leq n$, we have orthogonal designs

$$\text{OD}(n; x, b, n - x - b), \quad \text{OD}(2n; x, x, 2b, 2n - 2x - 2b),$$

and finally, $\text{OD}(2n; x, x + 2b, 2n - 2x - 2b)$, as required. \square

Corollary 4.4. *If x, y, z are nonnegative integers such that $x + y + z = 2^m$, then an orthogonal design $\text{OD}(2^m; x, y, z)$ exists.*

Proof. From the the array in Figure 4.1(a) and Lemma 4.2, the statement is true for $m = 2$. It then follows from Lemma 4.3 for all $m > 2$. \square

Corollary 4.5. *If x, y are nonnegative integers such that $x + y = 2^m$, then an orthogonal design $\text{OD}(2^m; x, y)$ exists.*

Proof. Apply case 1 of Lemma 4.2 to the $\text{OD}(2^m; x, y, z)$ obtained from the previous corollary. \square

4.2. An Existence Theorem for Hadamard Designs

We need one further result from number theory.

Theorem 4.6. *Let x and y be positive integers such that $(x, y) = 1$. Then every integer $N \geq (x - 1)(y - 1)$ can be written as a linear combination $N = ax + by$, where a and b are nonnegative integers.*

Corollary 4.7. *Let z be an odd integer. Then there exist nonnegative integers a and b such that*

$$a(z + 1) + b(z - 3) = n = 2^t$$

for some t .

Proof. If $z \geq 9$, let

$$d = (z + 1, z - 3) = \begin{cases} 2 & \text{if } z \equiv 1 \pmod{4}, \\ 4 & \text{if } z \equiv 3 \pmod{4}. \end{cases}$$

Let

$$N = \left(\frac{z+1}{d} - 1 \right) \left(\frac{z-3}{d} - 1 \right),$$

and choose m so that $2^{m-1} < N \leq 2^m$. By Theorem 4.6 there exist nonnegative integers a and b such that

$$\frac{a(z+1)}{d} + \frac{b(z-3)}{d} = 2^m,$$

and thus

$$a(z+1) + b(z-3) = 2^{m+s},$$

where

$$s = \begin{cases} 1 & \text{if } z \equiv 1 \pmod{4}, \\ 2 & \text{if } z \equiv 3 \pmod{4}, \end{cases}$$

and $t = m + s$. It is easy to verify that this result also holds for odd $3 \leq z \leq 9$. □

Lemma 4.8. *Let p be a prime, $p \geq 11$. Then there exists a positive integer t such that an Hadamard matrix of size $2^s p$ exists for every $s > t$.*

Proof. Let $x = p + 1$ and $y = p - 3$. By Corollary 4.7 there exist nonnegative integers a and b such that $ax + by = 2^t = n$ for some t . By Corollary 4.4 there exists an $OD(n; a, b, n - a - b)$ orthogonal design D on the variables x_1, x_2, x_3 .

The proof now divides into two cases.

Case 1 $p \equiv 3 \pmod{4}$. We replace each variable in D by a $p \times p$ $(1, -1)$ matrix: x_1 by J_p , x_2 by $J_p - 2I_p$, and x_3 by the back-circulant matrix N formed from the quadratic residues. This gives a $(1, -1)$ matrix E which is an Hadamard matrix of size $np = 2^t p$, and the Lemma follows for $p \equiv 3 \pmod{4}$.

Case 2 $p \equiv 1 \pmod{4}$. There exists an $OD(2n; 2a, 2b, n - a - b, n - a - b)$ orthogonal design F on the variables x_1, x_2, x_3, x_4 by identity 4 of Lemma 4.2. We replace each variable in F by a $p \times p$ $(1, -1)$ matrix: x_1 by J_p , x_2 by $J_p - 2I_p$, x_3 , and x_4 , respectively, by the circulant matrices $X = Q + I$ and $Y =$

$Q - I$ formed from the quadratic residue matrix Q . This gives an $np \times np$ $(1, -1)$ matrix G which is an Hadamard matrix of size $2np = 2^{t+1}p$, and the lemma also follows for $p \equiv 1 \pmod{4}$.

This completes the proof for all primes, except 2, 3, 5, and 7. \square

Lemma 4.9. *There exist Hadamard matrices of sizes 2^t for all $t \geq 1$, and $2^t p$ for all $t \geq 2$ and $p = 3, 5, 7$.*

Proof. There exists an Hadamard matrix of size 2^t for $t \geq 1$.

By Sylvester's multiplication theorem, if there exist Hadamard matrices of sizes 12, 20, and 28, then there exist Hadamard matrices of sizes $2^t p$ for all $t \geq 2$ and $p = 3, 5, 7$.

Hadamard matrices of these orders are obtained by the Paley construction. \square

Theorem 4.10. *Let q be any positive integer. Then there exists $t = t(q)$ such that an Hadamard matrix of size $2^s q$ exists for every $s \geq t$.*

Proof. We apply Lemma 4.8 and/or Lemma 4.9 to each prime factor of q . Since a Kronecker product of Hadamard matrices is an Hadamard matrix, the result follows. \square

Theorem 4.11 (Seberry Wallis [121]). *Let q be any positive integer, then there exists an Hadamard matrix of order $2^s q$ for every $s \geq [2\log_2(q-3)]$.*

Proof. By the proof of Corollary 4.7, we can choose t so that

$$2^t \geq \left(\frac{z+1}{d} - 1\right) \left(\frac{z-3}{d} - 1\right),$$

where z is an odd prime and $d = (z+1, z-3)$.

If $z \equiv 1 \pmod{4}$, then $d = 2$ and we must have

$$2^t \geq \frac{(z-1)(z-5)}{4}.$$

Since

$$(z-3)^2 > (z-1)(z-5),$$

it is sufficient to ensure that

$$2^{t+2} > (z-3)^2;$$

that is,

$$t + 2 > 2\log_2(z - 3).$$

Since t is an integer, we may choose

$$t = [2\log_2(z - 3)] - 1.$$

Similarly, if $z \equiv 3 \pmod{4}$, then $d = 4$, and we may choose

$$t = [2\log_2(z - 5)] - 3.$$

As in the proof of Lemma 4.8, these choices of t ensure the existence of an Hadamard matrix of size $2^t z$.

If $z = pq$ where p and q are primes, $p \equiv 1 \pmod{4}$, $q \equiv 1 \pmod{4}$, then there exists an Hadamard matrix of size $2^r pq$, where

$$r = [2\log_2(p - 3)] + [2\log_2(q - 3)] < [2\log_2(pq - 3)].$$

Analogously, if $z = \prod_i p_i$ for p_i prime and $p_i \equiv 1 \pmod{4}$, then

$$r = \sum_i 2\log_2(p_i - 3) < 2\log_2\left(\prod_i (p_i - 3)\right)$$

Since an integer z that is a product of primes congruent to 1 (mod 4) gives the greatest lower bound on the value of t for which we know an Hadamard matrix of size $2^t z$ exists, we have proved the theorem. \square

We note that better bounds (i.e., smaller r) can be obtained if not all primes in the decomposition of z are congruent to 1 (mod 4). We use the equivalence of Hadamard matrices and Hadamard designs to obtain the following corollary:

Corollary 4.12. *Let λ be any positive integer; then there exists an $s \geq 0$ so that an SBIBD($2^{s+2}\lambda - 1, 2^{s+1}\lambda - 1, 2^s\lambda - 1$) exists.*

In fact, as was indicated in Theorem 3.13, the value of s in Theorem 4.11 is slightly smaller if the proof is applied carefully.

4.3. Orthogonal Designs in Order 24

In this section, we discuss the particular case of orthogonal designs of order 24. In so doing, we demonstrate how the power of s in Theorem 4.11 can be reduced in specific cases.

The following is an OD(12; 1, 2, 3, 6) on the variables A, B, C, D :

| | | | | | | | | | | | |
|------|------|------|------|------|------|------|------|------|------|------|------|
| A | B | $-B$ | C | B | B | C | $-B$ | D | B | D | $-C$ |
| $-B$ | A | B | B | B | C | $-B$ | D | C | D | $-C$ | B |
| B | $-B$ | A | B | C | B | D | C | $-B$ | $-C$ | B | D |
| $-C$ | $-B$ | $-B$ | A | B | $-B$ | $-B$ | C | $-D$ | C | D | $-B$ |
| $-B$ | $-B$ | $-C$ | $-B$ | A | B | C | $-D$ | $-B$ | D | $-B$ | C |
| $-B$ | $-C$ | $-B$ | B | $-B$ | A | $-D$ | $-B$ | C | $-B$ | C | D |
| $-C$ | B | $-D$ | B | $-C$ | D | A | B | $-B$ | $-C$ | $-B$ | $-B$ |
| B | $-D$ | $-C$ | $-C$ | D | B | $-B$ | A | B | $-B$ | $-B$ | $-C$ |
| $-D$ | $-C$ | B | D | B | $-C$ | B | $-B$ | A | $-B$ | $-C$ | $-B$ |
| $-B$ | $-D$ | C | $-C$ | $-D$ | B | C | B | B | A | B | $-B$ |
| $-D$ | C | $-B$ | $-D$ | B | $-C$ | B | B | C | $-B$ | A | B |
| C | $-B$ | $-D$ | B | $-C$ | $-D$ | B | C | B | B | $-B$ | A |

Hence, there exists (equating variables) an OD(12; 4, 8).

Now, by identity 6 of Lemma 4.2, there are OD(24; 2, 4, 3, 3, 12), OD(24; 4, 4, 16), OD(24; 8, 8, 8), and OD(24; 1, 1, 4, 6, 12), giving

OD(24; 2, 4, 18);

OD(24; 3, a , $21 - a$), $a = 3, 4, 5, 6, 7$;

OD(24; 4, a , $20 - a$), $a = 4, 5, 6, 7, 8$;

OD(24; 8, 8, 8).

Robinson [72] has found OD(24; 1, 1, 1, 1, 1, 5, 5, 9) and OD(24; 1, 1, 1, 1, 1, 2, 8, 9) from which, by equating variables, all other OD(24; $x, y, 24 - x - y$) may be obtained.

Consider the following matrices, M_1 and M_2 : (we use the convention that $\bar{x} = -x$):

$$M_1 = \begin{array}{c} \begin{array}{|c|c|c|c|c|c|} \hline e & d\bar{h}f\bar{g} & gfhh & f\bar{g}hh & \bar{g}f\bar{h}h & gfhh \\ \hline \bar{d}h\bar{f}\bar{g} & \bar{e} & fghh & \bar{g}f\bar{h}h & \bar{g}f\bar{h}h & gfhh \\ \hline \bar{g}f\bar{h}h & \bar{f}\bar{g}hh & g & dhef & \bar{h}hgg & hh\bar{f}f \\ \hline fghh & gfhh & d\bar{h}\bar{e}f & \bar{g} & h\bar{h}ff & hhg\bar{g} \\ \hline gfhh & gfhh & h\bar{h}\bar{g}g & h\bar{h}ff & f & dhge \\ \hline \bar{g}f\bar{h}h & \bar{g}f\bar{h}h & h\bar{h}ff & h\bar{h}\bar{g}\bar{g} & d\bar{h}\bar{g}e & \bar{f} \\ \hline \end{array} \end{array}$$

$$M_2 = \begin{array}{c|c|c|c|c|c} e & \overline{dfhf} & hhgg & h\overline{hg}g & \overline{hg}h\overline{g} & hghg \\ \hline \overline{dfhf} & \overline{e} & hhgg & \overline{hhg}g & h\overline{g}h\overline{g} & h\overline{g}h\overline{g} \\ \hline h\overline{hg}g & h\overline{hg}g & g & dgeh & \overline{g}g\overline{hh} & hh\overline{ff} \\ \hline h\overline{hg}g & h\overline{hg}g & \overline{d}g\overline{eh} & \overline{g} & hh\overline{ff} & gg\overline{hh} \\ \hline hgh\overline{g} & \overline{hg}hg & g\overline{g}h\overline{h} & \overline{hhff} & g & dghe \\ \hline \overline{hg}h\overline{g} & \overline{hg}hg & \overline{hhff} & \overline{gg}h\overline{h} & \overline{dghe} & \overline{g} \end{array}$$

Let N_1 and N_2 be the matrices obtained from M_1 and M_2 by replacing the diagonal entries, y , of M_i by

$$\begin{array}{cccc} a & b & c & y \\ \overline{b} & a & y & \overline{c} \\ \overline{c} & \overline{y} & a & b \\ \overline{y} & c & \overline{b} & a \end{array}$$

and the off-diagonal block entries p, q, r, s of M_i by

$$\begin{array}{cccc} p & q & r & s \\ q & \overline{p} & s & \overline{r} \\ r & s & \overline{p} & q \\ s & \overline{r} & q & p. \end{array}$$

Then N_1 and N_2 give orthogonal designs of order 24 and types $(1, 1, 1, 1, 1, 5, 5, 9)$ and $(1, 1, 1, 1, 1, 2, 8, 9)$, respectively.

Hence, we have

Lemma 4.13 (P. Robinson [72]). *All three-tuples (x, y, z) , $x + y + z = 24$, are the types of orthogonal designs in order 24. That is, all $OD(24; x, y, 24 - x - y)$ exist.*

Proceeding as in Theorem 4.10 we obtain

Theorem 4.14. *Let q be a positive integer. Then there exists a $t = t(q)$ so that there is an Hadamard matrix of order $2^s \cdot 3 \cdot q$ for all $s \geq t$.*

Remark. A few other results of the kind in this section are known for orders $4 \cdot p \cdot q$ and $3 < p \leq 11$. The importance of this result lies in the fact that the power s will be smaller than the power t obtained from Theorem 3.13 (see [81]).

5 SEQUENCES

A special orthogonal design, the $OD(4t; t, t, t, t)$, is especially useful in constructing Hadamard matrices. An $OD(12; 3, 3, 3, 3)$ was first found by Baumert-Hall [6] and an $OD(20; 5, 5, 5, 5)$ by Welch. These were given in Section 3. $OD(4t; t, t, t, t)$ are sometimes called *Baumert-Hall arrays*. This chapter concentrates on the powerful construction techniques for these $OD(4t; t, t, t, t)$ using disjoint orthogonal matrices and sequences with zero autocorrelation.

Since we are concerned with orthogonal designs, we will consider sequences of commuting variables. Let $X = \{\{a_{11}, \dots, a_{1n}\}, \{a_{21}, \dots, a_{2n}\} \dots \{a_{m1}, \dots, a_{mn}\}\}$ be m sequences of commuting variables of length n . The *nonperiodic autocorrelation function of the family of sequences X* (denoted N_X) is a function defined by

$$N_X(j) = \sum_{i=1}^{n-j} (a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \dots + a_{m,i}a_{m,i+j}).$$

Early work of Golay [28, 29] was concerned with two $(1, -1)$ sequences with zero nonperiodic autocorrelation function, but Welti [123], Tseng [101], and Tseng and Liu [102] approached the subject from the point of view of two orthonormal vectors, each corresponding to one of two orthogonal waveforms. Later work, including Turyn's [108, 107], used four or more sequences.

Note that if the following collection of m matrices of order n is formed,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{11} & & a_{1,n-1} \\ & & \ddots & \\ 0 & & & a_{11} \end{bmatrix}, \begin{bmatrix} a_{21} & a_{22} & \dots & a_{2n} \\ & a_{21} & & a_{2,n-1} \\ & & \ddots & \\ 0 & & & a_{21} \end{bmatrix}, \dots, \\ \begin{bmatrix} a_{m1} & a_{m2} & \dots & a_{mn} \\ & a_{m1} & & a_{m,n-1} \\ & & \ddots & \\ 0 & & & a_{m1} \end{bmatrix},$$

then $N_X(j)$ is simply the sum of the inner products of rows 1 and $j+1$ of these matrices.

The *periodic autocorrelation function of the family of sequences X* (denoted P_X) is a function defined by

$$P_X(j) = \sum_{i=1}^n (a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \dots + a_{m,i}a_{m,i+j}),$$

where we assume the second subscript is actually chosen from the complete set of residues (mod n).

We can interpret the function P_X in the following way: Form the m circulant matrices that have first rows, respectively,

$$[a_{11}a_{12}\dots a_{1n}], [a_{21}a_{22}\dots a_{2n}], \dots, [a_{m1}a_{m2}\dots a_{mn}];$$

then $P_X(j)$ is the sum of the inner products of rows 1 and $j+1$ of these matrices. In these matrices, all a_{ij} are chosen from the set $\{0, 1, -1\}$.

We say the *weight* of a set of sequences X is the number of nonzero entries in X . If X is as above with $N_X(j) = 0$, $j = 1, 2, \dots, n-1$, then we will call X *m-complementary sequences* of length n . If

$$X = \{A_1, A_2, \dots, A_m\}$$

are m -complementary sequences of length n and weight $2k$ such that

$$Y = \left\{ \frac{(A_1 + A_2)}{2}, \frac{(A_1 - A_2)}{2}, \dots, \frac{(A_{2i-1} + A_{2i})}{2}, \frac{(A_{2i-1} - A_{2i})}{2}, \dots \right\}$$

are also m -complementary sequences (of weight k), then X will be said to be *m-complementary disjointable sequences* of length n . X will be said to be *m-complementary disjoint sequences* of length n if all $\binom{m}{2}$ pairs of sequences are disjoint.

For example $\{1\ 1\ 0\ 1\}$, $\{0\ 0\ 1\ 0\ -1\}$, $\{0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ -1\}$, $\{0\ 0\ 0\ 0\ 0\ 0\ 1\ -1\}$ are disjoint as they have zero nonperiodic autocorrelation function and precisely one $a_{ij} \neq 0$ for each j .

One more piece of notation is in order. If g_r denotes a sequence of integers of length r , then by xg_r we mean the sequence of integers of length r obtained from g_r by multiplying each member of g_r by x .

Proposition 5.1. *Let X be a family of m sequences of commuting variables. Then*

$$P_X(j) = N_X(j) + N_X(n-j), \quad j = 1, \dots, n-1.$$

Corollary 5.2. *If $N_X(j) = 0$ for all $j = 1, \dots, n-1$, then $P_X(j) = 0$ for all $j = 1, \dots, n-1$.*

Note: $P_X(j)$ may equal 0 for all $j = 1, \dots, n-1$, even though the $N_X(j)$ do not.

If $X = \{\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\}\}$ are two sequences where $a_i, b_j \in \{1, -1\}$ and $N_X(j) = 0$ for $j = 1, \dots, n-1$, then the sequences in X are called *Golay complementary sequences of length n* . For example, writing $-$ for minus 1, we

have

$$\begin{array}{ll}
 n = 2 & 11 \text{ and } 1- \\
 n = 10 & 1--1-1---1 \text{ and } 1-----11- \\
 n = 26 & 111--111-1-----1-11--1----- \text{ and} \\
 & ---11---1-11-1-1-11--1-----
 \end{array}$$

We note that if X is as above, if A is the circulant matrix with first row $\{a_1, \dots, a_n\}$, and, if B the circulant matrix with first row $\{b_1, \dots, b_n\}$, then

$$AA^T + BB^T = \sum_{i=1}^n (a_i^2 + b_i^2) I_n = 2nI_n.$$

Consequently, such matrices may be used to obtain Hadamard matrices constructed from two circulants.

We would like to use Golay sequences to construct other orthogonal designs, but first we consider some of their properties.

Lemma 5.3. *Let $X = \{\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\}\}$ be Golay complementary sequences of length n . Suppose that k_1 of the a_i are positive and k_2 of the b_i are positive. Then*

$$n = (k_1 + k_2 - n)^2 + (k_1 - k_2)^2,$$

and n is even.

Proof. Since $P_X(j) = 0$ for all j , we may consider the two sequences as $2 - \{n; k_1, k_2; \lambda\}$ supplementary difference sets with $\lambda = k_1 + k_2 - \frac{1}{2}n$. But the parameters (counting differences two ways) satisfy $\lambda(n-1) = k_1(k_1-1) + k_2(k_2-1)$. On substituting λ in this equation we obtain the result of the enunciation. \square

Geramita and Seberry [23, pp. 133–137], Andres [2] and James [38] have studied the smaller values of n, k_1, k_2 of the lemma, showing the only lengths ≤ 68 for which Golay sequences exist are 2, 4, 8, 10, 16, 20, 26, 32, 40, 52, and 64. Malcolm Griffin [30] has shown no Golay sequences can exist for lengths $n = 2 \cdot 9^t$. The value $n = 18$, which was previously excluded by a complete search, is now theoretically excluded by Griffin's theorem and independently by a result of Kruskal [62] and C. H. Yang [133, 134]. Andres [2] and James [38] have found greatly improved computer algorithms for studying these sequences.

Recent theoretical work of Koukouvinos, Kounias, and Sotirakoglou [50] and Eliahou, Kervaire, and Saffari [20] shows that Golay sequences do not exist for $n = 2p$ where p has any prime factor $\equiv 3 \pmod{4}$. This means the unresolved cases < 200 are $n = 74, 82, 106, 116, 122, 130, 136, 146, 148, 164, 170, 178, 194$.

Constraints can be found on the elements of a Golay sequence. One useful result (see Geramita and Seberry [23]) is

Lemma 5.4. For Golay sequences $X = \{x_i\}, \{y_i\}$ of length n ,

$$x_{n-i+1} = e_i x_i \Leftrightarrow y_{n-i+1} = -e_i y_i,$$

where $e_i = \pm 1$. That is,

$$x_{n-i+1} x_i = -y_{n-i+1} y_i.$$

Example 5.1. The sequences of length 10 are

$$1 - -1 - 1 - - - 1 \text{ and}$$

$$1 - - - - - 11 - .$$

Clearly, $e_1 = 1, e_2 = 1, e_3 = 1, e_4 = -1$, and $e_5 = -1$.

Proof (of Lemma 5.4). We use the fact that if x, y, z are ± 1 , $(x + y)z \equiv x + y \pmod{4}$ and $x + y \equiv xy + 1 \pmod{4}$.

Let $i = 1$. Clearly, the result holds. We proceed by induction. Suppose that the result is true for every $i \leq k - 1$. Now consider $N(k) = N(n - k) = 0$, and we have

$$\begin{aligned} 0 &= x_1 x_{n+1-k} + x_2 x_{n+2-k} + \cdots + x_k x_n + y_1 y_{n+1-k} + y_2 y_{n+2-k} + \cdots + y_k y_n \\ &= x_1 e_k x_k + x_2 e_{k-1} x_{k-1} + \cdots + x_k e_1 x_1 + y_1 y_{n+1-k} - y_2 e_{k-1} y_{k-1} \\ &\quad - \cdots - y_k e_1 y_1 \\ &\equiv e_1 + e_2 + \cdots + e_k + y_1 y_{n+1-k} - e_{k-1} - \cdots - e_2 - y_k e_1 y_1 \pmod{4} \\ &\equiv e_1 + e_k + y_1 y_{n+1-k} - y_k e_1 y_1 \pmod{4} \\ &\equiv e_k + y_k y_{n+1-k} \pmod{4} \\ &\equiv 0 \pmod{4}. \end{aligned}$$

So $y_{n+1-k} = -e_k y_k$. □

5.1. Summary of Golay Properties

Two sequences $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ are called *Golay complementary sequences* of length n if all their entries are ± 1 and

$$\sum_{i=1}^{n-j} (x_i x_{i+j} + y_i y_{i+j}) = 0 \quad \text{for every } j \neq 0, \quad j = 1, \dots, n-1,$$

that is, $N_X = 0$. These sequences have the following properties:

1. $\sum_{i=1}^n (x_i x_{i+j} + y_i y_{i+j}) = 0$ for every $j \neq 0$, $j = 1, \dots, n-1$ (where the subscripts are reduced modulo n), i.e., $P_X = 0$.
2. n is even and the sum of two squares.
3. $x_{n-i+1} = e_i x_i \Leftrightarrow y_{n-i+1} = -e_i y_i$, where $e_i = \pm 1$.
- 4.

$$\left[\sum_{i \in S} x_i \operatorname{Re}(\zeta^{2i+1}) \right]^2 + \left[\sum_{i \in D} x_i \operatorname{Im}(\zeta^{2i+1}) \right]^2 + \left[\sum_{i \in S} y_i \operatorname{Im}(\zeta^{2i+1}) \right]^2 + \left[\sum_{i \in D} y_i \operatorname{Re}(\zeta^{2i+1}) \right]^2 = \frac{1}{2}n,$$

where $S = \{i : 0 \leq i < n, e_i = 1\}$, $D = \{i : 0 \leq i < n, e_i = -1\}$, and ζ is a $2n$ th root of unity (Griffin [30]).

5. They exist for orders $2^a 10^b 26^c$, a, b, c nonnegative integers.
6. They do not exist for orders $2 \cdot 9^c$ (c a positive integer) (Griffin [30]), or for orders 34, 36, 50, 58, or 68.
7. They do not exist for orders $2 \cdot 49^c$ (c a positive integer) (Koukouvinos, Kounias, and Sotirakoglou [50]).
8. They do not exist for orders $2p$ where p has any prime factor $\equiv 3 \pmod{4}$ (Eliahou, Kervaire, and Saffari [20]).

We now discuss other sequences with zero autocorrelation function.

5.2. Other Sequences with Zero Autocorrelation Function

Lemma 5.5. *Suppose that $X = \{X_1, X_2, \dots, X_m\}$ is a set of $(0, 1, -1)$ sequences of length n for which $N_X = 0$ or $P_X = 0$. Further suppose that the weight of X_i is x_i and the sum of the elements of X_i is a_i . Then*

$$\sum_{i=1}^m a_i^2 = \sum_{i=1}^m x_i.$$

Proof. Form circulant matrices Y_i for each X_i . Then

$$Y_i J = a_i J \quad \text{and} \quad \sum_{i=1}^m Y_i Y_i^T = \sum_{i=1}^m x_i I.$$

Now considering

$$\sum_{i=1}^m Y_i Y_i^T J = \sum_{i=1}^m a_i^2 J = \sum_{i=1}^m x_i J,$$

we have the result. \square

Example 5.2. Suppose that X_1, X_2, X_3, X_4 have elements from +1 and -1 and lengths 19, 19, 18, 18. The total weight of these sequences is 74. The sum of the squares of the four row sums must be 74, so we could have

$$\begin{array}{ll} 3^2 + 1^2 + 8^2 + 0^2 & 1^2 + 1^2 + 6^2 + 6^2 \\ 7^2 + 5^2 + 0^2 + 0^2 & \text{or} \\ 7^2 + 3^2 + 4^2 + 0^2 & 5^2 + 3^2 + 6^2 + 2^2 \end{array}$$

A row sum of 8 and length 18 would require that there are 13 elements +1 and 5 elements -1 considerably shortening any search.

Now a few simple observations are in order. For convenience, we put them together as a lemma—though more has been observed by Whitehead [124].

Lemma 5.6. Let $X = \{A_1, A_2, \dots, A_m\}$ be m -complementary sequences of length n . Then

1. $Y = \{A_1^*, A_2^*, \dots, A_i^*, A_{i+1}, \dots, A_m\}$ are m -complementary sequences of length n where A_i^* means "reverse the elements of A_i ";
2. $W = \{A_1, A_2, \dots, A_i, -A_{i+1}, \dots, -A_m\}$ are m -complementary sequences of length n ;
3. $Z = \{\{A_1, A_2\}, \{A_1, -A_2\}, \dots, \{A_{2i-1}, A_{2i}\}, \{A_{2i-1}, -A_{2i}\}, \dots\}$ are m -(or $m+1$ - if m is odd, in which case we let A_{m+1} be n zeros) complementary sequences of length $2n$;
4. $U = \{\{A_1/A_2\}, \{A_1/-A_2\}, \dots, \{A_{2i-1}/A_{2i}\}, \{A_{2i-1}/-A_{2i}\}, \dots\}$, where A_j/A_k means that $a_{j1}, a_{k1}, a_{j2}, a_{k2}, \dots, a_{jn}, a_{kn}$, are m -(or $m+1$ - if m is odd, in which case we let A_{m+1} be n zeros) complementary sequences of length $2n$;
5. $V = \{A_1^+, A_2^+, \dots, A_m^+\}$, where $A_i^+ = \{a_{i1}, -a_{i2}, a_{i3}, -a_{i4}, \dots\}$ are m -complementary sequences of length n .

By a lengthy but straightforward calculation, it can be shown that

Theorem 5.7. Suppose that $X = \{A_1, \dots, A_{2m}\}$ are $2m$ -complementary sequences of length n and weight u and $Y = \{B_1, B_2\}$ are 2-complementary disjointable sequences of length t and weight $2k$. Then there are $2m$ -complementary sequences of length nt and weight ku .

The same result is true if X are $2m$ -complementary disjointable sequences of length n and weight $2u$ and Y are 2 -complementary sequences of weight k .

Proof. Write X^* for the sequence whose elements are the reverse of those in the sequence X . Using an idea of R. J. Turyn, we consider

$$A_{2i-1} \times \frac{(B_1 + B_2)}{2} + A_{2i} \times \frac{(B_1 - B_2)}{2} \quad \text{and}$$

$$A_{2i-1} \times \frac{(B_1^* - B_2^*)}{2} - A_{2i} \times \frac{(B_1^* + B_2^*)}{2},$$

for $i = 1, \dots, m$, which are the required sequences in the first case. While

$$\frac{(A_{2i-1} + A_{2i})}{2} \times B_1 + \frac{(A_{2i-1} - A_{2i})}{2} \times B_2^* \quad \text{and}$$

$$\frac{(A_{2i-1} + A_{2i})}{2} \times B_2 - \frac{(A_{2i-1} - A_{2i})}{2} \times B_1^*$$

for $i = 1, \dots, m$, are the required sequences for the second case. (Note here that \times is the normal Kronecker product.)

The proof now follows by an exceptionally tedious but straightforward verification. \square

Corollary 5.8. *Since there are Golay sequences of lengths 2, 10 and 26, there are Golay sequences of length $2^a 10^b 26^c$ for a, b, c nonnegative integers.*

Corollary 5.9. *There are 2-complementary sequences of lengths $2^a 6^b 10^c 14^d 26^e$ of weights $2^a 5^b 10^c 13^d 26^e$, where a, b, c, d, e are nonnegative integers.*

Proof. Use the sequences of Tables 5 and 6 of Appendix H of [23]. \square

5.3. T -Sequences and Base Sequences

The bulk of the remainder of this chapter will be devoted to obtaining T -sequences. We recall that T -sequences always yield T -matrices. If there are T -sequences of length t and Williamson matrices of order w there is an Hadamard matrix of order $4tw$.

Four sequences of elements $+1, -1$ of lengths $m + p, m + p, m, m$ where p is odd, and which have zero nonperiodic autocorrelation function, are called *base sequences*. In Table 5.1 base sequences are displayed for lengths $m + 1, m + 1, m, m$ for $m + 1 \in \{2, 3, \dots, 30\}$. If X and Y are Golay sequences, $\{1, X\}, \{1, -X\}, \{Y\}, \{Y\}$ are base sequences of lengths $m + 1, m + 1, m, m$. So base sequences exist for all $m = 2^a 10^b 26^c$, a, b, c nonnegative integers, $p = 1$. The cases for $m = 17$, $p = 1$, were found by A. Sproul and J. Seberry; for

$m = 23$, $p = 1$ by R. Turyn; and for $m = 22, 24, 26, 27, 28$, $p = 1$ by Koukouvinos, Kounias, and Sotirakoglou [51]. These sequences are also discussed in Geramita and Seberry [23, pp. 129–148].

Base sequences are crucial to Yang's [138, 135, 136, 137] constructions for finding longer T -sequences of odd length.

Lemma 5.10. Consider four $(1, -1)$ sequences $A = \{X, U, Y, W\}$, where

$$\begin{aligned} X &= \{x_1 = 1, x_2, x_3, \dots, x_m, h_m x_m, \dots, h_3 x_3, h_2 x_2, h_1 x_1 = -1\}, \\ U &= \{u_1 = 1, u_2, u_3, \dots, u_m, f_m u_m, \dots, f_3 u_3, f_2 u_2, f_1 u_1 = 1\}, \\ Y &= \{y_1, y_2, \dots, y_{m-1}, y_m, g_{m-1} y_{m-1}, \dots, g_3 y_3, g_2 y_2, g_1 y_1\}, \\ V &= \{v_1, v_2, \dots, v_{m-1}, v_m, e_{m-1} v_{m-1}, \dots, e_3 v_3, e_2 v_2, e_1 v_1\}. \end{aligned}$$

Then $N_A = 0$ implies that $h_i = f_i$ for $i \geq 2$ and that $g_j = e_j$ for $j \geq 1$. Here

$$\begin{aligned} 8m - 2 &= \left(\sum_{i=1}^m x_i + x_i h_i \right)^2 + \left(\sum_{i=1}^m u_i + u_i f_i \right)^2 + \left(y_m + \sum_{i=1}^{m-1} y_i + y_i g_i \right)^2 \\ &\quad + \left(v_m + \sum_{i=1}^{m-1} v_i + v_i e_i \right)^2. \end{aligned}$$

Corollary 5.11. Consider four $(1, -1)$ sequences $A = \{X, U, Y, V\}$, where

$$\begin{aligned} X &= \{x_1 = 1, x_2, x_3, \dots, x_m, -x_m, \dots, -x_3, -x_2, -x_1 = -1\}, \\ U &= \{u_1 = 1, u_2, u_3, \dots, u_m, f_m u_m, \dots, f_3 u_3, f_2 u_2, f_1 u_1 = 1\}, \\ Y &= \{y_1, y_2, \dots, y_{m-1}, y_m, y_{m-1}, \dots, y_3, y_2, y_1\}, \\ V &= \{v_1, v_2, \dots, v_{m-1}, v_m, e_{m-1} v_{m-1}, \dots, e_3 v_3, e_2 v_2, e_1 v_1\}. \end{aligned}$$

Then $N_A = 0$ implies that all $e_i = +1$ and that all f_i for $i \geq 2 = -1$. Here $8m - 6$ is the sum of two squares.

Corollary 5.12. Consider four $(1, -1)$ sequences $A = \{X, U, Y, V\}$, where

$$\begin{aligned} X &= \{x_1 = 1, x_2, x_3, \dots, x_m, x_{m+1}, x_m, \dots, x_3, x_2, x_1 = 1\}, \\ U &= \{u_1 = 1, u_2, u_3, \dots, u_m, u_{m+1}, f_m u_m, \dots, f_3 u_3, f_2 u_2, -1\}, \\ Y &= \{y_1, y_2, \dots, y_m, -y_m, \dots, -y_2, -y_1\}, \\ V &= \{v_1, v_2, \dots, v_m, e_m v_m, \dots, e_2 v_2, e_1 v_1\}. \end{aligned}$$

Then $N_A = 0$ implies that $e_i = -1$ for all i and that $f_i = +1$ for all i . Here $8m + 2$ is the sum of two squares.

TABLE 5.1 Base Sequences of Lengths $m + 1, m + 1, m, m$

| Length | Sums of squares | Sequences |
|--------------|-------------------------|-----------------------------------|
| $m + 1 = 2$ | $2^2 + 0^2 + 1^2 + 1^2$ | ++, +-, +, + |
| $m + 1 = 3$ | $3^2 + 1^2 + 0^2 + 0^2$ | +++ , +- , +- , +- |
| $m + 1 = 4$ | $2^2 + 0^2 + 3^2 + 1^2$ | +-+ , +-+ , +-+ , +-+ |
| $m + 1 = 5$ | $3^2 + 3^2 + 0^2 + 0^2$ | +-+ , +-+ , +-+ , +-+ |
| $m + 1 = 5$ | $3^2 + 1^2 + 2^2 + 2^2$ | +++ , +-+ , +-+ , +-+ , +-+ |
| $m + 1 = 6$ | $2^2 + 0^2 + 3^2 + 3^2$ | +-+ , +-+ , +-+ , +-+ , +-+ , +-+ |
| $m + 1 = 7$ | $3^2 + 1^2 + 4^2 + 0^2$ | +-+ , +-+ , +-+ , +-+ , +-+ , +-+ |
| $m + 1 = 7$ | $5^2 + 1^2 + 0^2 + 0^2$ | +++ , +-+ , +-+ , +-+ , +-+ , +-+ |
| $m + 1 = 8$ | $2^2 + 0^2 + 5^2 + 1^2$ | +++ , +-+ , +-+ , +-+ , +-+ , +-+ |
| $m + 1 = 8$ | $4^2 + 2^2 + 3^2 + 1^2$ | +++ , +-+ , +-+ , +-+ , +-+ , +-+ |
| $m + 1 = 9$ | $5^2 + 3^2 + 0^2 + 0^2$ | +++ , +-+ , +-+ , +-+ , +-+ , +-+ |
| $m + 1 = 10$ | $4^2 + 2^2 + 3^2 + 3^2$ | +++ , +-+ , +-+ , +-+ , +-+ , +-+ |
| $m + 1 = 11$ | $5^2 + 3^2 + 2^2 + 2^2$ | +++ , +-+ , +-+ , +-+ , +-+ , +-+ |
| $m + 1 = 11$ | $1^2 + 5^2 + 0^2 + 4^2$ | +++ , +-+ , +-+ , +-+ , +-+ , +-+ |
| $m + 1 = 12$ | $6^2 + 0^2 + 3^2 + 1^2$ | +++ , +-+ , +-+ , +-+ , +-+ , +-+ |
| $m + 1 = 12$ | $4^2 + 2^2 + 5^2 + 1^2$ | +++ , +-+ , +-+ , +-+ , +-+ , +-+ |
| $m + 1 = 13$ | $7^2 + 1^2 + 0^2 + 0^2$ | +++ , +-+ , +-+ , +-+ , +-+ , +-+ |
| $m + 1 = 13$ | $5^2 + 5^2 + 0^2 + 0^2$ | +++ , +-+ , +-+ , +-+ , +-+ , +-+ |

TABLE 5.1 Base Sequences of Lengths $m + 1, m + 1, m, m$ (continued)

| Length | Sums of squares | Sequences |
|--------------|-------------------------|---|
| $m + 1 = 13$ | $3^2 + 1^2 + 6^2 + 2^2$ | + + + + - + - - + + - - +, + + + + - + - - + + - - -, + + + + + - + - + - + +, + + + - - + - + - - + + |
| $m + 1 = 14$ | $6^2 + 4^2 + 1^2 + 1^2$ | + + + + + + - - + + - + - +, + - - - - - + + - - + - + -, + + + + - - - + - - + - - -, + - + - - + - - - + + + - |
| $m + 1 = 14$ | $2^2 + 0^2 + 7^2 + 1^2$ | - - - + + + - + + + + - + -, - - - + + - + - - + + - + +, + - + + + - + + + - + + +, + - + + - + - - - - + + + |
| $m + 1 = 14$ | $6^2 + 0^2 + 3^2 + 3^2$ | + + + - + + - + + + + + - -, + + + + - - + - - - - + - +, + + - - - + + + - + + - +, + - - + + - + - + - + + + |
| $m + 1 = 15$ | $7^2 + 3^2 + 0^2 + 0^2$ | + + - + + + - + - + + + - + +, + + + - + + - - - + + - + + -, + + + + - - + - + + - - - -, + - - - - + - + - + + + + - |
| $m + 1 = 16$ | $6^2 + 4^2 + 3^2 + 1^2$ | + + + + + - + + - - + + + - + -, + - - - - + - - + + - - - + - +, + - + + + + + - - + - + - - +, + + + - + - + + - - - - + + |
| $m + 1 = 16$ | $6^2 + 0^2 + 5^2 + 1^2$ | + - + - - + - - + + + + + + +, + + + - + - - + - + - - + + - -, + - + - + - + + + - - + + + +, + + - - - - + + + - + + - - + |
| $m + 1 = 17$ | $7^2 + 1^2 + 4^2 + 0^2$ | + + - - - - + - - - + - - - - -, + + + + - - + - - - - + - - - + - +, + - - - - + - + - + + - - - +, + + - + - - - - + + - - + - + + |
| $m + 1 = 17$ | $5^2 + 5^2 + 4^2 + 0^2$ | + - + + + - - + - + + + - + + + -, + + + - + + - + + + + - - + - +, + - + + - + + + + - + + + - - -, + - - - + + + + + - + - - + - - |
| $m + 1 = 17$ | $5^2 + 3^2 + 4^2 + 4^2$ | + + + + + - - + - + - + - - + + +, + - - + + - + - + - + + - - - - -, + + + + + - - + + - + - + + - - -, + + + + + - - + + - + - + + - - |
| $m + 1 = 17$ | $1^2 + 1^2 + 8^2 + 0^2$ | + - - - - + + + + - + - + - - - +, + + + - + + - - - + - - - - + + -, + + + + + - + + + + + - - + - +, + - - + - + - + + - - + - - + + |

TABLE 5.1 Base Sequences of Lengths $m + 1, m + 1, m, m$ (continued)

| Length | Sums of squares | Sequences |
|--------------|-------------------------|--|
| $m + 1 = 18$ | $4^2 + 2^2 + 7^2 + 1^2$ | $++-----+-++-----+-+--,$ $+++-----+---++-+-----+,$ $+--+--+---++-+++++++,$ $++++--++-+---+--+--+$ |
| $m + 1 = 18$ | $4^2 + 2^2 + 5^2 + 5^2$ | $+--+--+---++-+++++++,$ $+--+++-+---+-----++++++,$ $++++-++++-++++-+---+-,$ $++++-++++-++++-+---+-$ |
| $m + 1 = 19$ | $7^2 + 3^2 + 4^2 + 0^2$ | $+++--+++-+---++++-++-+,$ $+++--+++-+---++++-+-----,$ $+++--+++-+---+++++++,$ $+-----+++-+--+---++-+$ |
| $m + 1 = 19$ | $3^2 + 1^2 + 8^2 + 0^2$ | $+--+-----+--+--+---++-+++,$ $+++++--+--+-----+--+---+,$ $+++++--+--+-----+--+---+,$ $+---+--+--+-----+--+---+$ |
| $m + 1 = 19$ | $1^2 + 1^2 + 6^2 + 6^2$ | $++++-----+--+--+---++-+,$ $+--+-----+--+--+---++-+---,$ $+---+--+--+-----+--+---+,$ $+++++--+--+-----+--+---$ |
| $m + 1 = 20$ | $2^2 + 0^2 + 7^2 + 5^2$ | $+++-----+---+--+---++++-+,,$ $+++-----+---+--+---++++-+---,$ $+--+--+---+---+--+---+++++,$ $+++-----+---+--+---+++++$ |
| $m + 1 = 21$ | $7^2 + 5^2 + 2^2 + 2^2$ | $+++-----+---+--+---++++-+,,$ $-+---+---+---+---+---+---+---,$ $+---+---+---+---+---+---+---,$ $+---+---+---+---+---+---+$ |
| $m + 1 = 21$ | $3^2 + 1^2 + 6^2 + 6^2$ | $+++-----+---+--+---++++-+,,$ $-+---+---+---+---+---+---+---,$ $+---+---+---+---+---+---+---,$ $+---+---+---+---+---+---+$ |
| $m + 1 = 22$ | $6^2 + 0^2 + 7^2 + 1^2$ | $+++-----+---+--+---++++-+,,$ $+++-----+---+--+---++++-+---,$ $+--+--+---+---+--+---+++++,$ $+++-----+---+--+---+++++$ |
| $m + 1 = 23$ | $3^2 + 3^2 + 6^2 + 6^2$ | $+--+--+---+---+--+---++++-+,,$ $-+--+---+---+--+---++++-+---,$ $+++-----+---+--+---++++-+,,$ $-+---+---+---+---+---+---+---$ |

TABLE 5.1 Base Sequences of Lengths $m + 1, m + 1, m, m$ (continued)

| Length | Sums of squares | Sequences |
|--------------|--------------------------|--|
| $m + 1 = 24$ | $8^2 + 2^2 + 5^2 + 1^2$ | + - - + - - + - + + + - - - - + - - - + - - - , + - - - + - + - - - - + + + + + - - + - - - + , + + + - - - + + - - + - - - - + - - - - + - + , + + - - + - + + - + - + + + - + - - + - - - + |
| $m + 1 = 25$ | $7^2 + 7^2 + 0^2 + 0^2$ | - - - - + + + - + + - + + + + + + + - + - + - + , + - - + + + + + + + - - - - + + - + - + + + - + , - + - + - + + - + - - + + - - + + - + + + - - - , + + - + - - + - + + - - - - + + + - - + + - - - + |
| $m + 1 = 26$ | $8^2 + 6^2 + 1^2 + 1^2$ | + + + + + + + - - + + + - - + - - + + - + - + - + , + - - - - - - + + - - - + + - + + - - + - + - + - , + + + + + + - - - - + + - - + + - + - - + - + - - , + - + - + - - + - + + - - + + - - - - + + + + + - |
| $m + 1 = 27$ | $7^2 + 5^2 + 4^2 + 4^2$ | + + + + - - + + + - + - - + - + - + - - + + - + + + + , - + + + - - + + + - + - - + - + - + - - + + - + + + + , - - - + + - - - + - + + + + - + - - + + - + + + + , - - - + + - - - + - + + + + - + - - + + - + + + + |
| $m + 1 = 28$ | $4^2 + 2^2 + 3^2 + 9^2$ | - + + - + + - - - + + + + - - - - + - + - + + + - + + + , + + + - + - + + - + - + - - - + - - - + + - - + - + + + , + + - + - + + + + - - + - + + - - + + + - - + - - - , + + - + + - + + + - + + - + - - + + + + + + + - - - |
| $m + 1 = 29$ | $3^2 + 1^2 + 2^2 + 10^2$ | + - - + + + + + - + + + - - + - + - - + - - + - + + - - - + , + + - - - + - + - - - + - - + + + + + - - - + + + + - - + - , + + - + - + + + + - - + + - - + + + - + - - - - - - + , + + + + + - + + + - - - + - + + + + - + + - + - + - + - + |
| $m + 1 = 30$ | $8^2 + 6^2 + 3^2 + 3^2$ | + + + + + - + + + + - + - - + + - - + + + + - + - - - + - + , + - - - - + - - - - + - + + - - + + - - - - + - + + + - + - , + + + - + - - - + - + + - - - + - - + + + + - + + + + - - , + - + + + + - + + + + - + - - - + + - + - - - + - + + - |

TABLE 5.2 Turyn Sequences of Lengths $n + 1, n + 1, n, n$

| Length | Sequences |
|----------|--|
| $n = 1$ | $\{\{1 - 1\}, \{1 1\}, \{1\}, \{1\}\}$ |
| $n = 2$ | $\{\{1 1 1\}, \{1 1 - 1\}, \{1 - 1\}, \{1 - 1\}\}$ |
| $n = 3$ | $\{\{1 1 1 - 1 - 1\}, \{1 1 - 1 1\}, \{1 1 1\}, \{1 - 1 1\}\}$ |
| $n = 4$ | $\{\{1 1 1 - 1 1 1\}, \{1 1 1 1 - 1\}, \{1 1 - 1 - 1\}, \{1 - 1 1 - 1\}\}$ |
| $n = 5$ | $\{\{1 1 1 1 - 1 - 1 - 1\}, \{1 1 - 1 1 - 1 1\}, \{1 1 - 1 1 1\}, \{1 1 - 1 1 1\}\}$ |
| $n = 6$ | $\{\{1 1 1 1 - 1 1 1 1\}, \{1 1 - 1 - 1 - 1 1 - 1\}, \{1 1 - 1 1 - 1 - 1\}, \{1 1 - 1 1 - 1 - 1\}\}$ |
| $n = 7$ | $\{\{1 1 - 1 1 - 1 1 - 1 - 1\}, \{1 1 1 1 - 1 - 1 - 1 1\}, \{1 1 1 - 1 1 1 1\}, \{1 - 1 - 1 1 - 1 - 1 1\}\}$ |
| $n = 12$ | $\{\{1 1 1 1 - 1 1 - 1 1 - 1 1 1 1 1\}, \{1 1 1 1 - 1 - 1 1 - 1 1 - 1 1 1 1 - 1\}, \{1 1 1 - 1 1 1 - 1 - 1 1 - 1 - 1 - 1\}, \{1 1 1 - 1 - 1 1 - 1 1 1 - 1 - 1 - 1\}\}$ |
| $n = 14$ | $\{\{1 1 - 1 1 1 1 - 1 1 - 1 1 1 1 - 1 1 1\}, \{1 1 1 - 1 1 1 - 1 - 1 - 1 1 1 - 1 1 1 - 1\}, \{1 1 1 1 - 1 - 1 1 - 1 1 1 - 1 - 1 - 1 - 1\}, \{1 - 1 - 1 - 1 - 1 1 - 1 1 - 1 1 1 1 1 - 1\}\}$ |

lished that they do not exist for all lengths less than 42 aside from those listed here). The first unsettled case is $m + 1 = 43$.

A sequence $X = \{x_1, \dots, x_n\}$ will be called *skew* if n is even and $x_i = -x_{n-i+1}$, and *symmetric* if n is odd and $x_i = x_{n-i+1}$.

Theorem 5.13 (Turyn). *Suppose that $A = \{X, U, Y, V\}$ are Turyn sequences of lengths $m + 1, m + 1, m, m$. Then there are T -sequences of lengths $2m + 1$ and $4m + 3$.*

Proof. We use the notation A/B as before to denote the interleaving of two sequences $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_{m-1}\}$:

$$\frac{A}{B} = \{a_1, b_1, a_2, b_2, \dots, b_{m-1}, a_m\}.$$

Let 0_t be a sequence of zeros of length t . Then

$$T_1 = \{\{\frac{1}{2}(X + U), 0_m\}, \{\frac{1}{2}(X - U), 0_m\}, \{0_{m+1}, \frac{1}{2}(Y + V)\}, \{0_{m+1}, \frac{1}{2}(Y - V)\}\}$$

and

$$T_2 = \left\{ \{1, 0_{4m+2}\}, \left\{0, \frac{X}{Y}, 0_{2m+1}\right\}, \left\{0, 0_{2m+1}, \frac{U}{0_m}\right\}, \left\{0, 0_{2m+1}, \frac{0_{m+1}}{V}\right\} \right\}$$

are T -sequences of lengths $2m + 1$ and $4m + 3$, respectively. □

Theorem 5.14. *If X and Y are Golay sequences of length r , then writing 0_r for the vector of r zeros, we have that $T = \{\{1, 0_r\}, \{0, \frac{1}{2}(X + Y)\}, \{0, \frac{1}{2}(X - Y)\}, \{0_{r+1}\}\}$ are T -sequences of length $r + 1$.*

Corollary 5.15 (Turyn). *There exist T -sequences of lengths $1 + 2^a 10^b 26^c$, where a, b, c are nonnegative integers.*

Combining the two theorems we find

Corollary 5.16. *There exist T -sequences of lengths $3, 5, 7, \dots, 33, 41, 51, 53, 59, 65, 81, 101$.*

A desire to fill the gaps in the list in Corollary 5.7 leads to the following idea:

Lemma 5.17. *Suppose that $X = \{A, B, C, D\}$ are 4-complementary sequences of length $m + 1, m + 1, m, m$, respectively, and weight k . Then*

$$Y = \{\{A, C\}, \{A, -C\}, \{B, D\}, \{B, -D\}\}$$

are 4-complementary sequences of length $2m + 1$ and weight $2k$. Further, if $\frac{1}{2}(A + B)$ and $\frac{1}{2}(C + D)$ are also $(0, 1, -1)$ sequences, then, with 0_t the sequence of t zeros,

$$Z = \{\{\frac{1}{2}(A + B), 0_m\}, \{\frac{1}{2}(A - B), 0_m\}, \{0_{m+1}, \frac{1}{2}(C + D)\}, \{0_{m+1}, \frac{1}{2}(C - D)\}\}$$

are 4-complementary sequences of length $2m + 1$ and weight k . If A, B, C, D are $(1, -1)$ sequences, then Z consists of T -sequences of length $2m + 1$.

Lemma 5.18. *If there are Turyn sequences of length $m + 1, m + 1, m, m$, there are base sequences of lengths $2m + 2, 2m + 2, 2m + 1, 2m + 1$.*

Proof. Let X, U, Y, V be the Turyn sequences as in Table 5.2. Then

$$E = \left\{1, \frac{X}{Y}\right\}, \quad F = \left\{-1, \frac{X}{Y}\right\}, \quad G = \left\{\frac{U}{V}\right\}, \quad H = \left\{\frac{U}{-V}\right\}$$

are 4-complementary base sequences of lengths $2m + 2, 2m + 2, 2m + 1, 2m + 1$, respectively. \square

Corollary 5.19. *There are base sequences of lengths $m + 1, m + 1, m, m$ for m equal to*

1. $t, 2t + 1$, where there are Turyn sequences of length $t + 1, t + 1, t, t$;
2. $9, 11, 13, 25, 29$;
3. g , where there are Golay sequences of length g ;

4. 17 (Seberry-Sproul), 23 (Turyn), 22, 24, 26, 27, 28 (Koukouvinos, Kounias, Sotirakoglou) given in Table 5.1 and Table 5.3.

Corollary 5.20. *There are base sequences of lengths $m + 1, m + 1, m, m$ for $m \in \{1, 2, \dots, 29\} \cup G$, where $G = \{g : g = 2^a \cdot 10^b \cdot 26^c, a, b, c \text{ non-negative integers}\}$.*

Now Cooper-(Seberry)Wallis-Turyn have shown how 4 disjoint complementary sequences of length t and zero nonperiodic (or periodic) autocorrelation function can be used to form $OD(4t; t, t, t, t)$ (formerly called *Baumert-Hall arrays*) [12]. First, the sequences (variously called *T-sequences* or *Turyn sequences*, but the latter has two different usages) are turned into *T*-matrices and then the Cooper-(Seberry)Wallis construction can be applied (see Section 3). Thus, it becomes important to know for which lengths (and decomposition into squares) *T*-sequences exist. First,

Lemma 5.21. *If there are base sequences of length $m + 1, m + 1, m, m$, there are*

1. 4 (disjoint) *T*-sequences of length $2m + 1$,
2. 4-complementary sequences of length $2m + 1$.

Proof. Let X, U, Y, V be the base sequences of lengths $m + 1, m + 1, m, m$, then

$$\left\{ \frac{1}{2}(X + U), 0_m \right\}, \left\{ \frac{1}{2}(X - U), 0_m \right\}, \left\{ 0_{m+1}, \frac{1}{2}(Y + V) \right\}, \left\{ 0_{m+1}, \frac{1}{2}(Y - V) \right\}$$

are the *T*-sequences of length $2m + 1$ and

$$\{X, Y\}, \{X, -Y\}, \{U, V\}, \{U, -V\}$$

are 4-complementary sequences of length $2m + 1$. □

Corollary 5.22. *There are *T*-sequences of lengths t for the following $t < 106$:*

$$1, 3, \dots, 59, 65, 81, 101, 105.$$

5.4. On Yang's Theorems on *T*-Sequences

In an a series of papers in 1982 and 1983, Yang [135, 136, 137] found that base sequences can be multiplied by 3, 7, 13, and $2g + 1$, where $g = 2^a 10^b 26^c$, $a, b, c \geq 0$: These are instances of what are termed *Yang numbers*. If y is a Yang number and there are base sequences of lengths $m + p, m + p, m, m$, then there are (4-complementary) *T*-sequences of length $y(2m + p)$. This is of most interest when $2m + p$ is odd. (A new construction for the Yang number 57 is given in [58].)

Yang numbers currently exist for $y \in \{3, 5, \dots, 33, 37, 39, 41, 45, 49, 51, 53, 57, 59, 65, 81, \dots\}$, and $2g + 1 > 81, g \in G$, where

$$G = \{g : g = 2^a 10^b 26^c, a, b, c \text{ nonnegative integers}\}.$$

Base sequences currently exist for $p = 1$ and $m \in \{1, 2, \dots, 29\} \cup G$. We reprove and restate Yang's theorems from [138] to illustrate why they work.

Theorem 5.23 (Yang). *Let A, B, C, D be base sequences of lengths $m + p, m + p, m, m$, and let $F = (f_k)$ and $G = (g_k)$ be Golay sequences of length s . Then the following Q, R, S, T become 4-complementary sequences (i.e., the sum of nonperiodic autocorrelation functions is 0), using X^* to denote the reverse of X :*

$$Q = (Af_s, Cg_1; 0, 0; Af_{s-1}, Cg_2; 0, 0; \dots; Af_1, Cg_s; 0, 0; -B^*, 0);$$

$$R = (Bf_s, Dg_s; 0, 0; Bf_{s-1}, Dg_{s-1}; 0, 0; \dots; Bf_1, Dg_1; 0, 0; A^*, 0);$$

$$S = (0, 0; Ag_s, -Cf_1; 0, 0; Ag_{s-1}, -Cf_2; \dots; 0, 0; Ag_1, -Cf_s; 0, -D^*);$$

$$T = (0, 0; Bg_1, -Df_1; 0, 0; Bg_2, -Df_2; \dots; 0, 0; Bg_s, -Df_s; 0, C^*).$$

Furthermore, if we define sequences

$$X = (Q + R)/2, \quad Y = (Q - R)/2, \quad V = (S + T)/2, \quad W = (S - T)/2,$$

then these sequences become T -sequences of length $t(2s + 1)$, $t = 2m + p$.

Note: The interesting case for Yang's theorem is for base sequences of lengths $m + p, m + p, m, m$, where p is odd for then Yang's theorem produces T -sequences of odd length, for example, $3(2m + p)$.

Restatement 5.24 (Yang). *Suppose that E, F, G, H are base sequences of lengths $m + p, m + p, m, m$. Define $A = \frac{1}{2}(E + F)$, $B = \frac{1}{2}(E - F)$, $C = \frac{1}{2}(G + H)$, and $D = \frac{1}{2}(G - H)$ to be suitable sequences. Then the following sequences are disjoint T -sequences of length $3(2m + p)$:*

$$X = A, C; 0, 0'; B^*, 0';$$

$$Y = B, D; 0, 0'; -A^*, 0';$$

$$Z = 0, 0'; A, -C; 0, D^*;$$

$$W = 0, 0'; B, -D; 0, -C^*;$$

and

$$X = B^*, 0'; A, C; 0, 0';$$

$$Y = -A^*, 0'; B, D; 0, 0';$$

$$Z = 0, D^*; 0, 0'; A, -C;$$

$$W = 0, -C^*; 0, 0'; B, -D.$$

In these sequences 0 and 0' are sequences of zeros of lengths $m + p$ and m , respectively.

The next two theorems deal with multiplication by 7 and 13. They can be used recursively, but as the sequences produced are of equal lengths, the next recursive use of the theorems gives sequences of (equal) even length.

Theorem 5.25 (Yang [137]). Let (E, F, G, H) be the base sequences of length $m + p, m + p, m, m$. Let $t = 2m + p$ and define the suitable sequences $A = \frac{1}{2}(E + F)$, $B = \frac{1}{2}(E - F)$, $C = \frac{1}{2}(G + H)$, and $D = \frac{1}{2}(G - H)$ of lengths $m + p, m + p, m$ and m . Then the following X, Y, Z, W are 4-disjoint T -sequences of length $7t$ (where \bar{X} means negate all the elements of the sequence and X^* means reverse all the elements of the sequence):

$$X = (\bar{A}, C; 0, 0; A, D; 0, 0; A, C; 0, 0; \bar{B}^*, 0);$$

$$Y = (\bar{B}, D; 0, 0; B, \bar{C}; 0, 0; B, D; 0, 0; A^*, 0);$$

$$Z = (0, 0; A, \bar{C}; 0, 0; \bar{B}, \bar{C}; 0, 0; A, C; 0, \bar{D}^*);$$

$$W = (0, 0; B, \bar{D}; 0, 0; A, \bar{D}; 0, 0; B, D; 0, C^*).$$

Theorem 5.26 (Yang [137]). Let (E, F, G, H) be the base sequences of length $m + p, m + p, m, m$. Let $t = 2m + p$, and define the suitable sequences $A = \frac{1}{2}(E + F)$, $B = \frac{1}{2}(E - F)$, $C = \frac{1}{2}(G + H)$, and $D = \frac{1}{2}(G - H)$ of lengths $m + p, m + p, m$, and m . Then the following X, Y, Z, W are 4-disjoint T -sequences of length $13t$:

$$Q = (A, D^*; \bar{A}, \bar{C}; \bar{A}, D^*; \bar{A}, C; \bar{A}, D^*; A, \bar{C}; 0, C; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0);$$

$$R = (\bar{B}, C^*; B, D; B, C^*; B, \bar{D}; B, C^*; \bar{B}, D; 0, \bar{D}; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0);$$

$$S = (0, 0; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0; \bar{A}, 0; A, C; B^*, \bar{C}; \bar{A}, \bar{C}; B^*, \bar{C}; A, \bar{C}; B^*, C);$$

$$T = (0, 0; 0, 0; 0, 0; 0, 0; 0, 0; 0, 0; B, 0; \bar{B}, \bar{D}; A^*, D; B, D; A^*, D; \bar{B}, D; A^*, \bar{D}).$$

Yang [137] has also shown how to multiply by 11. The sequences obtained are not disjoint and so cannot be used in another iteration but still are

vital in that they give complementary sequences of length $11(2m + p)$, and hence Hadamard matrices of order $44(2m + p)$. Using the Yang numbers $y = 3, 5, 7, 9, 13, 17, 21, 33, 41, 53, 61, 65, 81$ with base sequences gives T -sequences, so

Corollary 5.27. *Yang numbers and base sequences of lengths $m + 1, m + 1, m, m$ can be used to give T -sequences of lengths $t = y(2m + 1)$ for the following $t < 200$:*

1, 3, ..., 41, 45, ..., 59, 61, 63, 65, 69, 75, 77, 81, 85, 91,
93, 95, 99, 101, 105, 111, 115, ..., 125, 133, 135, 141, ...,
147, 153, 155, 159, 161, 165, 169, 171, 175, 177, 183, 187, 189, 195.

The gaps in these sets can sometimes be filled by T -matrices. Thus, using Table 5.3 and Corollary 5.22 and noting that T -sequences give T -matrices, we have

Lemma 5.28. *T -matrices exist for the following $t < 196$:*

1, 3, ..., 71, 75, 77, 81, 85, 87, 91, 93, 95, 99, 101, 105, 111, 115, ...,
125, 129, ..., 135, 141, ..., 147, 153, 155, 159, ...,
165, 169, 171, 175, 177, 187, 189, 195.

These are given in more detail in Cohen, Rubie, Koukouvinos, Kounias, Seberry, and Yamada [10], Koukouvinos, Kounias, and Seberry [56], and Koukouvinos, Kounias and Sotirakoglou [51]. Further results, including multiplication and construction theorems, are given in recent work of Koukouvinos, Kounias, Seberry, C. H. Yang, and J. Yang [57, 58].

5.5. Koukouvinos and Kounias

We call κ a *Koukouvinos-Kounias number*, or *KK number*, if

$$\kappa = g_1 + g_2,$$

where g_1 and g_2 are both the lengths of Golay sequences. Then we have

Lemma 5.29. *Let κ be a KK number and y be a Yang number. Then there are T -sequences of length t and $\text{OD}(4t; t, t, t)$ for $t = y\kappa$.*

TABLE 5.3 T -Matrices Used

| Order | Sum of squares | T_i | Sets |
|-------|-------------------------|-------|---|
| 31 | $3^2 + 3^2 + 3^2 + 2^2$ | T_1 | {1, 5, -8, -9, 11, -14, 24, 25, 27} |
| | | T_2 | {2, 6, 10, -12, 19, -21, 26, -29, 30} |
| | | T_3 | {4, 7, -16, 17, -18, 20, 22, 23, -28} |
| | | T_4 | {3, 13, 15, -31} |
| 39 | $6^2 + 1^2 + 1^2 + 1^2$ | T_1 | {17, 20, -21, 23, 24, 26, 35, 38} |
| | | T_2 | {14, 15, -16, -18, 19, 22, 25, -34, -36, -37, 39} |
| | | T_3 | {-4, -7, -8, 10, 11, 13, 28, -29, -31, 32, 33} |
| | | T_4 | {1, 2, -3, -5, 6, -9, -12, 27, 30} |
| 43 | $4^2 + 3^2 + 3^2 + 3^2$ | T_1 | {1, 4, -5, 6, 7, 8, 9, -13, -14, 15, 16, -17, -18, 21} |
| | | T_2 | {-2, 3, 10, 11, -12, 19, 20} |
| | | T_3 | {-22, -23, 24, 26, 29, 31, 34, 36, -39, -41, 42} |
| | | T_4 | {-25, -27, 28, 30, 32, 33, -35, 37, -38, 40, 43} |
| 49 | $4^2 + 4^2 + 4^2 + 1^2$ | T_1 | {4, 6, -18, 19, 21, -32, 34, 44, 45, -46} |
| | | T_2 | {-8, 9, 10, 12, 14, 25, 26, 28, -36, 37, 38, -40, -42, -48} |
| | | T_3 | {11, 13, 22, -23, -24, 27, 39, -41, 47, 49} |
| | | T_4 | {-1, 2, 3, 5, 7, 15, -16, -17, -20, 29, -30, -31, 33, 35, -43} |
| 49 | $5^2 + 4^2 + 2^2 + 2^2$ | T_1 | {1, -2, -3, 5, 7, -15, 16, 17, 20, -29, 30, 31, 33, 35, -43} |
| | | T_2 | {11, -13, -22, 23, 24, -27, 39, 41, 47, 49} |
| | | T_3 | {-4, 6, 18, -19, -21, 32, 34, 44, 45, -46} |
| | | T_4 | {8, -9, -10, 12, 14, 25, 26, 28, 36, -37, -38, -40, -42, 48} |
| 55 | $5^2 + 5^2 + 2^2 + 1^2$ | T_1 | {1, 2, -5, 7, 8, -9, 10, 11, -23, -24, 27, 29, 30, -31, 32, 33, -45, -47, 48} |
| | | T_2 | {-14, 15, 17, -36, 37, 39, 51, 52, -53, 54, 55} |
| | | T_3 | {12, 13, -16, 18, 19, -20, 21, 22, 34, 35, -38, -40, -41, 42, -43, -44} |
| | | T_4 | {-3, 4, 6, 25, -26, -28, -46, 49, 50} |
| 57 | $4^2 + 4^2 + 4^2 + 3^2$ | T_1 | {-24, -25, 29, 30, -31, 32, 33, -35, 36, 37, 38, 53} |
| | | T_2 | {20, 21, 22, -23, -26, 27, -28, -34, 49, 50, 51, -52, 54, 55, -56, 57} |
| | | T_3 | {5, 6, -10, 11, -12, 13, 14, -16, 17, 18, 19, 40, -41, 42, 45, -46, -47, -48} |
| | | T_4 | {1, 2, 3, -4, -7, 8, -9, 15, -39, 43, 44} |

TABLE 5.3 *T*-Matrices Used (continued)

| Order | Sum of squares | T_i | Sets |
|-------|-------------------------|-------|---|
| 61 | $6^2 + 5^2$ | T_1 | {2, 7, 10, 17, 18, -26, 29, -30, 31, -32, 35, 40, -44, -51, 55, 61} |
| | | T_2 | {3, 4, -8, -11, -12, 13, 14, 15, 16, 19, 22, -25, 27, -28, 36, -37, -38, 41, -42, -47, 49, 52, 56, -57, 60} |
| | | T_3 | {-1, 5, 6, -9, -20, 21, -23, -24, -33, -34, 39, 43, 45, 46, 48, -50, -53, 54, -58, 59} |
| | | T_4 | { ϕ } |
| 67 | $8^2 + 1^2 + 1^2 + 1^2$ | T_1 | {-1, 5, 9, 13, 14, 15, 18, 25, 27, 29, -31, 32, -39, 43, 50, -67} |
| | | T_2 | {2, -8, -12, 16, 17, 23, -40, 41, 42, -45, -46, -47, -53, 54, -56, 65, 66} |
| | | T_3 | {-6, 7, 11, 19, 20, -21, 24, -26, -28, -37, 38, 44, -49, 57, -58, -59, 61} |
| | | T_4 | {-3, -4, 10, 22, 30, -33, 34, -35, 36, 48, 51, -52, -55, -60, -62, 63, 64} |
| 71 | $6^2 + 5^2 + 3^2 + 1^2$ | T_1 | {1, -2, -3, 4, 5, 6, -7, 8, 9, 10, -11, -12, -13, -14, 15, 16, -17, 18, 19, -20, 21, 22, 23, 24} |
| | | T_2 | {25, 26, 27, 28, -29, 30, 31, -32, 33, 34, 35, -36, 37, -38, 39, -40, 41, -42, -43, -44, -45, 46, 47} |
| | | T_3 | {48, 49, 50, 51, -52, -56, 57, 58, 60, -64, 65, -66, -71} |
| | | T_4 | {53, 54, -55, 59, -61, 62, -63, -67, -68, 69, 70} |
| 85 | $7^2 + 6^2$ | T_1 | {1, 2, 4, -5, -11, -12, 14, -15, 21, 22, 24, -25, 31, 32, -34, 35, -41, -42, -44, 45, 51, 52, -54, 55, 61, 62, 64, -65, 71, 72, -74, 75, -81} |
| | | T_2 | {3, -13, 23, 33, -43, 53, 63, 73, 82, 83} |
| | | T_3 | {-6, -7, -9, 10, 16, 17, -19, 20, -26, -27, -29, 30, -36, -37, 39, -40, -46, -47, -49, 50, 56, 57, -59, 60, 66, 67, 69, -70, 76, 77, -79, 80} |
| | | T_4 | {8, 18, 28, -38, -48, -58, 68, -78, -84, 85} |
| 87 | $7^2 + 6^2 + 1^2 + 1^2$ | T_1 | {-2, -3, 5, 6, 10, 11, 13, -14, 15, 16, -17, 20, 21, 24, -25, 28, 29, -62, -65, 66, 67, 70, -73} |
| | | T_2 | {30, -33, -36, -37, 38, 41, -47, 48, 51, 52, 55, -56, 74, 75, -78, 79, 82, 83, -86, 87} |
| | | T_3 | {1, -4, -7, -8, 9, 12, 18, -19, -22, -23, -26, 27, 59, -60, 61, 63, 64, 68, 69, -71, -72} |
| | | T_4 | {-31, -32, 34, 35, 39, 40, 42, -43, 44, -45, 46, -49, -50, -53, 54, -57, -58, -76, 77, 80, 81, 84, -85} |

TABLE 5.3 *T*-Matrices Used (continued)

| Order | Sum of squares | T_i | Sets |
|-------|-------------------------|-------|---|
| 91 | $5^2 + 5^2 + 5^2 + 4^2$ | T_1 | {-1, -2, 3, 5, -6, -8, 10, 11, 13, 27, 28, -29, -31, 32, 35, 38, 53, 54, -55, -57, 58, -60, 62, 63, 65, -79, -82} |
| | | T_2 | {-4, -7, 9, 12, 30, 33, 34, -36, -37, -39, 56, 59, 61, 64, 80, -81, -83, 84, 85} |
| | | T_3 | {17, 20, , -22, -25, 40, 41, -42, -44, 45, -48, -51, 69, 72, 74, 77, 86, 88, 89, -91} |
| | | T_4 | {-14, -15, 16, 18, -19, -21, 23, 24, 26, 43, 46, -47, 49, 50, 52, -66, -67, 68, 70, -71, 73, -75, -76, -78, 87, 90} |
| 93 | $6^2 + 5^2 + 4^2 + 4^2$ | T_1 | {2, 3, 4, 5, -6, 7, 8, -9, -10, 11, 12, 13, -14, 15, -16, 17, 19, 21, 23, -25, -27, -29, 31, -78} |
| | | T_2 | {1, 18, -20, -22, 24, -26, -28, 30, -63, 64, -65, 66, 67, 68, -69, -70, 71, 72, -73, 74, 75, 76, 77} |
| | | T_3 | {33, 34, 35, 36, -37, 38, 39, -40, -41, 42, 43, 44, -45, 46, -47, -48, -50, -52, -54, 56, 58, 60, -62, -80, 82, 84, -86, 88, 90, -92} |
| | | T_4 | {32, -49, 51, 53, -55, 57, 59, -61, 79, -81, -83, -85, 87, 89, 91, 93} |

This gives *T*-sequences of lengths

$$2 \cdot 101, 2 \cdot 109, 2 \cdot 113, 8 \cdot 127, 2 \cdot 129, 2 \cdot 131, 8 \cdot 151, \\ 8 \cdot 157, 16 \cdot 163, 2 \cdot 173, 4 \cdot 179, 4 \cdot 185, 4 \cdot 193, 2 \cdot 201.$$

6 AMICABLE HADAMARD MATRICES AND AOD

Two matrices $M = I + U$ and N will be called [complex] *amicable Hadamard matrices* if M is a (complex) skew Hadamard matrix and N a [complex] Hadamard matrix satisfying

$$N^T = N, \quad MN^T = NM^T \quad \text{if real,} \\ N^* = N, \quad MN^* = NM^* \quad \text{if complex.}$$

Amicable Hadamard matrices are useful in constructing skew Hadamard matrices: They are algebraically powerful and elegant. We will only use constructions with real matrices to construct (real) amicable Hadamard matrices. It is obvious, however, that if complex matrices are used, then complex amicable Hadamard matrices can be obtained.

We note that the truth of the conjecture implicit in Seberry [77] and Seberry-Yamada [86], that "amicable Hadamard matrices exist for every order 2 and $4n$, $n \geq 1$," would imply the two conjectures that "skew Hadamard matri-

ces exist for every order 2 and $4n, n \geq 1$ " (which appears to be hard to prove) and that "symmetric Hadamard matrices exist for every order 2 and $4n, n \geq 1$ " (which appears to be the easier to prove).

6.1. Other Amicable Matrices

M and N of order n are said to be *amicable orthogonal designs* of type $AOD(n; (m_1, \dots, m_p); (n_1, \dots, n_q))$ if M is an $OD(n; m_1, \dots, m_p)$, N is an *orthogonal design* $OD(n; n_1, \dots, n_q)$, and $MN^T = NM^T$. If M comprises the variables x_1, \dots, x_p and N comprises the variables y_1, \dots, y_q , then

$$MM^T = \sum_{i=1}^p m_i x_i^2 I_n, \quad NN^T = \sum_{j=1}^q n_j y_j^2 I_n$$

and

$$ZZ^T = (m_1 x_1^2 + \dots + m_p x_p^2)(n_1 y_1^2 + \dots + n_q y_q^2) I_n,$$

where $Z = MN^T$. Wolfe and Shapiro (see [23]) have studied and solved the algebraic necessary conditions for amicable orthogonal designs, but the sufficiency conditions are largely unresolved (see [71, 23, 79] for partial results).

Amicable orthogonal designs $AOD(n; (1, n-1); (n))$ give amicable Hadamard matrices (they are not the same since the orthogonal designs have no symmetry or skew symmetry conditions). *Normalized amicable Hadamard matrices* of order h can be written in the form

$$H = \begin{bmatrix} 1 & 1 & \dots & 1 \\ - & & & \\ \vdots & I + S & & \\ - & & & \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & & & \\ \vdots & P + R & & \\ 1 & & & \end{bmatrix},$$

where

$$\begin{aligned} S^T &= -S, & P^T &= P, & R^T &= R, & PR^T + RP^T &= 0, \\ RR^T &= I, & SJ &= PJ = 0, & RJ &= -J, & SP^T &= PS^T, \\ SR^T &= RS^T, & SS^T &= PP^T = (h-1)I - J. \end{aligned}$$

Amicable orthogonal designs, amicable Hadamard matrices, and skew Hadamard matrices have proved difficult to find. The Kronecker product of skew Hadamard matrices is not a skew Hadamard matrix. However, if h_1 and h_2 are the orders of amicable Hadamard matrices, then there are amicable Hadamard matrices of order $h_1 h_2$; further, if g is the order of a skew Hadamard matrix, there are skew Hadamard matrices of orders $h_1 g$ and $h_2 g$ [114]. We list the orders for which amicable matrices are known, but we do not prove these results here. The recent result of Seberry and Yamada [86], which is class AIII, indicate that powerful results may remain to be discovered.

6.2. Summary and Tables of Amicable Hadamard Matrices

| | | |
|------|--------------------|---|
| AI | 2^t | t a nonnegative integer; J. Wallis [110] |
| AII | $p^r + 1$ | p^r (prime power) $\equiv 3 \pmod{4}$; J. Wallis [110] |
| AIII | $(p - 1)^u + 1$ | p the order of normalized amicable Hadamard matrices, there are normalized amicable Hadamard matrices of order $(p - 1)^u + 1$, $u > 0$ an odd integer; Seberry and Yamada [86] |
| AIV | $2(q + 1)$ | $2q + 1$ is a prime power, q (prime) $\equiv 1 \pmod{4}$; J. Wallis [114, p. 304] |
| AV | $(t + 1)(q + 1)$ | q (prime power) $\equiv 5 \pmod{8} = s^2 + 4t^2$, $s \equiv 1 \pmod{4}$, and $ t + 1$ is the order of amicable orthogonal designs of type AOD($1 + t $; $(1, t)$; $(\frac{1}{2}(t + 1), \frac{1}{2}(t + 1))$); [23, §5.7] |
| | $2^r(q + 1)$ | q (prime power) $\equiv 5 \pmod{8} = s^2 + 4(2^r - 1)^2$, $s \equiv 1 \pmod{4}$, r some integer; [23, §5.7] |
| | $2(q + 1)$ | $q \equiv 5 \pmod{8}$; J. Wallis [116] |
| AVI | S | S is a product of the above orders; J. Wallis [110] |

Constructions for amicable orthogonal designs can be found in [23], [70], [69], [77], [79], [86], [96], [110], [116], [114], [119]. A summary of the orders for which skew Hadamard matrices are known can be found at the end of Section 7. Amicable Hadamard matrices appear in Table 6.1. In this table, a “.” means “unknown” and a blank means “2.”

TABLE 6.1 Orders $2^t q$ for Which Amicable Hadamard Matrices Exist

| q | t | q | t | q | t | q | t | q | t |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 | | 23 | 4 | 45 | | 67 | 5 | 89 | 4 |
| 3 | | 25 | 3 | 47 | 4 | 69 | 4 | 91 | 3 |
| 5 | | 27 | | 49 | 4 | 71 | | 93 | 3 |
| 7 | | 29 | 4 | 51 | 4 | 73 | 7 | 95 | |
| 9 | 3 | 31 | 3 | 53 | | 75 | 3 | 97 | 9 |
| 11 | | 33 | | 55 | 3 | 77 | | 99 | 4 |
| 13 | 3 | 35 | | 57 | | 79 | 3 | 101 | . |
| 15 | | 37 | . | 59 | . | 81 | 3 | 103 | 3 |
| 17 | | 39 | 3 | 61 | 3 | 83 | | 105 | |
| 19 | 3 | 41 | | 63 | | 85 | 4 | 107 | . |
| 21 | | 43 | 3 | 65 | 4 | 87 | | 109 | 9 |

TABLE 6.1 Orders $2^t q$ for Which Amicable Hadamard Matrices Exist (continued)

| q | t | q | t | q | t | q | t | q | t |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 111 | | 201 | 3 | 291 | | 381 | | 471 | 3 |
| 113 | 8 | 203 | | 293 | | 383 | | 473 | 5 |
| 115 | | 205 | 4 | 295 | 5 | 385 | 3 | 475 | 4 |
| 117 | | 207 | | 297 | | 387 | 5 | 477 | |
| 119 | 4 | 209 | 4 | 299 | 4 | 389 | . | 479 | . |
| 121 | 3 | 211 | 4 | 301 | 5 | 391 | 5 | 481 | 3 |
| 123 | | 213 | 4 | 303 | 3 | 393 | | 483 | |
| 125 | | 215 | | 305 | 5 | 395 | | 485 | 4 |
| 127 | . | 217 | 5 | 307 | . | 397 | 5 | 487 | 5 |
| 129 | 3 | 219 | 7 | 309 | 4 | 399 | 3 | 489 | 3 |
| 131 | | 221 | | 311 | . | 401 | . | 491 | . |
| 133 | 3 | 223 | 3 | 313 | 3 | 403 | 6 | 493 | 3 |
| 135 | 4 | 225 | 4 | 315 | 4 | 405 | | 495 | |
| 137 | | 227 | | 317 | 6 | 407 | | 497 | |
| 139 | 4 | 229 | 3 | 319 | 3 | 409 | 3 | 499 | 3 |
| 141 | | 231 | 3 | 321 | | 411 | 4 | 501 | |
| 143 | | 233 | 4 | 323 | | 413 | 4 | 503 | |
| 145 | 5 | 235 | 3 | 325 | 5 | 415 | 3 | 505 | . |
| 147 | | 237 | | 327 | | 417 | | 507 | |
| 149 | 4 | 239 | 4 | 329 | 6 | 419 | 4 | 509 | . |
| 151 | 5 | 241 | . | 331 | 3 | 421 | 7 | 511 | 5 |
| 153 | 3 | 243 | | 333 | 3 | 423 | 4 | 513 | 4 |
| 155 | | 245 | 4 | 335 | 7 | 425 | | 515 | 5 |
| 157 | 5 | 247 | 6 | 337 | . | 427 | 4 | 517 | 6 |
| 159 | 4 | 249 | 4 | 339 | 3 | 429 | 4 | 519 | 4 |
| 161 | | 251 | 6 | 341 | 5 | 431 | | 521 | |
| 163 | 3 | 253 | 6 | 343 | 6 | 433 | 3 | 523 | 7 |
| 165 | | 255 | | 345 | 4 | 435 | 4 | 525 | |
| 167 | 4 | 257 | 4 | 347 | . | 437 | | 527 | 4 |
| 169 | 5 | 259 | 5 | 349 | 3 | 439 | 3 | 529 | 3 |
| 171 | | 261 | 3 | 351 | 4 | 441 | 3 | 531 | 7 |
| 173 | | 263 | | 353 | 4 | 443 | 6 | 533 | |
| 175 | 3 | 265 | 4 | 355 | 4 | 445 | 3 | 535 | |
| 177 | . | 267 | 4 | 357 | | 447 | | 537 | 5 |
| 179 | 8 | 269 | 8 | 359 | 4 | 449 | . | 539 | 4 |
| 181 | 3 | 271 | 7 | 361 | 3 | 451 | 3 | 541 | 3 |
| 183 | 4 | 273 | | 363 | | 453 | | 543 | 5 |
| 185 | | 275 | 5 | 365 | | 455 | 5 | 545 | |
| 187 | 4 | 277 | 5 | 367 | | 457 | . | 547 | |
| 189 | 3 | 279 | 4 | 369 | 4 | 459 | 3 | 549 | 3 |
| 191 | . | 281 | | 371 | | 461 | . | 551 | |
| 193 | 3 | 283 | . | 373 | 7 | 463 | 7 | 553 | 3 |
| 195 | 3 | 285 | 4 | 375 | | 465 | 3 | 555 | 4 |
| 197 | | 287 | 4 | 377 | 7 | 467 | | 557 | . |
| 199 | 3 | 289 | 3 | 379 | . | 469 | 7 | 559 | 5 |

TABLE 6.1 Orders $2^t q$ for Which Amicable Hadamard Matrices Exist (continued)

| q | t | q | t | q | t | q | t | q | t |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 561 | | 649 | 7 | 737 | 7 | 825 | | 913 | 4 |
| 563 | | 651 | 5 | 739 | . | 827 | | 915 | |
| 565 | 3 | 653 | | 741 | | 829 | . | 917 | 4 |
| 567 | | 655 | 4 | 743 | | 831 | | 919 | 3 |
| 569 | 4 | 657 | 5 | 745 | 6 | 833 | | 921 | 6 |
| 571 | 3 | 659 | . | 747 | 4 | 835 | 3 | 923 | |
| 573 | 3 | 661 | . | 749 | . | 837 | | 925 | 4 |
| 575 | 4 | 663 | 3 | 751 | 3 | 839 | . | 927 | 4 |
| 577 | . | 665 | | 753 | | 841 | 8 | 929 | . |
| 579 | 5 | 667 | 8 | 755 | | 843 | | 931 | 5 |
| 581 | 4 | 669 | 3 | 757 | 5 | 845 | 7 | 933 | . |
| 583 | 3 | 671 | | 759 | 4 | 847 | 4 | 935 | |
| 585 | | 673 | 7 | 761 | . | 849 | 3 | 937 | 5 |
| 587 | | 675 | | 763 | 11 | 851 | . | 939 | 4 |
| 589 | 6 | 677 | | 765 | 3 | 853 | 3 | 941 | . |
| 591 | 4 | 679 | 3 | 767 | | 855 | 4 | 943 | 6 |
| 593 | | 681 | 4 | 769 | 3 | 857 | 4 | 945 | |
| 595 | 3 | 683 | | 771 | 5 | 859 | 3 | 947 | 6 |
| 597 | 4 | 685 | 3 | 773 | . | 861 | 4 | 949 | 3 |
| 599 | . | 687 | | 775 | 3 | 863 | 4 | 951 | |
| 601 | 5 | 689 | 5 | 777 | 4 | 865 | 4 | 953 | . |
| 603 | | 691 | 3 | 779 | 5 | 867 | | 955 | 3 |
| 605 | 4 | 693 | 4 | 781 | 3 | 869 | 4 | 957 | 5 |
| 607 | 5 | 695 | 4 | 783 | 3 | 871 | 3 | 959 | 4 |
| 609 | 3 | 697 | 4 | 785 | 7 | 873 | | 961 | 3 |
| 611 | 6 | 699 | 3 | 787 | 5 | 875 | | 963 | . |
| 613 | 3 | 701 | | 789 | 3 | 877 | . | 965 | 4 |
| 615 | | 703 | 3 | 791 | | 879 | 4 | 967 | . |
| 617 | | 705 | | 793 | 3 | 881 | 6 | 969 | 4 |
| 619 | 3 | 707 | 6 | 795 | 3 | 883 | . | 971 | 6 |
| 621 | 3 | 709 | . | 797 | | 885 | | 973 | 6 |
| 623 | 4 | 711 | | 799 | 6 | 887 | | 975 | 5 |
| 625 | 3 | 713 | | 801 | | 889 | 5 | 977 | |
| 627 | 4 | 715 | 4 | 803 | 9 | 891 | 3 | 979 | 5 |
| 629 | . | 717 | 4 | 805 | 4 | 893 | | 981 | |
| 631 | . | 719 | 4 | 807 | 4 | 895 | 3 | 983 | |
| 633 | | 721 | 5 | 809 | . | 897 | 5 | 985 | 4 |
| 635 | | 723 | 3 | 811 | 5 | 899 | 7 | 987 | |
| 637 | 5 | 725 | 6 | 813 | | 901 | 3 | 989 | 4 |
| 639 | 4 | 727 | . | 815 | | 903 | 4 | 991 | 3 |
| 641 | 6 | 729 | 4 | 817 | 6 | 905 | 4 | 993 | 4 |
| 643 | . | 731 | 5 | 819 | 3 | 907 | 5 | 995 | 4 |
| 645 | | 733 | . | 821 | 6 | 909 | 4 | 997 | . |
| 647 | . | 735 | | 823 | . | 911 | | 999 | 5 |

7 CONSTRUCTIONS FOR SKEW HADAMARD MATRICES

Some of the most powerful methods for constructing Hadamard matrices depend on the existence of skew Hadamard matrices. Skew Hadamard matrices are known to be equivalent to doubly regular tournaments. The analogue of a skew Hadamard matrix in orders $\equiv 2 \pmod{4}$ is a symmetric conference matrix, but very few symmetric conference matrices are known whose orders are not of the form prime power plus one or those derived from skew Hadamard matrices.

The properties of these matrices were noticed as long ago as 1933 and 1944 by Paley and Williamson, but it has only been recently when the talents of Szekeres, Seberry, and Whiteman (among others) were directed toward their study that significant understanding of their nature was achieved.

N. Ito has determined that for general skew Hadamard matrices, there is a unique matrix of each order less than 16, two of order 16, and 16 of order 24. Kimura has found 49 of order 28 [45] and 6 of order 32 [46].

For completeness, we will restate results given earlier that are corollaries of the stronger theorems on amicable Hadamard matrices. The smallest known skew Hadamard matrices are listed. The first rows of circulant matrices of small order that give skew Hadamard matrices are listed.

Jennifer Wallis [111] used a computer to obtain skew Hadamard matrices using the Williamson matrix

$$\begin{bmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{bmatrix}.$$

Those of order < 92 only took at most a few minutes to find, but the matrix of order 92 took many hours on an ICL 1904A. Subsequently, Szekeres and Hunt [35], using a bigger computer, developed indexing techniques that allowed the matrix of order 100 to be found in about one hour. Szekeres [100] has now extended these results and corrected minor errors. The number of inequivalent Hadamard matrices of this type depends on the decomposition into squares, but for order 12, he found one; for 20, one; for 28, three; for 36, one; for 44, three; for 52, six; for 60, eleven; for 68, two; for 76, eight; for 84, ten; for 92, six; for 100, nine; for 108, twelve; for 116, five; and for 124, three.

The following first rows for A, B, C, D generate the required matrices: The results for 21, 25 were found by Hunt; for 27, 29, 31 by Szekeres; and the remainder by (Seberry) Wallis:

- 3 1 -1 1
1 -1 -1
1 -1 -1
1 1 1
- 5 1 -1 -1 1 1
1 -1 -1 -1 -1
1 -1 -1 -1 -1
1 -1 1 1 -1
- 7 1 -1 -1 -1 1 1 1
1 -1 -1 -1 -1 -1 -1
1 -1 -1 1 1 -1 -1
1 -1 1 -1 -1 1 -1
- 9 1 -1 -1 -1 1 -1 1 1 1
1 -1 -1 -1 1 1 -1 -1 -1
1 1 -1 1 -1 -1 1 -1 1
1 1 1 -1 1 1 -1 1 1
- 11 1 -1 -1 -1 -1 1 -1 1 1 1 1
1 -1 -1 -1 -1 1 1 -1 -1 -1 -1
1 1 -1 1 -1 1 1 -1 1 -1 1
1 1 -1 1 1 -1 -1 1 1 -1 1
- 13 1 -1 -1 -1 -1 1 -1 1 -1 1 1 1 1
1 -1 1 1 1 -1 1 1 -1 1 1 1 -1
1 1 -1 -1 1 -1 1 1 -1 1 -1 -1 1
1 1 1 1 -1 -1 1 1 -1 -1 1 1 1
- 15 1 -1 -1 -1 -1 -1 1 -1 1 -1 1 1 1 1 1
1 -1 1 -1 -1 1 1 -1 -1 1 1 -1 -1 1 -1
1 1 1 -1 -1 1 -1 1 1 -1 1 -1 -1 1 1
1 1 -1 1 1 1 1 -1 -1 1 1 1 1 -1 1
- 17 1 -1 -1 -1 -1 -1 1 -1 -1 1 1 -1 1 1 1 1 1
1 1 -1 -1 -1 1 -1 -1 1 1 -1 -1 1 -1 -1 -1 1
1 1 -1 -1 -1 1 -1 1 -1 -1 1 -1 1 -1 -1 -1 1
1 -1 -1 1 -1 1 -1 -1 -1 -1 -1 -1 1 -1 1 -1 -1
- 19 1 -1 1 -1 -1 -1 -1 -1 1 1 -1 -1 1 1 1 1 1 -1 1
1 1 -1 1 1 1 1 1 -1 -1 -1 -1 1 1 1 1 1 -1 1
1 -1 1 -1 -1 -1 -1 1 1 -1 -1 1 1 -1 -1 -1 -1 1 -1
1 -1 -1 1 -1 1 -1 1 1 -1 -1 1 1 -1 1 -1 1 -1 -1
- 21 1 -1 -1 -1 -1 -1 1 -1 -1 1 -1 1 -1 1 1 1 1 1 1
1 1 1 -1 -1 1 -1 1 1 1 -1 -1 1 1 1 -1 1 -1 -1 1 1
1 1 -1 1 -1 -1 -1 -1 1 -1 -1 1 -1 -1 -1 -1 -1 1 -1 1
1 -1 1 -1 -1 1 1 -1 -1 -1 1 1 -1 -1 -1 1 1 -1 -1 1 -1
- 23 1 -1 -1 -1 -1 -1 -1 -1 1 1 -1 1 -1 1 -1 -1 1 1 1 1 1 1
1 1 -1 -1 1 -1 -1 1 1 1 1 -1 -1 1 1 1 1 -1 -1 1 -1 -1 1
1 1 -1 -1 -1 1 -1 1 -1 1 1 -1 1 -1 1 -1 1 -1 -1 -1 1 1
1 -1 -1 -1 -1 1 -1 -1 1 1 -1 -1 1 1 -1 -1 1 -1 -1 -1 -1

(continued)

25 1 1 -1 -1 -1 -1 -1 1 -1 1 -1 -1 -1 1 1 1 -1 -1 1 1 1 1 1 -1 -1
1 1 -1 1 1 1 -1 1 -1 1 1 -1 -1 1 1 -1 1 -1 1 1 1 1 -1 -1 1
1 -1 -1 -1 1 -1 -1 1 1 1 1 -1 -1 -1 -1 1 1 1 1 -1 -1 1 -1 -1 -1
1 -1 -1 1 -1 1 1 1 -1 -1 1 -1 -1 -1 -1 1 1 1 -1 1 -1 -1 -1 -1

27 1 1 -1 -1 -1 1 -1 1 -1 -1 1 1 -1 1 -1 1 -1 -1 1 1 1 -1 -1
1 1 1 1 -1 -1 -1 1 1 -1 1 1 -1 1 1 -1 1 1 -1 1 1 -1 -1 1 1 1
1 1 1 1 -1 1 1 1 -1 1 1 -1 -1 -1 -1 -1 1 1 -1 1 1 1 -1 1 1
1 1 -1 -1 -1 1 1 1 1 -1 1 -1 -1 1 -1 -1 1 1 1 1 -1 -1 -1 1

29 1 1 -1 -1 1 -1 -1 -1 -1 1 -1 1 1 -1 -1 1 -1 -1 1 1 1 -1 -1
1 1 -1 -1 -1 1 1 1 1 -1 1 1 -1 -1 1 1 -1 -1 1 1 1 1 -1 -1
1 1 -1 1 -1 -1 -1 1 1 -1 1 -1 -1 1 1 -1 -1 1 -1 -1 -1 -1 1
1 1 1 1 1 1 -1 -1 -1 1 -1 1 1 -1 1 -1 1 -1 -1 -1 -1 1 1 1

31 1 1 -1 -1 1 1 -1 -1 -1 1 -1 -1 -1 1 -1 1 -1 1 1 1 -1 -1 1
1 -1 1 1 -1 1 -1 -1 -1 1 -1 1 1 1 -1 -1 -1 1 1 1 -1 -1 1 -1
1 1 1 -1 -1 -1 1 -1 -1 1 1 1 -1 -1 1 1 1 -1 -1 1 -1 -1 -1 1
1 -1 1 1 1 1 1 1 1 -1 -1 1 -1 1 -1 -1 1 -1 1 1 1 1 1 -1

7.1. The Goethals-Seidel Type

Goethals and Seidel modified the Williamson matrix so that the matrix entries did not have to be circulant and symmetric. Their matrix, which has been valuable in constructing many new Hadamard matrices, was originally given to construct a skew Hadamard matrix of order 36 [27].

Theorem 7.1 (Goethals and Seidel [27]). *If A, B, C, D are square circulant matrices of order m , and $R = (r_{ij})$ is defined by $r_{i,m-i} = 1, i = 1, \dots, m$, then if A is skew type, and if*

$$AA^T + BB^T + CC^T + DD^T = 4mI, \tag{6}$$

then the array 7 in Section 3 is skew Hadamard of order $4m$.

This construction gave the first skew Hadamard matrices of orders 36 and 52.

Recently, Djokovic [17, 16] has carried out a computer search for circulant matrices that can be used in the Goethals-Seidel array and found matrices to give skew Hadamard matrices of order $4n, n = 37, 43, 49, 67, 113, 127, 157, 163, 181,$ and 241 .

The following two pairs of four sets are $4\text{-(}37; 18, 18, 16, 13; 28)$ and $4\text{-(}37, 18, 15, 15, 15; 26)$ supplementary difference sets, respectively, found by Djokovic [17], which may be used to construct circulant $(1, -1)$ matrices that give, using the Goethals-Seidel array, skew Hadamard matrices of order $4 \cdot 37 = 148$:

- 1, 3, 4, 10, 14, 17, 18, 21, 22, 24, 25, 26, 28, 29, 30, 31, 32, 35
- 1, 6, 8, 9, 10, 11, 12, 14, 16, 17, 22, 23, 26, 27, 29, 31, 35, 36
- 0, 5, 6, 7, 8, 11, 13, 18, 19, 23, 24, 27, 32, 33, 34, 36
- 0, 2, 5, 11, 13, 15, 17, 19, 20, 22, 27, 35, 36
- 1, 7, 9, 10, 12, 14, 16, 17, 18, 22, 24, 26, 29, 31, 32, 33, 34, 35
- 1, 5, 6, 7, 8, 10, 13, 18, 19, 23, 24, 26, 32, 33, 34
- 2, 5, 11, 13, 14, 15, 18, 19, 20, 24, 27, 29, 31, 32, 36
- 2, 5, 6, 8, 9, 12, 13, 14, 15, 16, 19, 20, 23, 29, 31

The following four sets, also found by Djokovic [17], give $4\text{-(}43; 21, 21, 21, 15; 35)$ supplementary difference sets and may be used similarly to form a skew Hadamard matrix of order $4 \cdot 43 = 172$:

- 2, 3, 5, 7, 8, 12, 18, 19, 20, 22, 26, 27, 28, 29, 30, 32, 33, 34, 36, 39, 42 (twice)
- 1, 3, 4, 5, 6, 10, 11, 12, 16, 19, 20, 21, 23, 24, 31, 33, 35, 36, 38, 40, 41
- 0, 6, 7, 10, 18, 23, 24, 26, 28, 29, 30, 31, 34, 38, 40.

7.2. An Adaption of Wallis-Whiteman

We note the following adaption of the Goethals-Seidel matrix that does not require the matrix entries to be circulant at all:

Theorem 7.2 (J. Wallis-Whiteman [113]). *Suppose that X , Y , and W are type one incidence matrices and that Z is a type two incidence matrix of $4\{v; k_1, k_2, k_3, k_4; \sum_{i=1}^4 k_i - v\}$ supplementary difference sets. If*

$$A = 2X - J, \quad B = 2Y - J, \quad C = 2Z - J, \quad D = 2W - J,$$

then

$$H = \begin{bmatrix} A & B & C & D \\ -B^T & A^T & -D & C \\ -C & D^T & A & -B^T \\ -D^T & -C & B & A^T \end{bmatrix} \quad (7)$$

is an Hadamard matrix of order $4v$.

Further, if A is skew-type ($C^T = C$ as Z is of type two) then H is skew Hadamard.

This matrix can be used when the sets are from any finite abelian group. We now show how Theorem 7.2 may be further modified to obtain useful results.

Theorem 7.3 (J. Wallis-Whiteman [113]). *Suppose that X , Y , and W are type one incidence matrices and that Z is a type two incidence matrix of $4\{2m + 1; m; 2(m - 1)\}$ supplementary difference sets. If*

$$A = 2X - J, \quad B = 2Y - J, \quad C = 2Z - J, \quad D = 2W - J,$$

and e is the $1 \times (2m + 1)$ matrix with every entry 1, then

$$H = \begin{bmatrix} -1 & -1 & -1 & -1 & e & e & e & e \\ 1 & -1 & 1 & -1 & -e & e & -e & e \\ 1 & -1 & -1 & 1 & -e & e & e & -e \\ 1 & 1 & -1 & -1 & -e & -e & e & e \\ e^T & e^T & e^T & e^T & A & B & C & D \\ -e^T & e^T & -e^T & e^T & -B^T & A^T & -D & C \\ -e^T & e^T & e^T & -e^T & -C & D^T & A & -B^T \\ -e^T & -e^T & e^T & e^T & -D^T & -C & B & A^T \end{bmatrix}$$

is an Hadamard matrix of order $8(m + 1)$. Further, if A is skew type, H is skew Hadamard.

Delsarte, Goethals, and Seidel's [15] important result states that if there exists a $W(n, n - 1)$ for $n \equiv 0 \pmod{4}$, then there exists a skew symmetric $W(n, n - 1)$. This is used in the next result which uses orthogonal designs and is due to Seberry. The results for skew Hadamard matrices are far less complete than for Hadamard matrices.

Theorem 7.4 (Seberry [77]). *Let $q \equiv 5 \pmod{8}$ be a prime power and $p = \frac{1}{2}(q + 1)$ be a prime. Then there is a skew Hadamard matrix of order $2^t p$, where $t \leq [2\log_2(p - 2)]$.*

7.3. Summary and Tables of Skew Hadamard Orders

Skew Hadamard matrices are known for the following orders (the reader should consult [114, pp. 451], [77] and Geramita and Seberry [23]):

| | | |
|-------|--------------------|---|
| SI | $2^t \Pi k_i$ | t, r_i , all nonnegative positive integers $k_i - 1 \equiv 3 \pmod{4}$ a prime power [66] |
| SII | $(p - 1)^u + 1$ | p the order of a skew Hadamard matrix, $u > 0$ an odd integer [105] |
| SIII | $2(q + 1)$ | $q \equiv 5 \pmod{8}$ a prime power [98] |
| SIV | $2(q + 1)$ | $q = p^t$ is a prime power with $p \equiv 5 \pmod{8}$ and $t \equiv 2 \pmod{4}$ [99, 125] |
| SV | $4m$ | $m \in \{\text{odd integers between 3 and 31 inclusive}\}$ [35, 100]; $m \in \{37, 39, 43, 49, 65, 67, 93, 113, 121, 127, 129, 133, 157, 163, 181, 217, 219, 241, 267\}$ [17, 16] |
| SVI | $mn(n - 1)$ | n the order of amicable orthogonal designs of types $((1, n - 1); (n))$ and nm the order of an orthogonal design of type $(1, m, mn - m - 1)$ [77] |
| SVII | $4(q + 1)$ | $q \equiv 9 \pmod{16}$ a prime power [113] |
| SVIII | $(t + 1)(q + 1)$ | $q = s^2 + 4t^2 \equiv 5 \pmod{8}$ a prime power, and $ t + 1$ the order of a skew Hadamard matrix [117] |
| SIX | $4(q^2 + q + 1)$ | q a prime power and $q^2 + q + 1 \equiv 3, 5, \text{ or } 7 \pmod{8}$ a prime power or $2(q^2 + q + 1) + 1$ a prime power [94] |
| SX | $2^t q$ | $q = s^2 + 4r^2 \equiv 5 \pmod{8}$ a prime power, and an orthogonal design $\text{OD}(2^t; 1, a, b, c, c + r)$ exists where $1 + a + b + 2c + r = 2^t$ and $a(q + 1) + b(q - 4) = 2^t$ [77] |
| SXI | hm | h the order of a skew Hadamard matrix; m the order of amicable Hadamard matrices [121] |

Spence [95] has found a new construction for *SIV* and Whiteman [125] a new construction for *SI* when $k_i - 1 \equiv 3 \pmod{8}$. These are of considerable interest because of the structure involved and have use in the construction of orthogonal designs.

In Table 7.1, the lowest power of two for which a skew Hadamard matrix is known is indicated. For example, the entry (193,3) means a skew Hadamard matrix of order $2^3 \cdot 193$ is known, the entry (59,.) means a skew Hadamard matrix of order $2^t \cdot 59$ is not yet known for any t . Also, a blank represents 2.

8 M-STRUCTURES

Named after Mieko Yamada and Masahiko Miyamoto, *M*-structures have proved to be very powerful in attacking the question "if there is an Hadamard matrix of order $4t$, is there an Hadamard matrix of order $8t + 4$?" *M*-structures provide another variety of "plug in" matrices that have yet to be fully exploited.

Table A.1 gives the present knowledge of Williamson matrices. The theorems were applied to get the table.

Definition 8.1. An orthogonal matrix of order $4t$ can be divided into $16 t \times t$ blocks M_{ij} . This partitioned matrix is said to be an *M-structure*. If the orthogonal matrix can be partitioned into $64 s \times s$ blocks M_{ij} , it will be called a *64 block M-structure*.

An Hadamard matrix made from (symmetric) Williamson matrices W_1, W_2, W_3, W_4 is an *M-structure* with

$$\begin{aligned} W_1 &= M_{11} = M_{22} = M_{33} = M_{44}, \\ W_2 &= M_{12} = -M_{21} = M_{34} = -M_{43}, \\ W_3 &= M_{13} = -M_{31} = -M_{24} = M_{42}, \\ W_4 &= M_{14} = -M_{41} = M_{23} = -M_{32}. \end{aligned}$$

An Hadamard matrix made from four (4) circulant (or type 1) matrices A_1, A_2, A_3, A_4 of order n [where R is the matrix that makes all of the $A_i R$ back circulant (or type 2)] is an *M-structure* with

$$\begin{aligned} A_1 &= M_{11} = M_{22} = M_{33} = M_{44}, \\ A_2 &= M_{12}R = -M_{21}R = RM_{34}^T = -RM_{43}^T, \\ A_3 &= M_{13}R = -M_{31}R = -RM_{24}^T = RM_{42}^T, \\ A_4 &= M_{14}R = -M_{41}R = RM_{23}^T = -RM_{32}^T. \end{aligned}$$

TABLE 7.1 Orders for Which Skew Hadamard Matrices Exist

| q | t | q | t | q | t | q | t | q | t |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 | | 89 | 4 | 177 | . | 265 | 4 | 353 | 4 |
| 3 | | 91 | | 179 | 8 | 267 | | 355 | |
| 5 | | 93 | | 181 | | 269 | 8 | 357 | |
| 7 | | 95 | | 183 | | 271 | | 359 | 4 |
| 9 | | 97 | 9 | 185 | | 273 | | 361 | 3 |
| 11 | | 99 | | 187 | | 275 | 4 | 363 | |
| 13 | | 101 | 10 | 189 | | 277 | 5 | 365 | |
| 15 | | 103 | 3 | 191 | . | 279 | | 367 | |
| 17 | | 105 | | 193 | 3 | 281 | | 369 | 4 |
| 19 | | 107 | | 195 | | 283 | . | 371 | |
| 21 | | 109 | 9 | 197 | | 285 | 3 | 373 | 7 |
| 23 | | 111 | | 199 | | 287 | 4 | 375 | |
| 25 | | 113 | | 201 | 3 | 289 | 3 | 377 | 6 |
| 27 | | 115 | | 203 | | 291 | | 379 | |
| 29 | | 117 | | 205 | 3 | 293 | | 381 | |
| 31 | | 119 | 4 | 207 | | 295 | 5 | 383 | |
| 33 | | 121 | | 209 | 4 | 297 | | 385 | 3 |
| 35 | | 123 | | 211 | | 299 | 4 | 387 | |
| 37 | | 125 | | 213 | 4 | 301 | 3 | 389 | 15 |
| 39 | | 127 | | 215 | | 303 | 3 | 391 | 4 |
| 41 | | 129 | | 217 | | 305 | 4 | 393 | |
| 43 | | 131 | | 219 | | 307 | | 395 | |
| 45 | | 133 | | 221 | | 309 | 3 | 397 | 5 |
| 47 | 4 | 135 | | 223 | 3 | 311 | . | 399 | |
| 49 | | 137 | | 225 | 4 | 313 | | 401 | . |
| 51 | | 139 | | 227 | | 315 | | 403 | 5 |
| 53 | | 141 | | 229 | 3 | 317 | 6 | 405 | |
| 55 | | 143 | | 231 | | 319 | 3 | 407 | |
| 57 | | 145 | 5 | 233 | 4 | 321 | | 409 | 3 |
| 59 | | 147 | | 235 | 3 | 323 | | 411 | |
| 61 | | 149 | 4 | 237 | | 325 | 5 | 413 | 4 |
| 63 | | 151 | 5 | 239 | 4 | 327 | | 415 | |
| 65 | | 153 | 3 | 241 | | 329 | 6 | 417 | |
| 67 | | 155 | | 243 | | 331 | 3 | 419 | 4 |
| 69 | 3 | 157 | | 245 | 4 | 333 | | 421 | |
| 71 | | 159 | | 247 | 6 | 335 | 7 | 423 | 4 |
| 73 | | 161 | | 249 | 4 | 337 | 18 | 425 | |
| 75 | | 163 | | 251 | 6 | 339 | | 427 | |
| 77 | | 165 | | 253 | 4 | 341 | 4 | 429 | 3 |
| 79 | | 167 | 4 | 255 | | 343 | 6 | 431 | |
| 81 | 3 | 169 | 5 | 257 | 4 | 345 | 4 | 433 | 3 |
| 83 | | 171 | | 259 | 5 | 347 | . | 435 | 4 |
| 85 | | 173 | | 261 | 3 | 349 | 3 | 437 | |
| 87 | | 175 | | 263 | | 351 | | 439 | |

TABLE 7.1 Orders for Which Skew Hadamard Matrices Exist (continued)

| q | t | q | t | q | t | q | t | q | t |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 441 | 3 | 529 | 3 | 617 | | 705 | | 793 | 3 |
| 443 | 6 | 531 | | 619 | | 707 | 4 | 795 | 3 |
| 445 | 3 | 533 | | 621 | 3 | 709 | . | 797 | |
| 447 | | 535 | | 623 | 4 | 711 | | 799 | |
| 449 | . | 537 | 5 | 625 | 3 | 713 | | 801 | |
| 451 | 3 | 539 | 4 | 627 | 4 | 715 | | 803 | 9 |
| 453 | | 541 | 3 | 629 | . | 717 | 4 | 805 | 4 |
| 455 | 4 | 543 | 5 | 631 | . | 719 | 4 | 807 | |
| 457 | . | 545 | | 633 | | 721 | 5 | 809 | . |
| 459 | 3 | 547 | | 635 | | 723 | 3 | 811 | |
| 461 | 17 | 549 | 3 | 637 | 4 | 725 | 6 | 813 | |
| 463 | 7 | 551 | | 639 | | 727 | | 815 | |
| 465 | 3 | 553 | 3 | 641 | 6 | 729 | 4 | 817 | 5 |
| 467 | | 555 | | 643 | . | 731 | 5 | 819 | |
| 469 | 3 | 557 | . | 645 | | 733 | . | 821 | 6 |
| 471 | | 559 | | 647 | . | 735 | | 823 | 3 |
| 473 | 5 | 561 | | 649 | 7 | 737 | 7 | 825 | |
| 475 | 4 | 563 | | 651 | | 739 | . | 827 | |
| 477 | | 565 | 3 | 653 | . | 741 | | 829 | . |
| 479 | . | 567 | | 655 | 4 | 743 | | 831 | |
| 481 | 3 | 569 | 4 | 657 | 5 | 745 | 6 | 833 | |
| 483 | | 571 | 3 | 659 | . | 747 | | 835 | |
| 485 | 4 | 573 | 3 | 661 | . | 749 | . | 837 | |
| 487 | 5 | 575 | 4 | 663 | | 751 | 3 | 839 | . |
| 489 | 3 | 577 | . | 665 | | 753 | | 841 | 8 |
| 491 | . | 579 | 5 | 667 | 6 | 755 | | 843 | |
| 493 | 3 | 581 | 4 | 669 | 3 | 757 | | 845 | 6 |
| 495 | | 583 | 3 | 671 | | 759 | 4 | 847 | |
| 497 | | 585 | | 673 | 7 | 761 | . | 849 | 3 |
| 499 | | 587 | | 675 | | 763 | 11 | 851 | . |
| 501 | | 589 | 5 | 677 | | 765 | 4 | 853 | 3 |
| 503 | | 591 | | 679 | 3 | 767 | | 855 | |
| 505 | . | 593 | | 681 | 4 | 769 | 3 | 857 | 4 |
| 507 | | 595 | 3 | 683 | | 771 | | 859 | 3 |
| 509 | . | 597 | 4 | 685 | | 773 | . | 861 | 4 |
| 511 | | 599 | . | 687 | | 775 | | 863 | 4 |
| 513 | 4 | 601 | 5 | 689 | 4 | 777 | 4 | 865 | 4 |
| 515 | 5 | 603 | | 691 | | 779 | 4 | 867 | |
| 517 | 6 | 605 | 4 | 693 | 4 | 781 | 3 | 869 | 4 |
| 519 | 4 | 607 | | 695 | 4 | 783 | | 871 | |
| 521 | | 609 | 3 | 697 | 4 | 785 | 7 | 873 | |
| 523 | 7 | 611 | 6 | 699 | 3 | 787 | 5 | 875 | |
| 525 | | 613 | 3 | 701 | | 789 | 3 | 877 | |
| 527 | 4 | 615 | | 703 | 3 | 791 | | 879 | |

TABLE 7.1 Orders for Which Skew Hadamard Matrices Exist (continued)

| q | t | q | t | q | t | q | t | q | t |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 881 | 6 | 905 | 4 | 929 | . | 953 | . | 977 | |
| 883 | . | 907 | 5 | 931 | . | 955 | 3 | 979 | 5 |
| 885 | | 909 | 4 | 933 | . | 957 | 4 | 981 | |
| 887 | | 911 | | 935 | | 959 | 4 | 983 | |
| 889 | 5 | 913 | 4 | 937 | 5 | 961 | 3 | 985 | 3 |
| 891 | 3 | 915 | | 939 | | 963 | | 987 | |
| 893 | | 917 | 4 | 941 | 6 | 965 | 4 | 989 | 4 |
| 895 | | 919 | 3 | 943 | 4 | 967 | | 991 | 3 |
| 897 | 5 | 921 | 4 | 945 | | 969 | 4 | 993 | |
| 899 | 6 | 923 | | 947 | 6 | 971 | 6 | 995 | 4 |
| 901 | 3 | 925 | 3 | 949 | 3 | 973 | 4 | 997 | . |
| 903 | 4 | 927 | 4 | 951 | | 975 | | 999 | |

8.1. Multiplication Theorems Using M -Structures

In this section, the reader wishing more details of constructions is referred to Seberry and Yamada [87]. As shown in Section 3, the power of M -structures comprising wholly circulant or type one blocks permits them to be multiplied by the order of T -matrices.

Theorem 8.1. *Suppose that there is an M -structure orthogonal matrix of order $4m$ with each block circulant or type one. Then there is an M -structure orthogonal matrix of order $4mt$ where t is the order of T -matrices.*

Further,

Theorem 8.2. *Let $N = (N_{ij})$, $i, j = 1, 2, 3, 4$, be an Hadamard matrix of order $4n$ of M -structure. Further, let T_{ij} , $i, j = 1, 2, 3, 4$, be 16 $(0, +1, -1)$ type 1 or circulant matrices of order t that satisfy*

1. $T_{ij} * T_{ik} = 0$, $T_{ji} * T_{ki} = 0$, $j \neq k$ ($*$ is the Hadamard product);
 2. $\sum_{k=1}^4 T_{ik}$ is a $(1, -1)$ matrix;
 3. $\sum_{k=1}^4 T_{ik} T_{ik}^T = tI_t = \sum_{k=1}^4 T_{ki} T_{ki}^T$;
 4. $\sum_{k=1}^4 T_{ik} T_{jk}^T = 0 = \sum_{k=1}^4 T_{ki} T_{kj}^T$, $i \neq j$.
- (8)

Then there is an M -structure Hadamard matrix of order $4nt$.

Corollary 8.3. *If there exists an Hadamard matrix of order $4h$ and an orthogonal design $OD(4u; u_1, u_2, u_3, u_4)$, then an $OD(8hu; 2hu_1, 2hu_2, 2hu_3, 2hu_4)$ exists. In particular, the u_i 's can be equal.*

This gives the theorem of Agayan and Sarukhanyan [1] as a corollary by setting all variables equal to one:

Corollary 8.4. *If there exist Hadamard matrices of orders $4h$ and $4u$, then there exists an Hadamard matrix of order $8hu$.*

We now give as a corollary a result motivated by (and a little stronger than) that of Agayan and Sarukhanyan [1]:

Corollary 8.5. *Suppose that there are Williamson or Williamson-type matrices of orders u and v . Then there are Williamson-type matrices of order $2uv$. If the matrices of orders u and v are symmetric, the matrices of order $2uv$ are also symmetric. If the matrices of orders u and v are circulant and/or type one, the matrices of order $2uv$ are type 1.*

Proof. Suppose A, B, C, D are (symmetric) Williamson or Williamson type matrices of order u , then they are pairwise amicable. Define

$$E = \frac{1}{2}(A + B), \quad F = \frac{1}{2}(A - B), \quad G = \frac{1}{2}(C + D), \quad H = \frac{1}{2}(C - D),$$

then E, F, G, H are pairwise amicable (and symmetric) and satisfy

$$EE^T + FF^T + GG^T + HH^T = 2uI_u.$$

Now define

$$T_1 = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \quad T_2 = \begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & G \\ G & 0 \end{bmatrix},$$

$$\text{and } T_4 = \begin{bmatrix} 0 & H \\ H & 0 \end{bmatrix},$$

so that

$$T_1 = T_{11} = T_{22} = T_{33} = T_{44},$$

$$T_2 = T_{12} = -T_{21} = T_{34} = -T_{43},$$

$$T_3 = T_{13} = -T_{31} = -T_{24} = T_{42},$$

$$T_4 = T_{14} = -T_{41} = T_{23} = -T_{32},$$

in the theorem. Note that T_1, T_2, T_3, T_4 are pairwise amicable. If A, B, C, D were circulant (or type 1) they would be type 1 of order $2u$.

Let X, Y, Z, W be the Williamson or Williamson-type (symmetric) matrices of order v . Then X, Y, Z, W are pairwise amicable and

$$XX^T + YY^T + ZZ^T + WW^T = 4vI_v.$$

Then

$$L = T_1 \times X + T_2 \times Y + T_3 \times Z + T_4 \times W,$$

$$M = -T_1 \times Y + T_2 \times X + T_3 \times W - T_4 \times Z,$$

$$N = -T_1 \times Z - T_2 \times W + T_3 \times X + T_4 \times Y,$$

$$P = -T_1 \times W + T_2 \times Z - T_3 \times Y + T_4 \times X,$$

are 4 Williamson type (symmetric) matrices of order $2uv$. If the matrices of orders u and v were circulant or type 1, these matrices are type 1. \square

8.2. Miyamoto's Theorem and Corollaries via M-Structures

In this section, we reformulate Miyamoto's [64] results so that symmetric Williamson-type matrices can be obtained. The results given here are due to Miyamoto, Seberry, and Yamada.

Lemma 8.6 (Miyamoto's Lemma Reformulated by Seberry-Yamada [87]). *Let $U_i, V_j, i, j = 1, 2, 3, 4$, be $(0, +1, -1)$ matrices of order n that satisfy*

1. U_i, U_j are pairwise amicable, $i \neq j$;
2. V_i, V_j are pairwise amicable, $i \neq j$;
3. $U_i \pm V_i$ are $(+1, -1)$ matrices, $i = 1, 2, 3, 4$;
4. the row sum of U_1 is 1, and the row sum of U_j is zero, $i = 2, 3, 4$;
5. $\sum_{i=1}^4 U_i U_i^T = (2n + 1)I - 2J$, $\sum_{i=1}^4 V_i V_i^T = (2n + 1)I$.

Then there are four Williamson type matrices of order $2n + 1$. Hence, there is a Williamson-type Hadamard matrix of order $4(2n + 1)$. If U_i and V_i are symmetric, $i = 1, 2, 3, 4$, then the Williamson type matrices are symmetric.

Proof. Let S_1, S_2, S_3, S_4 be 4 $(+1, -1)$ -matrices of order $2n$ defined by

$$S_j = U_j \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_j \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

So the row sum of $S_1 = 2$ and of $S_i = 0, i = 2, 3, 4$. Now define

$$X_1 = \begin{bmatrix} 1 & -e_{2n} \\ -e_{2n}^T & S_1 \end{bmatrix} \quad \text{and} \quad X_i = \begin{bmatrix} 1 & e_{2n} \\ e_{2n}^T & S_i \end{bmatrix}, \quad i = 2, 3, 4.$$

First, note that since $U_i, U_j, i \neq j$, and $V_i, V_j, i \neq j$, are pairwise amicable,

$$\begin{aligned} S_i S_j^T &= \left(U_i \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_i \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \left(U_j^T \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_j^T \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \\ &= U_i U_j^T \times \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + V_i V_j^T \times \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \\ &= S_j S_i^T. \end{aligned}$$

(Note that this relationship is valid if and only if conditions (1) and (2) of the theorem are valid.)

$$\begin{aligned} \sum_{i=1}^4 S_i S_i^T &= \sum_{i=1}^4 U_i U_i^T \times \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \sum_{i=1}^4 V_i V_i^T \times \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \\ &= 2 \begin{bmatrix} 2(2n+1)I - 2J & -2J \\ -2J & 2(2n+1)I - 2J \end{bmatrix} \\ &= 4(2n+1)I_{2n} - 4J_{2n}. \end{aligned}$$

Next, we observe that

$$X_1 X_i^T = \begin{bmatrix} 1-2n & e_{2n} \\ e_{2n}^T & -J + S_1 S_i^T \end{bmatrix} = X_i X_1^T, \quad i = 2, 3, 4,$$

and

$$X_i X_j^T = \begin{bmatrix} 1+2n & e_{2n} \\ e_{2n}^T & J + S_i S_j^T \end{bmatrix} = X_j X_i^T, \quad i \neq j, \quad i, j = 2, 3, 4.$$

Further,

$$\begin{aligned} \sum_{i=1}^4 X_i X_i^T &= \begin{bmatrix} 1+2n & -3e_{2n} \\ -3e_{2n}^T & J + S_1 S_1^T \end{bmatrix} + \sum_{i=2}^4 \begin{bmatrix} 1+2n & e_{2n} \\ e_{2n}^T & J + S_i S_i^T \end{bmatrix} \\ &= \begin{bmatrix} 4(2n+1) & 0 \\ 0 & 4J + 4(2n+1)I - 4J \end{bmatrix}. \end{aligned}$$

Thus, we have shown that X_1, X_2, X_3, X_4 are 4 Williamson-type matrices of order $2n+1$. Hence, there is a Williamson-type Hadamard matrix of order $4(2n+1)$. \square

Many powerful corollaries which give many new results exist by suitable choices in the theorem. For example,

Corollary 8.7. *Let $q \equiv 1 \pmod{4}$ be a prime power. Then there are symmetric Williamson-type matrices of order $q + 2$ whenever $\frac{1}{2}(q + 1)$ is a prime power or $\frac{1}{2}(q + 3)$ is the order of a symmetric conference matrix. Also, there exists an Hadamard matrix of Williamson type of order $4(q + 2)$.*

Corollary 8.8. *Let $q \equiv 1 \pmod{4}$ be a prime power. Then*

1. *if there are Williamson type matrices of order $(q - 1)/4$ or an Hadamard matrix of order $\frac{1}{2}(q - 1)$, there exist Williamson type matrices of order q ;*
2. *if there exist symmetric conference matrices of order $\frac{1}{2}(q - 1)$ or a symmetric Hadamard matrix of order $\frac{1}{2}(q - 1)$, then there exist symmetric Williamson type matrices of order q .*

Hence, there exists an Hadamard matrix of Williamson type of order $4q$.

Corollary 8.9. *Let $q \equiv 1 \pmod{4}$ be a prime power or $q + 1$ be the order of a symmetric conference matrix. Let $2q - 1$ be a prime power. Then there exist symmetric Williamson type matrices of order $2q + 1$ and an Hadamard matrix of Williamson type of order $4(2q + 1)$.*

Note that this last corollary is a modified version of Miyamoto's Corollary 5 (original manuscript).

Theorem 8.10 (Miyamoto's Theorem [64] reformulated by Seberry-Yamada [87]). *Let $U_{ij}, V_{ij}, i, j = 1, 2, 3, 4$, be $(0, +1, -1)$ matrices of order n that satisfy*

1. *U_{ki}, U_{kj} are pairwise amicable, $k = 1, 2, 3, 4, i \neq j$;*
2. *V_{ki}, V_{kj} are pairwise amicable, $k = 1, 2, 3, 4, i \neq j$;*
3. *$U_{ki} \pm V_{ki}$ are $(+1, -1)$ matrices, $i, k = 1, 2, 3, 4$;*
4. *the row sum of U_{ii} is 1, and the row sum of U_{ij} is zero, $i \neq j, i, j = 1, 2, 3, 4$;*
5. *$\sum_{i=1}^4 U_{ji} U_{ji}^T = (2n + 1)I - 2J, \sum_{i=1}^4 V_{ji} V_{ji}^T = (2n + 1)I, j = 1, 2, 3, 4$;*
6. *$\sum_{i=1}^4 U_{ji} U_{ki}^T = 0, \sum_{i=1}^4 V_{ji} V_{ki}^T = 0, j \neq k, j, k = 1, 2, 3, 4$.*

If conditions 1 to 5 hold, there are four Williamson-type matrices of order $2n + 1$ and thus a Williamson type Hadamard matrix of order $4(2n + 1)$. Furthermore, if the matrices U_{ki} and V_{ki} are symmetric for all $i, j = 1, 2, 3, 4$, the Williamson matrices obtained of order $2n + 1$ are also symmetric.

If conditions 3 to 6 hold, there is an M-structure Hadamard matrix of order $4(2n + 1)$.

Proof. We prove the first assertion. Let $S_{ij}, i, j = 1, 2, 3, 4$, be $16 (+1, -1)$ -matrices of order $2n$ defined by

$$S_{ij} = U_{ij} \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_{ij} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

So the row sum of $S_{ii} = 2$ and of $S_{ij} = 0$, $i \neq j$, $i, j = 1, 2, 3, 4$. Now define

$$\begin{aligned} X_{11} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{11} \end{bmatrix}, & X_{12} &= \begin{bmatrix} 1 & e \\ e^T & S_{12} \end{bmatrix}, & X_{13} &= \begin{bmatrix} 1 & e \\ e^T & S_{13} \end{bmatrix}, & X_{14} &= \begin{bmatrix} -1 & e \\ e^T & S_{14} \end{bmatrix}, \\ X_{21} &= \begin{bmatrix} 1 & e \\ e^T & S_{21} \end{bmatrix}, & X_{22} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{22} \end{bmatrix}, & X_{23} &= \begin{bmatrix} 1 & e \\ e^T & S_{23} \end{bmatrix}, & X_{24} &= \begin{bmatrix} -1 & e \\ e^T & S_{24} \end{bmatrix}, \\ X_{31} &= \begin{bmatrix} 1 & e \\ e^T & S_{31} \end{bmatrix}, & X_{32} &= \begin{bmatrix} 1 & e \\ e^T & S_{32} \end{bmatrix}, & X_{33} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{33} \end{bmatrix}, & X_{34} &= \begin{bmatrix} -1 & e \\ e^T & S_{34} \end{bmatrix}, \\ X_{41} &= \begin{bmatrix} -1 & e \\ e^T & -S_{41} \end{bmatrix}, & X_{42} &= \begin{bmatrix} 1 & e \\ e^T & -S_{42} \end{bmatrix}, & X_{43} &= \begin{bmatrix} -1 & e \\ e^T & -S_{43} \end{bmatrix}, & X_{44} &= \begin{bmatrix} -1 & -e \\ -e^T & -S_{44} \end{bmatrix}. \end{aligned}$$

Thus, $X_{41}, X_{42}, X_{43}, X_{44}$ are 4 Williamson-type matrices of order $2n + 1$, and thus a Williamson-type Hadamard matrix of order $4(2n + 1)$ exists. \square

Note that if we write our M -structure from the theorem as

$$\begin{array}{cccccccc} -1 & 1 & 1 & -1 & -e & e & e & e \\ 1 & -1 & 1 & -1 & e & -e & e & e \\ 1 & 1 & -1 & -1 & e & e & -e & e \\ 1 & 1 & 1 & 1 & -e & -e & -e & e \\ -e^T & e^T & e^T & e^T & S_{11} & S_{12} & S_{13} & S_{14} \\ e^T & -e^T & e^T & e^T & S_{21} & S_{22} & S_{23} & S_{24} \\ e^T & e^T & -e^T & e^T & S_{31} & S_{32} & S_{33} & S_{34} \\ -e^T & -e^T & -e^T & e^T & S_{41} & S_{42} & S_{43} & S_{44} \end{array}$$

then we can see Yamada's matrix with trimming [131] or the J. Wallis-Whiteman [113] matrix with a border embodied in the construction.

Corollary 8.11. *Suppose that there exists a symmetric conference matrix of order $q + 1 = 4t + 2$ and an Hadamard matrix of order $4t = q - 1$. Then there is an Hadamard matrix with M -structure of order $4(4t + 1) = 4q$. Further, if the Hadamard matrix is symmetric, the Hadamard matrix of order $4q$ is of the form*

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix},$$

where X, Y are amicable and symmetric.

In a similar fashion, we consider the following lemma so symmetric 8-Williamson-type matrices can be obtained.

Lemma 8.12 (Seberry-Yamada [87]). *Let $U_i, V_j, i, j = 1, \dots, 8$, be $(0, +1, -1)$ matrices of order n that satisfy*

1. $U_i, U_j, i \neq j$ are pairwise amicable;
2. $V_i, V_j, i \neq j$ are pairwise amicable;
3. $U_i \pm V_i$ are $(+1, -1)$ matrices, $i = 1, \dots, 8$;
4. the row (column) sums of U_1 and U_2 are both 1, and the row sum of $U_i, i = 3, \dots, 8$ is zero;
5. $\sum_{i=1}^8 U_i U_i^T = 2(2n+1)I - 4J, \sum_{i=1}^8 V_i V_i^T = 2(2n+1)I$.

Then there are 8-Williamson-type matrices of order $2n+1$. Furthermore, if the U_i and V_i are symmetric, $i = 1, \dots, 8$, then the 8-Williamson-type matrices are symmetric. Hence, there is a block-type Hadamard matrix of order $8(2n+1)$.

Proof. Let S_1, \dots, S_8 be 8 $(+1, -1)$ -matrices of order $2n$ defined by

$$S_j = U_j \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_j \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

So the row sums of S_1 and S_2 are both 2 and those of S_i are 0, $i = 3, \dots, 8$. Now define

$$X_j = \begin{bmatrix} 1 & -e_{2n} \\ -e_{2n}^T & S_j \end{bmatrix}, \quad j = 1, 2, \quad \text{and}$$

$$X_i = \begin{bmatrix} 1 & e_{2n} \\ e_{2n}^T & S_i \end{bmatrix}, \quad i = 3, \dots, 8.$$

Thus, we have that X_1, \dots, X_8 are 8-Williamson type matrices of order $2n+1$.

Hence, there is a block-type Hadamard matrix of order $8(2n+1)$ obtained by replacing the variables of an orthogonal design $OD(8; 1, 1, 1, 1, 1, 1, 1, 1)$ by the 8-Williamson-type matrices. \square

Some very powerful corollaries are

Corollary 8.13 [87]. *Let $q+1$ be the order of amicable Hadamard matrices $I+S$ and P . Suppose that there exist 4 Williamson-type matrices of order q . Then there exist Williamson-type matrices of order $2q+1$. Furthermore, there exists a 64 block M-structure Hadamard matrix of order $8(2q+1)$.*

Corollary 8.14. *Let q be a prime power and let $(q-1)/2$ be the order of (symmetric) 4 Williamson-type matrices. Then there exist (symmetric) 8 Williamson-type matrices of order q and a 64-block M-structure Hadamard matrix of order $8q$.*

Corollary 8.15. *Let $q \equiv 1 \pmod{4}$ be a prime power or $q + 1$ be the order of a symmetric conference matrix. Suppose that there exist (symmetric) 4 Williamson-type matrices of order q . Then there exist (symmetric) 8-Williamson-type matrices of order $2q + 1$ and a 64-block M -structure Hadamard matrix of order $8(2q + 1)$.*

Proof. Form the core Q . Thus, we choose

$$U_1 = I + Q, \quad U_2 = I - Q, \quad U_3 = U_4 = Q, \quad U_5 = U_6 = U_7 = U_8 = 0,$$

$$\text{and} \quad V_1 = V_2 = 0, \quad V_3 = V_4 = I, \quad V_{i+4} = W_i,$$

$i = 1, 2, 3, 4$, where W_i are (symmetric) Williamson-type matrices. Then

$$\sum_{i=1}^8 U_i U_i^T = 2(2q + 1)I - 4J, \quad \sum_{i=1}^8 V_i V_i^T = 2(2q + 1)I.$$

These U_i and V_i are then used in Lemma 8.12 to obtain the (symmetric) 8-Williamson-type matrices. \square

This corollary gives 8-Williamson-type matrices for many new orders, but it does not give new Hadamard matrices for these orders.

Corollary 8.16 [87]. *Let $q = 9^t$, $t > 0$. There exist (symmetric) 4 Williamson-type matrices of order 9^t , $t > 0$. Hence, there exist (symmetric) 8-Williamson type matrices of order $2 \cdot 9^t + 1$, $t > 0$, and an Hadamard matrix of block structure of order $8(2 \cdot 9^t + 1)$.*

Also we have the following theorem:

Theorem 8.17 (Seberry-Yamada [87]). *Let U_{ij}, V_{ij} , $i, j = 1, \dots, 8$, be $(0, +1, -1)$ matrices of order n that satisfy*

1. U_{ki}, U_{kj} are pairwise amicable, $k = 1, \dots, 8$, $i \neq j$;
2. V_{ki}, V_{kj} are pairwise amicable, $k = 1, \dots, 8$, $i \neq j$;
3. $U_{ki} \pm V_{ki}$ are $(+1, -1)$ matrices, $i, k = 1, \dots, 8$;
4. the row (column) sum of U_{ab} is 1 for $(a, b) \in \{(i, i), (i, i + 1), (i + 1, i)\}$, $i = 1, 3, 5, 7$; the row (column) sum of U_{aa} is -1 for $a = 2, 4, 6, 8$; and otherwise, the row (column) sum of U_{ij} , $i \neq j$ is zero;
5. $\sum_{i=1}^8 U_{ji} U_{ji}^T = 2(2n + 1)I - 4J$, $\sum_{i=1}^8 V_{ji} V_{ji}^T = 2(2n + 1)I$, $j = 1, \dots, 8$;
6. $\sum_{i=1}^8 U_{ji} U_{ki}^T = 0$, $\sum_{i=1}^8 V_{ji} V_{ki}^T = 0$, $j \neq k$, $j, k = 1, \dots, 8$.

If conditions 1 to 5 hold, there are 8-Williamson-type matrices of order $2n + 1$ and thus a block-type Hadamard matrix of order $8(2n + 1)$. Further, if U_{7i}, V_{7i} are symmetric, $1 \leq i \leq 8$, then the 8-Williamson-type matrices are symmetric.

If conditions 3 to 6 hold, there is a 64-block M -structure Hadamard matrix of order $8(2n + 1)$.

Proof. Let S_{ij} be 64 $(+1, -1)$ -matrices of order $2n$ defined by

$$S_{ij} = U_{ij} \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_{ij} \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

So the row (column) sum of S_{ii} , $S_{i,i+1}$, $S_{i+1,i}$ $i = 1, 3, 5, 7$, is 2, the row (column) sum of S_{ii} is -2 for (i, i) , $i = 2, 4, 6, 8$, and otherwise, the row (column) sum of $S_{ij} = 0$, $i \neq j$. Now define

$$\begin{aligned} X_{11} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{11} \end{bmatrix}, & X_{12} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{12} \end{bmatrix}, & X_{13} &= \begin{bmatrix} 1 & e \\ e^T & S_{13} \end{bmatrix}, & X_{14} &= \begin{bmatrix} 1 & e \\ e^T & S_{14} \end{bmatrix}, \\ X_{15} &= \begin{bmatrix} 1 & e \\ e^T & S_{15} \end{bmatrix}, & X_{16} &= \begin{bmatrix} 1 & e \\ e^T & S_{16} \end{bmatrix}, & X_{17} &= \begin{bmatrix} -1 & e \\ e^T & S_{17} \end{bmatrix}, & X_{18} &= \begin{bmatrix} -1 & e \\ e^T & S_{18} \end{bmatrix}, \\ X_{21} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{21} \end{bmatrix}, & X_{22} &= \begin{bmatrix} 1 & e \\ e^T & S_{22} \end{bmatrix}, & X_{23} &= \begin{bmatrix} 1 & e \\ e^T & S_{23} \end{bmatrix}, & X_{24} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{24} \end{bmatrix}, \\ X_{25} &= \begin{bmatrix} 1 & e \\ e^T & S_{25} \end{bmatrix}, & X_{26} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{26} \end{bmatrix}, & X_{27} &= \begin{bmatrix} -1 & e \\ e^T & S_{27} \end{bmatrix}, & X_{28} &= \begin{bmatrix} 1 & -e \\ -e^T & S_{28} \end{bmatrix}, \\ X_{31} &= \begin{bmatrix} 1 & e \\ e^T & S_{31} \end{bmatrix}, & X_{32} &= \begin{bmatrix} 1 & e \\ e^T & S_{32} \end{bmatrix}, & X_{33} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{33} \end{bmatrix}, & X_{34} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{34} \end{bmatrix}, \\ X_{35} &= \begin{bmatrix} 1 & e \\ e^T & S_{35} \end{bmatrix}, & X_{36} &= \begin{bmatrix} 1 & e \\ e^T & S_{36} \end{bmatrix}, & X_{37} &= \begin{bmatrix} -1 & e \\ e^T & S_{37} \end{bmatrix}, & X_{38} &= \begin{bmatrix} -1 & e \\ e^T & S_{38} \end{bmatrix}, \\ X_{41} &= \begin{bmatrix} 1 & e \\ e^T & S_{41} \end{bmatrix}, & X_{42} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{42} \end{bmatrix}, & X_{43} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{43} \end{bmatrix}, & X_{44} &= \begin{bmatrix} 1 & e \\ e^T & S_{44} \end{bmatrix}, \\ X_{45} &= \begin{bmatrix} 1 & e \\ e^T & S_{45} \end{bmatrix}, & X_{46} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{46} \end{bmatrix}, & X_{47} &= \begin{bmatrix} -1 & e \\ e^T & S_{47} \end{bmatrix}, & X_{48} &= \begin{bmatrix} 1 & -e \\ -e^T & S_{48} \end{bmatrix}, \\ X_{51} &= \begin{bmatrix} 1 & e \\ e^T & S_{51} \end{bmatrix}, & X_{52} &= \begin{bmatrix} 1 & e \\ e^T & S_{52} \end{bmatrix}, & X_{53} &= \begin{bmatrix} 1 & e \\ e^T & S_{53} \end{bmatrix}, & X_{54} &= \begin{bmatrix} 1 & e \\ e^T & S_{54} \end{bmatrix}, \\ X_{55} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{55} \end{bmatrix}, & X_{56} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{56} \end{bmatrix}, & X_{57} &= \begin{bmatrix} -1 & e \\ e^T & S_{57} \end{bmatrix}, & X_{58} &= \begin{bmatrix} -1 & e \\ e^T & S_{58} \end{bmatrix}, \\ X_{61} &= \begin{bmatrix} 1 & e \\ e^T & S_{61} \end{bmatrix}, & X_{62} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{62} \end{bmatrix}, & X_{63} &= \begin{bmatrix} 1 & e \\ e^T & S_{63} \end{bmatrix}, & X_{64} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{64} \end{bmatrix}, \\ X_{65} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{65} \end{bmatrix}, & X_{66} &= \begin{bmatrix} 1 & e \\ e^T & S_{66} \end{bmatrix}, & X_{67} &= \begin{bmatrix} -1 & e \\ e^T & S_{67} \end{bmatrix}, & X_{68} &= \begin{bmatrix} 1 & -e \\ -e^T & S_{68} \end{bmatrix}, \\ X_{71} &= \begin{bmatrix} 1 & -e \\ -e^T & S_{71} \end{bmatrix}, & X_{72} &= \begin{bmatrix} 1 & -e \\ -e^T & S_{72} \end{bmatrix}, & X_{73} &= \begin{bmatrix} 1 & -e \\ -e^T & S_{73} \end{bmatrix}, & X_{74} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{74} \end{bmatrix}, \\ X_{75} &= \begin{bmatrix} 1 & -e \\ -e^T & S_{75} \end{bmatrix}, & X_{76} &= \begin{bmatrix} 1 & -e \\ -e^T & S_{76} \end{bmatrix}, & X_{77} &= \begin{bmatrix} 1 & e \\ e^T & S_{77} \end{bmatrix}, & X_{78} &= \begin{bmatrix} 1 & e \\ e^T & S_{78} \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
 X_{81} &= \begin{bmatrix} 1 & -e \\ -e^T & S_{81} \end{bmatrix}, & X_{82} &= \begin{bmatrix} -1 & e \\ e^T & S_{82} \end{bmatrix}, & X_{83} &= \begin{bmatrix} 1 & -e \\ -e^T & S_{83} \end{bmatrix}, & X_{84} &= \begin{bmatrix} -1 & e \\ e^T & S_{84} \end{bmatrix}, \\
 X_{85} &= \begin{bmatrix} 1 & -e \\ -e^T & S_{85} \end{bmatrix}, & X_{86} &= \begin{bmatrix} -1 & e \\ e^T & S_{86} \end{bmatrix}, & X_{87} &= \begin{bmatrix} 1 & e \\ e^T & S_{87} \end{bmatrix}, & X_{88} &= \begin{bmatrix} -1 & -e \\ -e^T & S_{88} \end{bmatrix}.
 \end{aligned}$$

Then provided conditions 1 to 5 hold, and $S_{7i}^T = S_{7i}$, $i = 1, \dots, 8$, are symmetric, X_{7i} , $i = 1, \dots, 8$, are symmetric 8-Williamson-type matrices. Otherwise, X_{7i} , $i = 1, \dots, 8$, are 8-Williamson-type matrices. This can be verified by straightforward checking. Hence, there is an Hadamard matrix of block structure of order $8(2n + 1)$.

If conditions 3 to 6 hold, then straightforward verification shows the 64-block M -structure X_{ij} is an Hadamard matrix of order $8(2n + 1)$. \square

Corollary 8.18. *Let q be an odd prime power, and suppose that there exist Williamson-type matrices of order $\frac{1}{2}(q - 1)$. Then there exists an M -structure Hadamard matrix of order $8q$.*

Corollary 8.19. *Let $q = 2m + 1 \equiv 9 \pmod{16}$ be a prime power. Suppose that there are Williamson-type matrices of order q . Then there is a M -structure Hadamard matrix of order $8(2q + 1)$.*

The analogous Yamada-J. Wallis-Whiteman structure to Theorem 8.17 is

| | | | | | | | | | | | | | | | | |
|--------|--------|--------|--------|--------|--------|-------|--------|----------|----------|----------|----------|----------|----------|----------|----------|----|
| -1 | -1 | 1 | 1 | 1 | 1 | -1 | -1 | -e | -e | e | e | e | e | e | e | e |
| -1 | 1 | 1 | 1 | -1 | 1 | -1 | 1 | -e | e | e | -e | e | -e | e | -e | -e |
| 1 | 1 | -1 | -1 | 1 | 1 | -1 | -1 | e | e | -e | -e | e | e | e | e | e |
| 1 | -1 | -1 | 1 | 1 | -1 | -1 | 1 | e | -e | -e | e | e | -e | e | -e | -e |
| 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | e | e | e | e | -e | -e | e | e | e |
| 1 | -1 | 1 | -1 | -1 | 1 | -1 | 1 | e | -e | e | -e | -e | e | e | -e | -e |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | -e | -e | -e | -e | -e | -e | -e | e | e |
| 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | -e | e | -e | -e | e | e | -e | -e | e |
| $-e^T$ | $-e^T$ | e^T | e^T | e^T | e^T | e^T | e^T | S_{11} | S_{12} | S_{13} | S_{14} | S_{15} | S_{16} | S_{17} | S_{18} | |
| $-e^T$ | e^T | e^T | $-e^T$ | e^T | $-e^T$ | e^T | $-e^T$ | S_{21} | S_{22} | S_{23} | S_{24} | S_{25} | S_{26} | S_{27} | S_{28} | |
| e^T | e^T | $-e^T$ | $-e^T$ | e^T | e^T | e^T | e^T | S_{31} | S_{32} | S_{33} | S_{34} | S_{35} | S_{36} | S_{37} | S_{38} | |
| e^T | $-e^T$ | $-e^T$ | e^T | e^T | $-e^T$ | e^T | $-e^T$ | S_{41} | S_{42} | S_{43} | S_{44} | S_{45} | S_{46} | S_{47} | S_{48} | |
| e^T | e^T | e^T | e^T | $-e^T$ | $-e^T$ | e^T | e^T | S_{51} | S_{52} | S_{53} | S_{54} | S_{55} | S_{56} | S_{57} | S_{58} | |
| e^T | $-e^T$ | e^T | $-e^T$ | $-e^T$ | e^T | e^T | $-e^T$ | S_{61} | S_{62} | S_{63} | S_{64} | S_{65} | S_{66} | S_{67} | S_{68} | |
| $-e^T$ | $-e^T$ | $-e^T$ | $-e^T$ | $-e^T$ | $-e^T$ | e^T | e^T | S_{71} | S_{72} | S_{73} | S_{74} | S_{75} | S_{76} | S_{77} | S_{78} | |
| $-e^T$ | e^T | $-e^T$ | e^T | $-e^T$ | e^T | e^T | $-e^T$ | S_{81} | S_{82} | S_{83} | S_{84} | S_{85} | S_{86} | S_{87} | S_{88} | |

With some trimming, we can see Yamada's matrix [131] or the J. Wallis-Whiteman [113] matrix with a border embodied in the construction. Miyamoto has done further work using the quaternions rather than the complex numbers to build bigger M -structures [64]. This work is probably further extendable.

9 WILLIAMSON AND WILLIAMSON-TYPE MATRICES

In the previous section, we saw many constructions for Williamson-type matrices using M -structures. *Williamson matrices* and *Williamson-type matrices* were defined in Section 3. They are the most used "plug in" matrices and give many previously unknown Hadamard matrices.

Williamson's famous theorem is

Theorem 9.1 (Williamson [128]). *Suppose that there exist four symmetric $(1, -1)$ matrices A, B, C, D of order n that commute in pairs. Further, suppose that*

$$A^2 + B^2 + C^2 + D^2 = 4nI_n.$$

Then

$$H = \begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix} \quad (9)$$

is an Hadamard matrix of order $4n$ of Williamson type or quaternion type.

Theorem 9.2 (Williamson). *If there exist solutions to the equations*

$$\mu_i = 1 + 2 \left\{ \sum_{j=1}^s t_{ij} (w^j + w^{n-j}) \right\}, \quad i = 1, 2, 3, 4$$

where $s = \frac{1}{2}(n-1)$, w is an n th root of unity, exactly one of $t_{1j}, t_{2j}, t_{3j}, t_{4j}$ is nonzero and equals ± 1 for each $j = 1, 2, \dots, s$, and

$$\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 = 4n,$$

then there exist matrices A, B, C, D satisfying Theorem 9.1 of the form

$$A = \sum_{i=0}^{n-1} a_i T^i, \quad a_0 = 1, \quad a_i = a_{n-i} = \pm 1;$$

$$B = \sum_{i=0}^{n-1} b_i T^i, \quad b_0 = 1, \quad b_i = b_{n-i} = \pm 1;$$

$$C = \sum_{i=0}^{n-1} c_i T^i, \quad c_0 = 1, \quad c_i = c_{n-i} = \pm 1;$$

$$D = \sum_{i=0}^{n-1} d_i T^i, \quad d_0 = 1, \quad d_i = d_{n-1-i} = \pm 1.$$

where T is the matrix whose (i, j) entry is 1 if $j - i \equiv 1 \pmod{n}$ and 0 otherwise. Hence, there exists an Hadamard matrix of order $4n$.

Table 9.1 shows the μ_i found by Williamson [128], Baumert and Hall [5], Djokovic [18], Koukouvinos and Kounias [52], and Sawade [74]. We write w_j for $w^j + w^{n-j}$ and w_{2j} for $w^{2j} + w^{n-2j}$. Williamson found the results for 148 and 172, Baumert and Hall for 92, Baumert for 116, Sawade for 100 and 108, Koukouvinos and Kounias for 132, and Djokovic for 156. Results have also appeared in Baumert [3, 4], Koukouvinos [49], and Yamada [130].

Note: The sums of squares in Table 9.1 are not necessarily those of the corresponding ± 1 matrix. For example, the ± 1 matrices corresponding to $92 = 1^2 + 1^2 + (-3)^2 + 9^2$ have row sums 3, 3, 7, -5.

Example 9.1. How to turn the formulas in Table 9.1 into Williamson matrices? Let $t = 13$, $n = 52$, $\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 = 1^2 + 1^2 + 1^2 + 7^2$. Form four sums:

$$\sigma_1 = -\mu_1 + \mu_2 + \mu_3 + \mu_4 = 2 + 2w_1 - 2w_2 - 2w_3 - 2w_4 + 2w_5 - 2w_6,$$

$$\sigma_2 = \mu_1 - \mu_2 + \mu_3 + \mu_4 = 2 + 2w_1 - 2w_2 - 2w_3 - 2w_4 + 2w_5 - 2w_6,$$

$$\sigma_3 = \mu_1 + \mu_2 - \mu_3 + \mu_4 = 2 - 2w_1 - 2w_2 - 2w_3 + 2w_4 - 2w_5 + 2w_6,$$

$$\sigma_4 = \mu_1 + \mu_2 + \mu_3 - \mu_4 = 2 + 2w_1 + 2w_2 + 2w_3 - 2w_4 + 2w_5 - 2w_6.$$

Then, recalling $w_i = w^i + w^{n-i}$, we use $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ to form the first rows (coefficients of T^i) of the circulant matrices A, B, C, D , respectively. σ_1 gives a_0, a_2, \dots, a_{12} as

$$a_0 = 1, \quad a_1 = a_{12} = 1, \quad a_2 = a_{11} = -1, \quad a_3 = a_{10} = -1,$$

$$a_4 = a_9 = -1, \quad a_5 = a_8 = 1, \quad a_6 = a_7 = -1$$

so the first row of A is

$$1 \ 1 \ \dots \ 1 \ \dots \ 1 \ \dots \ 1 \quad \text{and} \quad \frac{1}{4}\sigma_1^2 = (-3)^2.$$

For B, C, D , we have

$$\begin{array}{lcl} 1 & 1 & \text{and } \frac{1}{4}\sigma_2^2 = (-3)^2, \\ \text{---} & \text{---} & \\ 1 & \text{---} & \\ \text{---} & 1 & \text{and } \frac{1}{4}\sigma_3^2 = (-3)^2, \\ \text{---} & \text{---} & \\ 1 & 1 & \\ 1 & \text{---} & \text{and } \frac{1}{4}\sigma_4^2 = 5^2, \\ 1 & \text{---} & \\ 1 & 1 & \end{array}$$

where $4n = 52 = 3^2 + 3^2 + 3^2 + 5^2$.

We now introduce some matrices that were first used by Seberry and Whiteman [85] in the construction of conference matrices. Matrices obeying the same equations are constructed using auxiliary matrices from projective planes in [80].

Suppose that B_1, B_2, \dots, B_s are square $(1, -1)$ matrices of order b that satisfy

$$\begin{aligned} B_i^2 &= B_i B_j = J, & i, j \in \{1, 2, \dots, s\}; \\ B_i B_j^T &= B_j^T B_i = J, & i \neq j, \quad i, j \in \{1, 2, \dots, s\}; \\ B_i J &= aJ, & a \in \mathbb{Z}^+; \end{aligned} \quad (10)$$

$$\sum_{i=1}^s B_i B_i^T + B_i^T B_i = 2sbI_b.$$

Call s matrices satisfying equations (10) a *regular s -set* of matrices. Define, in particular,

$$\begin{aligned} A_i &= B_i \times \frac{1}{2}(B + B^T) + B_{i+1} \times \frac{1}{2}(B - B^T), & i = 1, 3, \dots, s-1, \\ A_{i+1} &= -B_i \times \frac{1}{2}(C - C^T) + B_{i+1} \times \frac{1}{2}(C + C^T), \end{aligned}$$

where B, C is a regular 2-set and $B_j, j = 1, \dots, s$, is a regular s -set of matrices. Then A_1, \dots, A_s is a regular s -set of matrices. Thus, we have

Lemma 9.3. *If there exists a regular s -set of matrices of order a , and a regular 2-set of order b , then there exists a regular s -set of order ab .*

So in the special case $s = t = 2$, if A_1, A_2 is a regular 2-set of order a and B_1, B_2 is a regular 2-set of order b , then C_1, C_2 is a regular 2-set of order $c = ab$.

In Seberry and Whiteman [85], it is shown that

Theorem 9.4 (Seberry-Whiteman). *If $n \equiv 3 \pmod{4}$ is a prime power, then there exists a regular $\frac{1}{2}(n+1)$ -set of matrices of order n^2 .*

In particular, if $n = 3$, there is a regular 2-set of matrices of order 9. Hence, using Lemma 9.3, we have a regular 2-set of matrices of order $9^t, t > 0$. Thus, we have another proof of Turyn's theorem.

TABLE 9.1 Hadamard Matrices from Williamson Matrices

| t | n | $\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2$ | μ_1 | μ_2 | μ_3 | μ_4 |
|-----|-----|---|---------------------------------|--------------------------|---------------------------------|--|
| 3 | 12 | $1^2 + 1^2 + 1^2 + 3^2$ | 1 | 1 | 1 | $1 - 2w_1$ |
| 5 | 20 | $1^2 + 1^2 + 3^2 + 3^2$ | 1 | 1 | $1 - 2w_1$ | $1 - 2w_2$ |
| 7 | 28 | $1^2 + 3^2 + 3^2 + 3^2$ | 1 | $1 - 2w_1$ | $1 - 2w_2$ | $1 - 2w_3$ |
| 7 | 28 | $1^2 + 1^2 + 1^2 + 5^2$ | 1 | 1 | $1 + 2w_1 - 2w_2$ | $1 + 2w_3$ |
| 9 | 36 | $3^2 + 3^2 + 3^2 + 3^2$ | $1 - 2w_1$ | $1 - 2w_2$ | $1 - 2w_3$ | $1 - 2w_4$ |
| 9 | 36 | $1^2 + 1^2 + 3^2 + 5^2$ | 1 | $1 + 2w_1 - 2w_2$ | $1 - 2w_3$ | $1 + 2w_4$ |
| 11 | 44 | $1^2 + 3^2 + 3^2 + 5^2$ | $1 + 2w_1 - 2w_2$ | 1 | $1 - 2w_3$ | $1 + 2w_4 + 2w_5 - 2w_6$ |
| 13 | 52 | $1^2 + 1^2 + 1^2 + 7^2$ | 1 | $1 - 2w_4$ | $1 + 2w_1 - 2w_4 + 2w_5 - 2w_6$ | $1 - 2w_2 - 2w_3$ |
| 13 | 52 | $3^2 + 3^2 + 3^2 + 5^2$ | 1 | $1 + 2w_4 - 2w_5$ | $1 - 2w_1 - 2w_6$ | $1 - 2w_2 - 2w_3$ |
| 13 | 52 | $1^2 + 1^2 + 5^2 + 5^2$ | $1 - 2w_2$ | $1 - 2w_4$ | $1 - 2w_1 - 2w_3 + 2w_5$ | $1 + 2w_6$ |
| 15 | 60 | $1^2 + 3^2 + 5^2 + 5^2$ | $1 - 2w_3 + 2w_4$ | $1 - 2w_2 + 2w_6$ | $1 + 2w_1$ | $1 + 2w_5$ |
| 15 | 60 | $1^2 + 3^2 + 5^2 + 5^2$ | 1 | $1 - 2w_5$ | $1 + 2w_6$ | $1 + 2w_1 - 2w_2 + 2w_3 + 2w_4 - 2w_7$ |
| 15 | 60 | $1^2 + 1^2 + 3^2 + 7^2$ | $1 - 2w_1 + 2w_7$ | $1 - 2w_3$ | $1 + 2w_2$ | $1 + 2w_4 + 2w_5 - 2w_6$ |
| 17 | 68 | $3^2 + 3^2 + 5^2 + 5^2$ | $1 - 2w_4 + 2w_6$ | $1 - 2w_1 - 2w_3 + 2w_5$ | $1 + 2w_7$ | $1 + 2w_2$ |
| 17 | 68 | $1^2 + 3^2 + 3^2 + 7^2$ | 1 | 1 | $1 - 2w_1 - 2w_5 + 2w_7$ | $1 + 2w_2 - 2w_3 - 2w_4 - 2w_6$ |
| 17 | 68 | $1^2 + 3^2 + 3^2 + 7^2$ | $1 - 2w_2$ | $1 - 2w_8$ | $1 - 2w_1 + 2w_5 + 2w_6$ | $1 + 2w_3 - 2w_4 + 2w_7$ |
| 17 | 68 | $1^2 + 3^2 + 3^2 + 7^2$ | $1 - 2w_3 - 2w_5 + 2w_6 + 2w_7$ | $1 - 2w_2$ | $1 - 2w_8$ | $1 - 2w_1 - 2w_4$ |
| 17 | 68 | $1^2 + 3^2 + 3^2 + 7^2$ | 1 | $1 - 2w_4 - 2w_5 + 2w_6$ | $1 - 2w_1 - 2w_3 + 2w_7$ | $1 - 2w_2 - 2w_8$ |
| 17 | 68 | $1^2 + 3^2 + 3^2 + 7^2$ | 1 | $1 - 2w_2 - 2w_4 + 2w_5$ | $1 - 2w_1 + 2w_3 - 2w_8$ | $1 - 2w_6 - 2w_7$ |

| | | | | | | | |
|----|-----|-------------------------|--------------------------|---|--|--|---|
| 19 | 76 | $1^2 + 5^2 + 5^2 + 5^2$ | 1 | $1 - 2w_3 - 2w_4 + 2w_5 + 2w_9$ | $1 + 2w_1 - 2w_2 + 2w_4$ $1 + 2w_2 - 2w_7 + 2w_2$ | $1 - 2w_3 + 2w_6 + 2w_8$ $1 + 2w_6$ | $1 - 2w_5 + 2w_7 + 2w_9$ $1 + 2w_1$ |
| 19 | 76 | $3^2 + 3^2 + 3^2 + 7^2$ | None | $1 - 2w_3 + 2w_8 + 2w_9$ | 1 | $1 + 2w_4 - 2w_5 + 2w_7$ | $1 + 2w_1 - 2w_2 + 2w_6$ |
| 19 | 76 | $1^2 + 1^2 + 5^2 + 7^2$ | 1 | $1 - 2w_2 + 2w_8$ | $1 - 2w_4 + 2w_7$ | $1 + 2w_1 - 2w_3 + 2w_8$ | $1 + 2w_2 - 2w_4 - 2w_5 + 2w_6 - 2w_7 - 2w_9$ |
| 21 | 84 | $3^2 + 5^2 + 5^2 + 5^2$ | $1 - 2w_7$ | $1 + 2w_3 + 2w_5 - 2w_8$ | $1 - 2w_4 + 2w_7$ | $1 + 2w_3 + 2w_6 - 2w_9$ | $1 - 2w_1 - 2w_5$ |
| 21 | 84 | $1^2 + 1^2 + 1^2 + 9^2$ | $1 + 2w_2 - 2w_3$ | $1 - 2w_6 + 2w_{10}$ | $1 + 2w_2 - 2w_5$ | $1 + 2w_8 - 2w_9$ | $1 + 2w_1 + 2w_9 - 2w_{10}$ $1 + 2w_1 + 2w_4 + 2w_5 - 2w_7$ |
| 21 | 84 | $1^2 + 3^2 + 5^2 + 7^2$ | 1 | $1 - 2w_3 + 2w_9$ | $1 + 2w_8 - 2w_{10}$ | $1 - 2w_5 - 2w_6 + 2w_7 + 2w_9$ | $1 + 2w_1 + 2w_2 - 2w_3 + 2w_4 + 2w_8 - 2w_{10}$ $1 + 2w_1 + 2w_2 - 2w_6 + 2w_7$ |
| 23 | 92 | $1^2 + 1^2 + 3^2 + 9^2$ | $1 - 2w_4 + 2w_5$ | $1 - 2w_2 - 2w_6 - 2w_8 - 2w_9 + 2w_{10}$ | $1 + 2w_5 - 2w_6 - 2w_8 - 2w_9 + 2w_{10}$ | $1 + 2w_1$ | $1 - 2w_3 - 2w_7$ |
| 23 | 92 | $3^2 + 3^2 + 5^2 + 7^2$ | None | $1 - 2w_5 + 2w_9$ | $1 + 2w_2 - 2w_4 - 2w_{10}$ | $1 + 2w_6 + 2w_7 - 2w_8$ | $1 - 2w_1 - 2w_3$ |
| 25 | 100 | $1^2 + 3^2 + 3^2 + 9^2$ | $1 + 2w_6 - 2w_{11}$ | $1 - 2w_6 + 2w_8$ | $1 + 2w_2 - 2w_4 - 2w_{10}$ | $1 + 2w_5 + 2w_7 - 2w_9$ | $1 - 2w_1 - 2w_3$ |
| 25 | 100 | $5^2 + 5^2 + 5^2 + 5^2$ | $1 + 2w_1 - 2w_6 + 2w_9$ | $1 - 2w_1 + 2w_3 - 2w_{12}$ | $1 + 2w_5 - 2w_7$ | $1 + 2w_1 - 2w_3 - 2w_{10}$ | $1 + 2w_2 + 2w_5 - 2w_8 + 2w_{10}$ $1 - 2w_3 + 2w_{10} + 2w_{11}$ |

TABLE 9.1 Hadamard Matrices from Williamson Matrices (continued)

| t | n | $\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2$ | μ_1 | μ_2 | μ_3 | μ_4 |
|-----|-----|---|---|---|--|--|
| 25 | 100 | $1^2 + 1^2 + 7^2 + 7^2$ | 1 | 1 | $1 - 2w_2 - 2w_3 - 2w_5 + 2w_6 - 2w_7 + 2w_{12}$ | $1 - 2w_1 - 2w_4 + 2w_8 + 2w_9 - 2w_{10} - 2w_{11}$ |
| | | $1 + 2w_3 - 2w_7$ | | $1 - 2w_1 + 2w_4$ | $1 + 2w_8 - 2w_9 - 2w_{10} - 2w_{11}$ | $1 - 2w_2 - 2w_5 + 2w_6 - 2w_{12}$ |
| | | $1 + 2w_3 - 2w_9$ | | $1 + 2w_4 - 2w_{12}$ | $1 - 2w_1 - 2w_7$ | $1 + 2w_6 + 2w_8 - 2w_{11} - 2w_{10} - 2w_5 - 2w_2$ |
| 25 | 100 | $1^2 + 5^2 + 5^2 + 7^2$ | $1 + 2w_5 - 2w_{10}$ | $1 + 2w_6 + 2w_{11} - 2w_2$ | $1 + 2w_{12} + 2w_9 + 2w_4 - 2w_7 - 2w_8$ | $1 - 2w_1 - 2w_3$ |
| 27 | 108 | $1^2 + 1^2 + 9^2 + 5^2$ | 1 | 1 | $1 - 2w_3 + 2w_4 + 2w_5 + 2w_7 - 2w_9 + 2w_{12}$ | $1 - 2w_1 - 2w_2 + 2w_6 + 2w_8 + 2w_{10} - 2w_{11} + 2w_{13}$ |
| 27 | 108 | $1^2 + 3^2 + 7^2 + 7^2$ | $1 + 2w_5 + 2w_2 - 2w_8 - 2w_7$ | $1 + 2w_9 - 2w_{10} - 2w_{11}$ | $1 + 2w_3 - 2w_4 - 2w_{13} - 2w_6$ | $1 - 2w_1 - 2w_{12}$ |
| 27 | 108 | $3^2 + 3^2 + 3^2 + 9^2$ | None | | | |
| 27 | 108 | $3^2 + 5^2 + 5^2 + 7^2$ | $1 + 2w_1 - 2w_4 - 2w_6$ | $1 + 2w_{10} + 2w_{13} - 2w_{11}$ | $1 + 2w_5 + 2w_2 - 2w_{12}$ | $1 + 2w_7 - 2w_8 - 2w_3 - 2w_9$ |
| 29 | 116 | $1^2 + 3^2 + 5^2 + 9^2$ | $1 + 2w_2 - 2w_4 + 2w_6 - 2w_9 - 2w_{11} + 2w_{12}$ | $1 - 2w_3 - 2w_5 + 2w_7 - 2w_8 + 2w_{10}$ | $1 + 2w_1$ | $1 + 2w_{13} + 2w_{14}$ |
| 31 | 124 | $1^2 + 1^2 + 1^2 + 11^2$ | 1 | 1 | $1 + 2w_3 + 2w_4 + 2w_5 - 2w_6 - 2w_8 - 2w_{12}$ | $1 - 2w_1 - 2w_2 + 2w_7 - 2w_{11} - 2w_{13} - 2w_{14} + 2w_{15}$ |
| 31 | 124 | $3^2 + 3^2 + 5^2 + 9^2$ | $1 - 2w_2 + 2w_{13} - 2w_{14}$ | $1 + 2w_4 - 2w_{10} - 2w_{15}$ | $1 + 2w_1 + 2w_3 - 2w_5 - 2w_6 + 2w_7$ | $1 - 2w_4 - 2w_{10} - 2w_{15}$ |
| 33 | 132 | $1^2 + 1^2 + 3^2 + 11^2$ | $1 + 2w_2 + 2w_5 - 2w_6 - 2w_8 - 2w_9 + 2w_{11}$ | $1 + 2w_1 - 2w_{13} + 2w_{14} - 2w_{16}$ | $1 - 2w_3 - 2w_7 + 2w_{12}$ | $1 + 2w_4 - 2w_7 + 2w_9 + 2w_{11} - 2w_{12}$ |
| 33 | 132 | $1^2 + 1^2 + 7^2 + 9^2$ | $1 - 2w_6 - 2w_8 + 2w_{11} + 2w_{16}$ | $1 - 2w_2 + 2w_3 - 2w_{10} + 2w_{14}$ | $1 + 2w_1 - 2w_5 - 2w_{12} - 2w_{15}$ | $1 + 2w_4 - 2w_7 + 2w_9 + 2w_{13}$ |

| | | | | | | |
|-----------------|-----|---------------------------|---|---|---|--|
| 33 | 132 | $1^2 + 5^2 + 9^2$ | $1 + 2w_5 - 2w_7 + 2w_{12} - 2w_{15}$ | $1 - 2w_2 + 2w_{10} + 2w_{16}$ | $1 - 2w_1 + 2w_4 + 2w_6 + 2w_9 - 2w_{13}$ | $1 + 2w_3 + 2w_8 + 2w_{11} - 2w_{14}$ |
| 33 | 132 | $1^2 + 5^2 + 9^2$ | $1 + 2w_1 - 2w_8 + 2w_{10} - 2w_{15}$ | $1 + 2w_4 - 2w_7 + 2w_{13}$ | $1 + 2w_7 - 2w_{12} - 2w_{14}$ | $1 + 2w_3 + 2w_5 - 2w_6 - 2w_9 + 2w_{11} + 2w_{16}$ |
| 33 | 132 | $3^2 + 5^2 + 7^2 + 7^2$ | $1 - 2w_7 - 2w_{11} + 2w_{12} - 2w_{15}$ | $1 - 2w_5 + 2w_{14} + 2w_{15}$ | $1 + 2w_1 - 2w_3 - 2w_6 - 2w_8 + 2w_9 - 2w_{13}$ | $1 + 2w_2 - 2w_4 - 2w_{10} - 2w_{16}$ |
| 35 | 140 | Any decomposition | None | | | |
| 37 | 148 | $1^2 + 1^2 + 5^2 + 11^2$ | 1 | 1 | $1 - 2w_2 - 2w_6 - 2w_7 - 2w_8 + 2w_{11} - 2w_{13} + 2w_{14}$ | $1 + 2w_1 + 2w_3 - 2w_4 + 2w_5 - 2w_9 + 2w_{10} - 2w_{12} - 2w_{15} - 2w_{16} + 2w_{17} + 2w_{18}$ |
| 37 | 148 | $5^2 + 5^2 + 7^2 + 7^2$ | $1 - 2w_1 + 2w_3 + 2w_4 + 2w_7 - 2w_{11}$ | $1 + 2w_5 - 2w_6 - 2w_8 + 2w_{13} + 2w_{18}$ | $1 + -2w_2 - 2w_9 - 2w_{10} + 2w_{16}$ | $1 - 2w_{12} - 2w_{14} + 2w_{15} - 2w_{17} + 2w_{18}$ |
| 37 ^a | 148 | $1^2 + 7^2 + 7^2 + 7^2$ | 1 | $1 - 2\alpha_0 - 2\alpha_1 - 2\alpha_5$ | $1 - 2\alpha_3 - 2\alpha_4 + 2\alpha_8$ | $1 + 2\alpha_2 - 2\alpha_6 - 2\alpha_7$ |
| 39 | 156 | $3^2 + 7^2 + 7^2 + 7^2$ | $1 - 2w_{13}$ | $1 - 2w_3 + 2w_5 - 2w_{10} - 2w_{11} + 2w_{17} - 2w_{18}$ | $1 - 2w_6 + 2w_7 - 2w_8 - 2w_{12} - 2w_{14} + 2w_{16}$ | $1 + 2w_1 + 2w_2 - 2w_4 - 2w_9 - 2w_{15} - 2w_{19}$ |
| 43 ^b | 172 | $1^2 + 1^2 + 1^2 + 13^2$ | $1 + 2\alpha_0 - 2\alpha_2$ | $1 - 2\alpha_1 + 2\alpha_3$ | $1 + 2\alpha_4 - 2\alpha_6$ | $1 + 2\alpha_5$ |
| 61 | 244 | $1^2 + 1^2 + 11^2 + 11^2$ | 1 | 1 | $1 + 2w_1 - 2w_6 + 2w_7 - 2w_9 - 2w_{10} + 2w_{13} + 2w_{17} + 2w_{18} - 2w_{19} - 2w_{21} - 2w_{22} - 2w_{26} + 2w_{27} - 2w_{28} - 2w_{29} - 2w_{30}$ | $1 - 2w_2 - 2w_3 + 2w_4 - 2w_5 + 2w_8 + 2w_{11} - 2w_{12} - 2w_{14} + 2w_{15} + 2w_{16} - 2w_{20} + 2w_{21} - 2w_{23} - 2w_{25} - 2w_{26}$ |

^a $\alpha_j = w_{2j} + w_{29+j}$.

^b $\alpha_j = w_{3j} + w_{37+j} + w_{314+j}$.

Corollary 9.5 (Turyn [109]). *There are Williamson-type matrices of order 9^t , $t > 0$, that pairwise satisfy $XY = XY^T = J$, $XJ = 3J$.*

Example 9.2. The regular 2-set of matrices of order 9 can be written as B, C where writing a, b, c, W for the circulant matrices with first rows

$$[0 + +] [- + -] [- - +] [0 + -],$$

respectively, we have

$$c = b^T, \quad b + c = -2I.$$

The matrix B is

$$\begin{bmatrix} -c & a - I & -b \\ a - I & -b & -c \\ -b & -c & a - I \end{bmatrix}.$$

It should be noted that B is a block back-circulant matrix whose elements are circulant matrices. Hence, B is neither a type one nor a type two matrix over $Z_3 \times Z_3$ (perhaps it should be referred to as a type three matrix over $Z_3 \times Z_3$), but it can still be defined as a group matrix over $Z_3 \times Z_3$.

The matrix B may also be written in the form

$$B = \begin{bmatrix} M & MT & MT^2 \\ MT & MT^2 & M \\ MT^2 & M & MT \end{bmatrix} \quad \text{or} \quad M \begin{bmatrix} I & T & T^2 \\ T & T^2 & I \\ T^2 & I & T \end{bmatrix},$$

where $M = I + W$, W is as before, and T is the circulant matrix (shift matrix) with first row $[0 + 0]$. Note that

$$T^2 = T^T, \quad T^3 = I, \quad I + T + T^2 = J.$$

The matrix C is constructed as follows:

$$\begin{bmatrix} + + - & + + - & + + - \\ + + - & + + - & + + - \\ + + - & + + - & + + - \\ - + + & - + + & - + + \\ - + + & - + + & - + + \\ - + + & - + + & - + + \\ + - + & + - + & + - + \\ + - + & + - + & + - + \\ + - + & + - + & + - + \end{bmatrix}.$$

The construction of the matrix C is an ingenious idea of Mathon. Note that C is *not* composed of circulants or back circulants.

The matrix C may also be written in the form

$$C = \begin{bmatrix} N & N & N \\ NT & NT & NT \\ NT^2 & NT^2 & NT^2 \end{bmatrix} \quad \text{or} \quad N \begin{bmatrix} I & I & I \\ T & T & T \\ T^2 & T^2 & T^2 \end{bmatrix},$$

where

$$N = \begin{bmatrix} + & + & - \\ + & + & - \\ + & + & - \end{bmatrix}.$$

Note that each row of N is the same as the top row of M .

Corollary 9.6. *Since there is a regular 4-set of regular matrices of order 49 and a regular 2-set of regular matrices of order 9^t , $t > 0$, there is a regular 4-set of regular matrices of order $49 \cdot 9^t$. Hence, there are 8-Williamson-type matrices of order $49 \cdot 9^t$, $t \geq 0$.*

Using the OD(8;1,1,1,1,1,1) and the Plotkin OD(24;3,3,3,3,3,3), we have

Corollary 9.7. *There is an Hadamard matrix of order $8 \cdot 49 \cdot 3^t$, $t \geq 0$.*

In general, we have

Corollary 9.8. *If $n \equiv 3 \pmod{4}$ is a prime power, there is a regular $\frac{1}{2}(n+1)$ -set of regular matrices of order n^2 . Hence, there are $(n+1)$ -Williamson-type matrices of order $n^2 \cdot 9^t$, $t \geq 0$ each with row sum $3^t n$.*

This also means that we have

Corollary 9.9. *If $n \equiv 3 \pmod{4}$ is a prime power, there is an Hadamard matrix of order $n^2(n+1) \cdot 9^t$, $t \geq 0$.*

Proof. Choose a Latin square of size $n+1$ and an Hadamard matrix $H = (h_{ij})$ of order $n+1$. Replace the 1, 2, 3, ..., $\frac{1}{2}(n+1)$ th elements of the Latin square by $B_1, B_2, \dots, B_{(n+1)/2}$ and the $\frac{1}{2}(n+3)$ rd, ..., $(n+1)$ th elements by $B_1^T, B_2^T, \dots, B_{(n+1)/2}^T$. We now have a block matrix (B_{ij}) . The required Hadamard matrix is $(h_{ij}B_{ij})$. \square

This method is considered further in [80], where it is used to show

Theorem 9.10 (Seberry). *Let q be a prime power. Then there are Williamson-type matrices of order*

1. $\frac{1}{2}q^2(q+1)$ when $q \equiv 1 \pmod{4}$,
2. $\frac{1}{4}q^2(q+1)$ when $q \equiv 3 \pmod{4}$, and there are Williamson-type matrices of order $\frac{1}{4}(q+1)$.

Example 9.3. Let B_1, B_2, \dots, B_6 be the matrices constructed by Seberry and Whiteman [85] or Seberry [80] of order 121. Write $S_1 = B_1$, $S_7 = B_1^T$, $S_2 = B_2$, $S_8 = B_2^T, \dots, S_6 = B_6$, $S_0 = B_6^T$.

Let

$$W_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad W_2 = W_3 = W_4 = \begin{bmatrix} 1 & - & - \\ - & 1 & - \\ - & - & 1 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} S_1 & S_2 & S_3 \\ S_3 & S_1 & S_2 \\ S_2 & S_3 & S_1 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} S_4 & S_5 & S_6 \\ S_6 & S_4 & S_5 \\ S_5 & S_6 & S_4 \end{bmatrix},$$

$$Y_3 = \begin{bmatrix} S_7 & S_8 & S_9 \\ S_9 & S_7 & S_8 \\ S_8 & S_9 & S_7 \end{bmatrix}, \quad Y_4 = \begin{bmatrix} S_{10} & S_{11} & S_0 \\ S_0 & S_{10} & S_{11} \\ S_{11} & S_0 & S_{10} \end{bmatrix},$$

and

$$X_1 = Y_1, \quad X_2 = \begin{bmatrix} S_4 & -S_5 & -S_6 \\ -S_6 & S_4 & -S_5 \\ -S_5 & -S_6 & S_4 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} S_7 & -S_8 & -S_9 \\ -S_9 & S_7 & -S_8 \\ -S_8 & -S_9 & S_7 \end{bmatrix}, \quad X_4 = \begin{bmatrix} S_{10} & -S_{11} & -S_0 \\ -S_0 & S_{10} & -S_{11} \\ -S_{11} & -S_0 & S_{10} \end{bmatrix}.$$

Now the S_i are 12 $(1, -1)$ matrices of order 11^2 , satisfying

$$S_i S_j^T = J, \quad i \neq j,$$

$$\sum_{i=0}^{11} S_i S_i^T = 11^2 \cdot 12I \times I.$$

Thus, $X_1 X_j^T = -J \times J$, $j = 2, 3, 4$,

$$X_i X_j^T = \begin{bmatrix} 3J & -J & -J \\ -J & 3J & -J \\ -J & -J & 3J \end{bmatrix}, \quad i, j = 2, 3, 4,$$

and

$$\sum_{i=1}^4 X_i X_i^T = \sum_{j=0}^{11} S_j S_j^T \times I = 11^2 \cdot 12I \times I.$$

Hence, X_1, X_2, X_3, X_4 are Williamson-type matrices of order 363.

9.1. New Difference Sets

M. Xia and G. Liu [129] have recently announced the existence of 4 - $\{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ supplementary difference sets for $q \equiv 1 \pmod{4}$ a prime power. A. L. Whiteman has also given the following set of 4 - $\{9; 3; 3\}$ supplementary difference sets:

$$\{0, 1, 2\}, \quad \{0, x, 2x\}, \quad \{0, x+1, 2x+2\}, \quad \{0, x+2, 2x+1\},$$

whose incidence matrices A_i , $i = 1, 2, 3, 4$, satisfy $A_i A_j = J$, $i \neq j$, and he has given 4 - $\{25; 10; 15\}$ supplementary difference sets

$$\{2, 3, x+1, x+2, x+3, 2x+4, 3x+1, 4x+2, 4x+3, 4x+4\},$$

$$\{1, 2, 3, 4, x, x+4, 2x+4, 3x+1, 4x, 4x+1\},$$

$$\{1, 4, x+2, 2x+1, 2x+2, 2x+4, 3x+1, 3x+3, 3x+4, 4x+3\},$$

$$\{1, 2, 3, 4, x+2, 2x, 2x+3, 3x, 3x+2, 4x+3\}.$$

The Xia-Liu result means the following:

Theorem 9.11 (Xia-Liu). *There exist four Williamson matrices of order q^2 for all $q \equiv 1 \pmod{4}$ a prime power. The negation of each matrix has row sum q .*

This also gives Williamson matrices of orders p^4 for $p \equiv 3 \pmod{4}$ a prime because then $p^2 \equiv 1 \pmod{4}$. Thus,

Corollary 9.12. *There exist four Williamson matrices of orders $3^4, 5^4$, and p^4 , $p \equiv 3 \pmod{4}$ a prime.*

Now $OD(4t; t, t, t, t)$ exist for $t = 3, 9, 27, 5, 25, 125, 7, 49, 11, 121$, for all $t \equiv 1 \pmod{4}$, t prime $\in \{13, 17, 29, 37, 41, 53, 61, 101, \dots\}$, and for t prime of the form $1 + 2^a 10^b 26^c$, $a, b, c \geq 0$. This gives

Corollary 9.13. *There exist Hadamard matrices of order $4 \cdot 3^r, 4 \cdot 5^r, 4 \cdot 13^r, 4 \cdot 17^r, 4 \cdot 29^r, 4 \cdot 37^r, 4 \cdot 41^r, 4 \cdot 53^r, 4 \cdot 61^r, 4 \cdot 101^r$, $r \geq 0$; $4 \cdot g^{4i}, 4 \cdot g^{4i+1}, 4 \cdot g^{4i+2}, 8 \cdot g^{4i+3}$, $i \geq 0$, $g = 7, 11$; and $4 \cdot p^r$ whenever $p = 1 + 2^a 10^b 26^c$ is prime, $a, b, c \geq 0$.*

9.2. Other Results

We define a *complete regular 4-set of regular matrices* of order q^2 as four matrices satisfying

$$\begin{aligned} A_i^T &= A_i, \\ A_i A_j &= pJ, \quad p \text{ constant}, \quad i \neq j, \quad i, j = 1, 2, 3, 4, \\ \sum_{i=1}^4 A_i A_i^T &= 4q^2 I, \\ A_i J &= qJ. \end{aligned}$$

These are a special form of Williamson type matrices and exist for at least orders 9^i , $i = 1, 2$.

As with regular 2-sets of regular matrices, we have

Theorem 9.14 (Seberry). *If there exist complete regular 4-sets of regular matrices of orders s^2 and t^2 respectively there exists a complete regular 4-set of regular matrices of order $s^2 t^2$.*

Proof. Let the complete regular 4-sets of regular matrices of order s^2 and t^2 be A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 , respectively. Then

$$\begin{aligned} C_1 &= \frac{1}{2}[A_1 \times (B_1 + B_2) + A_2 \times (B_1 - B_2)], \\ C_2 &= \frac{1}{2}[-A_1 \times (B_3 - B_4) + A_2 \times (B_3 + B_4)], \\ C_3 &= \frac{1}{2}[A_3 \times (B_1 + B_2) - A_4 \times (B_1 - B_2)], \\ C_4 &= \frac{1}{2}[A_3 \times (B_3 - B_4) + A_4 \times (B_3 + B_4)], \end{aligned}$$

is a complete regular 4-set of regular matrices of order $s^2 t^2$. □

Corollary 9.15. *If there exist complete regular 4-sets of regular matrices of orders q_1, q_2, \dots , then there exists a complete regular 4-set of regular matrices of order $q_1 \cdot q_2 \cdot q_3 \dots$, and Williamson-type matrices.*

Many authors have found suitable and near suitable matrices of Williamson type, and this will be pursued in a later article. Appendix A.2 gives a summary of orders for which Williamson and Williamson-type matrices exist plus a list of known orders < 2000 .

10 SBIBD AND THE EXCESS OF HADAMARD MATRICES

10.1. SBIBD(4t, 2t - 1, t - 1)

Every Hadamard matrix H of order $4t$ is associated in a natural way with an SBIBD with parameters $(4t - 1, 2t - 1, t - 1)$, and with its complement, an SBIBD $(4t - 1, 2t, t)$. To obtain the SBIBD, we first normalize H and write the resultant matrix in the form

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & & & \\ \vdots & & A & \\ 1 & & & \end{bmatrix}.$$

Then

$$AJ = JA = -J \quad \text{and} \quad AA^T = 4tI - J.$$

So $B = \frac{1}{2}(A + J)$ satisfies

$$BJ = JB = (2t - 1)J \quad \text{and} \quad BB^T = tI + (t - 1)J.$$

Thus, B is a $(0, 1)$ matrix satisfying the equations for the incidence matrix of an SBIBD with parameters $(4t - 1, 2t - 1, t - 1)$. Similarly, $C = \frac{1}{2}(J - A)$ is the incidence matrix of an SBIBD with parameters $(4t - 1, 2t, t)$. Clearly, if we start with the incidence matrix of an SBIBD with parameters $(4t - 1, 2t - 1, t - 1)$ or $(4t - 1, 2t, t)$ and replace all the 0 elements by -1 , we form either A or $-A$. Thus,

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & & & \\ \vdots & & A & \\ 1 & & & \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & -1 & \dots & -1 \\ -1 & & & \\ \vdots & & -A & \\ -1 & & & \end{bmatrix}$$

are Hadamard matrices of order $4t$ obtained from these SBIBD.

Thus, we have shown

Theorem 10.1. *There exists an Hadamard matrix of order $4t$ if and only if there exists an SBIBD $(4t - 1, 2t - 1, t - 1)$.*

Since a $(4t - 1, 2t - 1, t - 1)$ difference set yields an SBIBD we have

Corollary 10.2. *If there exists a $(4t - 1, 2t - 1, t - 1)$ difference set, then there exists an Hadamard matrix of order $4t$.*

In view of the Seberry theorem [121] (see Section 3) we have that

Theorem 10.3. *Let q be any odd natural number. Then there exists a t ($\leq [2\log_2(q-3)]$) so that there is an SBIBD($2^t q - 1, 2^{t-1}q - 1, 2^{t-2}q - 1$).*

Constructions given above indicate that for small q ($< 10,000$) $t = 2$ in about 97% of cases, and $t = 3, 4, 5$ in about 2% of further cases. So for $q < 10,000$ most SBIBD($4q - 1, 2q - 1, q - 1$) exist. Table A.2 in Appendix A.3 illustrates this point.

10.2. The Equivalence Theorem

The main theorem of this section deals with the equivalence among Hadamard matrices with maximal excess, regular Hadamard matrices, and certain SBIBDs. We begin with the definition of excess of a Hadamard matrix.

Definition 10.1. Let H be an Hadamard matrix of order n . The sum $\sigma(H)$ of the elements of H is called the *excess* of H . The maximum excess of H , over all Hadamard matrices of order n , is denoted by $\sigma(n)$; i.e.,

$$\sigma(n) = \max\{\sigma(H) : H \text{ an Hadamard matrix of order } n\}.$$

An equivalent notion is the *weight* of H , denoted $w(H)$, which is defined as the number of 1's in H . It follows that $\sigma(H) = 2w(H) - n^2$ and $\sigma(n) = 2w(n) - n^2$ (see [8]).

Theorem 10.4. *There is an Hadamard matrix of order $n = 4s^2$ with maximal excess $n\sqrt{n} = 8s^3$ if and only if there is an SBIBD($4s^2, 2s^2 + s, s^2 + s$).*

In (Seberry) Wallis [114, p. 343], it is pointed out that Goethals and Seidel [25] and Shrikhande and Singh [92] have established

Theorem 10.5. *If there exists a BIBD($2k^2 - k, 4k^2 - 1, 2k + 1, k, 1$), then there exists a symmetric Hadamard matrix of order $4k^2$ with constant diagonal.*

Moreover, Shrikhande [90] has studied these designs and shown they exist for all $k = 2^t$, $t \geq 1$. They are also known for $k = 3, 5, 6, 7$ [114].

In (Seberry) Wallis [114, pp. 344–346], it is established that symmetric Hadamard matrices of order h with constant diagonal exist for $h = 2^{2t}$ for all $t \geq 1$, and for $h = 36, 100, 144, 196$ (after Theorem 5.15 of [114]). Using results of (Seberry) Wallis-Whiteman [113] and Szekeres [99], they are shown to exist with the extra property of regularity (constant row sum) for $h = 4 \cdot 5^2, 4 \cdot 13^2, 4 \cdot 29^2, 4 \cdot 51^2$, and $h = 4(2((p-3)/4) + 1)^2$, for $p \equiv 3 \pmod{4}$ a prime power (after Theorem 5.15 of [114]).

Remark 10.1. A theorem of Goethals and Seidel [25] (see Geramita and Seberry [23]) tells us that if there is an Hadamard matrix with constant diagonal of order $4k$, then there is a regular symmetric Hadamard matrix with constant diagonal of order $4(2k)^2$. So an Hadamard matrix of order $4t$ gives a regular symmetric Hadamard matrix with constant diagonal of order $4k^2$, $k = 2t$. In particular, known results give these matrices for $2t \leq 210$.

Remark 10.2. We note that regular symmetric Hadamard matrices with constant diagonal of orders $4s^2$ and $4t^2$ give a regular symmetric Hadamard matrix with constant diagonal with order $(2st)^2$.

Theorem 10.6 (J. Wallis [114]). *A regular Hadamard matrix H of order $4k^2$ with row sum $\pm 2k$ exists if and only if there exists an SBIBD($4k^2, 2k^2 \pm k, k^2 \pm k$).*

We observe that the stipulation that the row sum is $\pm 2k$ is unnecessary for the following reason: If the matrix is regular, it must have constant row sum, say x . Thus, $eH^T = (x, \dots, x)$, where e is the $1 \times 4k^2$ matrix of ones. Now $H^T H = 4k^2 I$, so

$$16k^4 = 4k^2 ee^T = eH^T H e^T = (x, \dots, x)(x, \dots, x)^T = 4k^2 x^2.$$

Thus, $x = \pm 2k$. The matrix with constant row sum $-2k$ is the negative of the matrix with constant row sum $2k$.

We can now combine the results obtained so far as

Theorem 10.7 (Equivalence Theorem). *The following are equivalent:*

1. *There exists an Hadamard matrix of order $4k^2$ with maximal excess $8k^3$.*
2. *There exists a regular Hadamard matrix of order $4k^2$.*
3. *There is an SBIBD($4k^2, 2k^2 + k, k^2 + k$) (and its complement the SBIBD($4k^2, 2k^2 - k, k^2 - k$)).*

Part of this result was also observed by Brown and Spencer [9] and Best [8].

We also note the following consequence of the Liu-Xia result mentioned in Section 9. In the next theorem, we need the notion of a proper n -dimensional Hadamard matrix. This is defined to be an n -dimensional array (with entries -1 and 1) such that every two-dimensional face is an Hadamard matrix.

Theorem 10.8. *Suppose that there exist $4\text{-}\{q^2; \frac{1}{2}q(q-1); q(q-2)\}$ supplementary difference sets. Then*

1. *there is a regular symmetric Hadamard matrix with constant diagonal of order $4q^2$ with maximal excess $8q^3$;*
2. *there is an SBIBD($4q^2, 2q^2 \pm q, q^2 \pm q$);*
3. *there is a proper n -dimensional Hadamard matrix of order $(4q^2)^n$.*

10.3. Excess

In this section, we present several results dealing with the excess of a Hadamard matrix and the excess of an orthogonal design. We begin with an example.

Example 10.1. The excess of the following Hadamard matrices

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{bmatrix}, \quad R_4 = \begin{bmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{bmatrix},$$

is easily determined. We see that $\sigma(H_2) = 2$, $\sigma(H_4) = 4$, $\sigma(R_4) = 8$. Since R_4 has the maximal excess of all Hadamard matrices of order 4, $\sigma(4) = 8$. We can find the Hadamard matrix of maximal excess of order 8 quite easily. We note that if H and K are Hadamard matrices, then so is

$$\begin{bmatrix} H & H \\ K & -K \end{bmatrix}$$

and, in particular,

$$H_8 = \begin{bmatrix} R_4 & R_4 \\ H_4 & -H_4 \end{bmatrix}, \quad \sigma(H_8) = 16.$$

Now H_8 has its fifth column $(-1, 1, 1, 1, -, -, -, -)^T$. Negating this column gives R_8 where $\sigma(R_8) = 20$.

This construction yields

Lemma 10.9. $\sigma(2n) \geq 2\sigma(n) + 4$.

Noting that the Kronecker product of two Hadamard matrices is an Hadamard matrix, we have

Lemma 10.10. $\sigma(mn) \geq \sigma(m)\sigma(n)$.

We define the *excess of the orthogonal design* $D = x_1A_1 + \cdots + x_uA_u$ as

$$\sigma(D) = \sigma(A_1) + \cdots + \sigma(A_u),$$

where $\sigma(A_i)$ is the sum of the entries of A_i . This is equivalent to putting all the variables equal to +1.

The concept of excess of orthogonal designs is used by Hammer-Levingston-Seberry [34] to obtain bounds on the excess of Hadamard matrices and by Seberry [82], Koukouvinos and Kounias [54] and Koukouvinos, Kounias, and Seberry [55] to find Hadamard matrices of order $4k^2$ with maximal excess and equivalently SBIBD($4k^2, 2k^2 \pm k, k^2 \pm k$).

Example 10.2. The excesses of the OD(4; 1, 1, 1, 1)

$$D_1 = \begin{bmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{bmatrix}, \quad D_2 = \begin{bmatrix} -A & B & C & D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{bmatrix},$$

are

$$\begin{aligned} \sigma(D_1) &= \sigma \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \sigma \begin{bmatrix} 0 & 1 & 0 & 0 \\ - & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & - & 0 \end{bmatrix} \\ &+ \sigma \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & - \\ - & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \sigma \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & - & 0 & 0 \\ - & 0 & 0 & 0 \end{bmatrix} \\ &= 4 + 0 + 0 + 0 = 4, \\ \sigma(D_2) &= \sigma \begin{bmatrix} - & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \sigma \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & - \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &+ \sigma \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & - & 0 & 0 \end{bmatrix} + \sigma \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & - & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ &= 2 + 2 + 2 + 2 = 8. \end{aligned}$$

Constructions that give OD's of larger order with large excess could lead to a construction such as that of Seberry Wallis [121] for Hadamard matrices of large excess.

10.4. Bounds on the Excess of Hadamard Matrices

Many authors, including Brown and Spencer [9], Best [8], Enomoto and Miyamoto [21], Farmakis and Kounias [22, 61], Hammer, Levingston, and Seberry [34], Jenkins, Koukouvinos, Kounias, J. Seberry, and R. Seberry [39], Khara-ghani [41], Koukouvinos and Kounias [54], Koukouvinos, Kounias, and Seberry [56], Koukouvinos and Seberry [59], Sathe and Shenoy [73], Schmidt and Wang [76], Seberry [82], Wallis [122] and Yamada [131] have found the excess of Hadamard matrices for particular orders or families of orders. Lower and upper bounds have been given [8, 61, 34, 56]. Here, we are interested in the upper bound, which is surveyed in Jenkins et al. [39].

The most encompassing upper bound is that of Brown and Spencer [9] and later by Best [8].

Brown-Spencer-Best Bound: $\sigma(n) \leq n\sqrt{n}$ Now, in the case of $n = 4k^2$, we can restate this bound as $\sigma(4k^2) \leq 8k^3$. Hadamard matrices with maximal excess meeting this bound have been found by Koukouvinos, Kounias, Seberry, and Yamada [54, 56, 82, 131] for $n = 4k^2$ with even k when there is an Hadamard matrix of order $2k$ (in particular, for all $2k \leq 210$) and also for $k \in \{1, 3, 5, \dots, 45, 49, \dots, 69, 73, 75, 81, \dots, 101, 105, 109, 125, 625\} \cup \{3^{2m}, 25 \cdot 3^{2m} : m \geq 0\}$.

Let a_i , $1 \leq i \leq n$, be the i th row sum of an Hadamard matrix of order n . Denote the integer part of z by $[z]$. Then, with

$$\begin{aligned} a_1 &= a_2 = \dots = a_i = t, \\ a_{i+1} &= a_{i+2} = \dots = a_n = t + 4, \end{aligned}$$

where $t = [\sqrt{n}]$ when $[\sqrt{n}]$ is even and $t = [\sqrt{n}] - 3$ when $[\sqrt{n}]$ is odd, and i is the integer part of $(n((t+4)^2 - n))/8(t+2)$, the Brown-Spencer-Best bound can be refined to the HLS bound (see [34]).

Hammer-Levingston-Seberry (HLS) Bound: $\sigma(n) \leq n(t+4) - 4i$ Jenkins et al. [39] lists a number of cases where this bound is satisfied. The HLS bound has been improved for some orders by Farmakis and Kounias [22]. Write $n = (2x+1)^2 + 3$. Then $[\sqrt{n}] = 2x+1$. From HLS bound, putting $t = [\sqrt{n}] - 3 = 2x - 2$, $i = x^2 + x + 1$,

$$\sigma(n) \leq n(2x+2) - 4(x^2 + x + 1) = n(2x+1) = n\sqrt{n-3}.$$

Thus, we have the Farmakis-Kounias bound.

Farmakis-Kounias (KF) Bound: $\sigma(n) \leq n\sqrt{n-3}$ for $n = (2x+1)^2 + 3$ In some special cases, the HLS and KF bound are identical. If $n = (2x+1)^2 + 3$, both give $\sigma(n) \leq n\sqrt{n-3}$. Hadamard matrices of order $n = (2x+1)^2 + 3$

satisfying the bound $\sigma(n) \leq n\sqrt{n-3}$ with equality are known for

$$x = 0, 1, \dots, 7, 9, 11, 16, 18, 22, 25, 26, 29, 36, 37, 49.$$

There is also the Kharaghani-Kounias-Farmakis bound.

Kharaghani-Kounias-Farmakis Bound: $\sigma(n) \leq 4(m-1)^2(2m+1)$ for $n = 4m(m-1)$. Hadamard matrices are known that meet this bound for some values of m where m is the order of a skew Hadamard matrix, the order of a conference matrix, or the order of a skew complex Hadamard matrix [60, 56]. The precise details of the constructions used to find the Hadamard matrices of maximal excess and order $4k^2$ can be found in Koukouvinos, Kounias, and Sotirakoglou [51], Koukouvinos, Kounias, and Seberry [56], and Seberry [83].

Using all the known results we have the following:

Theorem 10.11. *Hadamard matrices of order $4k^2$ with maximal excess $8k^3$ exist for*

1. k even, $k \leq 210$, or if an Hadamard matrix of order $2k$ exists;
2. $k \in \{1, 3, 5, \dots, 45, 49, \dots, 57, 61, \dots, 69, 75, 81, \dots, 95, 99, 115, 117, 625\} \cup \{3^{2m}, 5^2 \cdot 3^{2m} : m \geq 0\}$;
3. $k = qs$, $q \in \{q : q \equiv 1 \pmod{4} \text{ is a prime power}\}$, $s \in \{1, \dots, 33, 37, \dots, 41, 45, \dots, 59\} \cup \{2g+1 : g \text{ the length of a Golay sequence}\}$.

It follows from the equivalence theorem (Theorem 10.7) that regular Hadamard matrices of order $4k^2$ and $SBIBD(4k^2, 2k^2 \pm k, k^2 \pm k)$ also exist for these k values.

11 COMPLEX HADAMARD MATRICES

Complex Hadamard matrices were first introduced by Richard J. Turyn [104] who showed how they could be used to construct Hadamard matrices. These matrices are very important for they exist for orders for which symmetric conference matrices cannot exist. Complex Hadamard matrices also give powerful "multiplication" theorems. They are conjectured to exist for all even orders [114], a conjecture that implies the Hadamard conjecture.

Known small orders and a list of classes of complex Hadamard matrices are given in this section. This section is not a complete study of complex Hadamard matrices; it just gives some interesting constructions.

Theorem 11.1 (Turyn [104]). *If C is a complex Hadamard matrix of order c and H is a real Hadamard matrix of order h , then there exists a real Hadamard matrix of order hc .*

We note a connection between complex Hadamard matrices and matrices to “plug into.”

Lemma 11.2. *If there is a complex Hadamard matrix, $C = H + iK$ of order n , then H and K are amicable, disjoint, suitable matrices of total weight n .*

Lemma 11.3. *If there is a complex Hadamard matrix, $C = H + iK$ of order n , then there is an orthogonal design $OD(2n; n, n)$ and amicable orthogonal designs $AOD(2n; (n, n); (n, n))$.*

Proof. Let a, b be commuting variables and use

$$\begin{bmatrix} aH - bK & aH + bK \\ -aH - bK & aH - bK \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} aH + bK & aH - bK \\ aH - bK & -aH - bK \end{bmatrix}. \quad \square$$

11.1. Constructions for Complex Hadamard Matrices

Theorem 11.4 (Turyn [104]). *If C and D are complex Hadamard matrices of orders r and q , then $C \times D$ (where \times is the Kronecker product) is a complex Hadamard matrix of order rq .*

Proof. $CC^* = rI$ and $DD^* = qI$, so $(C \times D)(C^* \times D^*) = rqI$. \square

Theorem 11.5 (Turyn [104]). *If $I + N$ is a symmetric conference matrix, then $iI + N$ is a (symmetric) complex Hadamard matrix and $I + iN$ is a complex skew Hadamard matrix.*

Adapting a theorem of Turyn [104], Kharaghani and Seberry [43] showed

Theorem 11.6. *There is an Hadamard matrix of order $4m$ of the form*

$$\begin{bmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{bmatrix}$$

if and only if there is a complex Hadamard matrix of order $2m$ of the form

$$\begin{bmatrix} S & T \\ -\bar{T} & \bar{S} \end{bmatrix},$$

where \bar{T} denotes the complex conjugate of T .

This theorem and the next lemma show complex Hadamard matrices are also related to matrices to “plug in.”

Lemma 11.7 (Kharaghani and Seberry [42]). *Suppose that A, B, C, D are four Williamson-type matrices of order m with constant row and column sum a, a, b, b . Then there exists a regular complex Hadamard matrix of order $2m$, with row sum $a + ib$.*

Proof. We form $X = \frac{1}{2}(A + B)$, $Y = \frac{1}{2}(A - B)$, $W = \frac{1}{2}(C + D)$ and $V = \frac{1}{2}(C - D)$, which have row sums $a, 0, b, 0$. Then

$$E = \begin{bmatrix} X + iY & V + iW \\ -V + iW & X - iY \end{bmatrix}$$

is the required regular complex Hadamard matrix with row and column sum $a + ib$. \square

Lemma 11.8 (Kharaghani and Seberry [42]). *Let g be the length of a pair of Golay sequences U and V . Suppose that the row sums of U and V are a and b , so $2g = a^2 + b^2$. Then there is a regular complex Hadamard matrix of order $2g$, with row sum $a + ib$.*

Proof. Use U and V as the first rows of circulant matrices X and Y of order g . Then

$$C = \begin{bmatrix} X & iY \\ iY^T & X^T \end{bmatrix}$$

is the required regular complex Hadamard matrix. \square

Lemma 11.9 (Kharaghani and Seberry [42]). *Suppose that there is a regular complex Hadamard matrix C of order $4c$, with row sum $a + ib$ and of the form*

$$\begin{bmatrix} A & iB \\ iB & A \end{bmatrix},$$

where A and B are real. Then $D = \frac{1}{2}(-i + 1)(A + iB)$ is a regular complex Hadamard matrix of order $2c$ with row sum $\frac{1}{2}(a + b) + \frac{1}{2}(a - b)i$.

Lemma 11.10 (Kharaghani-Seberry [42]). *Let c_1, c_2, \dots, c_{2c} be the columns of a complex Hadamard matrix C . Define C_i to be the $2c \times 2c$ matrix $C_i = c_i c_i^*$ (where $*$ is the hermitian conjugate). Then*

1. $C_i = C_i^*$; $C_i C_j = 0$, $i \neq j$;
2. $\sum_{i=1}^{2c} C_i = 2c I_{2c}$; $\sum_{i=1}^{2c} C_i C_i^* = 4c^2 I_{2c}$.

The next four results, found by Kharaghani and Seberry [42], are based on the work of Kharaghani:

Theorem 11.11 (Kharaghani-Seberry [42]). *Let C be a complex Hadamard matrix of order c . Then there is a regular complex Hermitian Hadamard matrix, D of order c^2 with constant diagonal and with row (and column) sum c . Hence D has element sum c^3 .*

Proof. Form C_1, \dots, C_c of order c as in the Lemma 11.5. Now from condition 1, $\sum_{i=1}^c C_i = cI_c$, and from condition 2, $C_i C_j^* = 0$.

Form the block back-circulant complex Hadamard matrix

$$D = \begin{bmatrix} C_1 & C_2 & \cdots & C_c \\ C_2 & C_3 & \cdots & C_1 \\ \vdots & & & \vdots \\ C_c & C_1 & \cdots & C_{c-1} \end{bmatrix}$$

of order c^2 which has row and column sum c and hence element sum c^3 . The diagonal of each C_j , $j = 1, \dots, c$, is one by condition 1 of Lemma 11.10, so D has diagonal one. Moreover, each C_j is hermitian, $C_j^* = C_j$, so D is hermitian. \square

Lemma 11.12 (Kharaghani-Seberry [42]). *Let H, C_1, C_2, \dots, C_n be $(1, -1, i, -i)$ matrices of order n satisfying*

1. $HH^* = nI_n$; $HC_j = C_jH^*$;
2. $C_j^* = C_j$; $C_j C_k = 0$, $k \neq j$; $\sum_{j=1}^n C_j^2 = n^2 I_n$.

Then there is a complex Hadamard matrix of order $2n(n+1)$ of the form

$$D = \begin{bmatrix} A & iB \\ iB^* & A^* \end{bmatrix}$$

where A and B are block circulant. Furthermore, if H, C_1, C_2, \dots, C_n are real and H is regular, then D is regular.

Corollary 11.13. *For each positive integer n , there is a regular complex Hadamard matrix of order $4^n(4^n + 1)$.*

The next result is based on a similar theorem for real Hadamard matrices by Mukopadhyay [65].

Theorem 11.14 (Kharaghani-Seberry). *Suppose that there exists a skew-type complex Hadamard matrix $C = I + U$ of order $p+1$, where $U^* = -U$. Further, suppose that there exist two $(1, -1, i, -i)$ matrices A_r, B_r of order q satisfying*

1. $A_r B_r^* = B_r A_r^*$,
2. $A_r A_r^* + p B_r B_r^* = q(1+p)I_q$.

Then there are two $(1, -1, i, -i)$ matrices of order $p^j q$, $j \geq 0$, satisfying

$$A_{r+j} B_{r+j}^* = B_{r+j} A_{r+j}^*,$$

$$A_{r+j} A_{r+j}^* + p B_{r+j} B_{r+j}^* = q p^j (p+1) I.$$

Also, there exists a complex Hadamard matrix of order $q p^j (p+1)$ for every $j \geq 0$.

Corollary 11.15. *Let $n+1$ be the order of a symmetric conference matrix. Then there is a complex Hadamard matrix of order $n^j (n+1)$ for every $j \geq 0$.*

A result analogous to the next one was also found by R. Turyn [104].

Lemma 11.16 (Miyamoto [64]). *If there is an Hadamard matrix of order $4t$ with structure*

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix},$$

then there is a complex Hadamard matrix of order $2t$.

Proof. From the Hadamard matrix $AB^T = BA^T$ and $AA^T + BB^T = 4tI_{2t}$. Let

$$E = \frac{1}{2}(A+B) - \frac{i}{2}(A-B).$$

Then the elements of E are $1, -1, i, -i$ and

$$EE^* = \frac{1}{2}(AA^T + BB^T) + \frac{i}{2}(-AB^T + BA^T) = 2tI_{2t}.$$

Thus, E is the desired complex Hadamard matrix. Clearly, E will be a real matrix if and only if $A = B$. \square

This lemma, in view of many recent results on Williamson-type matrices gives us many new complex Hadamard matrices:

Corollary 11.17. *Let w be the order of a Williamson-type matrix. Then there exists a complex Hadamard matrix of order $2w$. In particular, there are complex Hadamard matrices for orders $2c$, $c \in \{33, 39, 53, 73, 81, 83, 89, 93, 101, 105, 109, 113, 125, 137, 149, 153, 173, 189, 193, 197, 233, 241, 243, 257, 277, 281, 293\}$.*

Kharaghani and Seberry went on to show how certain complex Hadamard matrices were extremely powerful in the construction of real Hadamard matrices with large excess.

Seberry and Whiteman [84] have also found complex weighing matrices analogous to the real matrices of Goethals and Seidel [25], and these matrices give some of the unsolved complex orthogonal designs of Geramita and Geramita [24].

11.2. Constructions Using Amicable Hadamard Matrices

Theorem 11.18 (Seberry-Wallis [114]). *Let $W = I + C$ be a complex skew Hadamard matrix of order w . Let $M = I + U$ and N be complex amicable orthogonal designs $\text{CAOD}(m; (1, m-1), (m))$ of order m satisfying $U^* = -U$ and $N^* = N$. Further, let X, Y, Z be pairwise amicable complex matrices of order p that are suitable matrices for a complex orthogonal design, $\text{COD}(wm; 1, m-1, (w-1)m)$:*

$$XX^* + (m-1)YY^* + (w-1)mZZ^* = wpmI.$$

Then there is a complex Hadamard matrix of order wpm .

Proof. Use $K = I \times I \times X + I \times U \times Y + C \times N \times Z$. □

Corollary 11.19. *Let $I + C$ be a complex skew Hadamard matrix of order w . Let X, Z be amicable complex matrices of order p that are suitable matrices for a $\text{COD}(w; 1, w-1)$. Then there is a complex Hadamard matrix of order pw .*

Proof. Put $m = 1$ in the theorem. □

Corollary 11.20 [89]. *Let $S = I + C$ be a complex skew Hadamard matrix of order w . Then there is a complex Hadamard matrix of order $w(w-1)$.*

We can use this corollary to form complex Hadamard matrices. In Table 11.1, the * signifies that a symmetric conference matrix for this order is not possible as $w(w-1)$ is not the sum of two squares. A number of other similar constructions are discussed in Seberry-Wallis [114, pp. 349-353], but we will not pursue them here.

TABLE 11.1

| w | Complex Hadamard order | Comment |
|-----|----------------------------|---------|
| 18 | 306 | |
| 26 | $650 = 59 \times 11 + 1$ | * |
| 30 | $870 = 789 \times 11 + 1$ | * |
| 38 | $1406 = 281 \times 5 + 1$ | |
| 50 | $2450 = 79 \times 31 + 1$ | * |
| 62 | $3782 = 199 \times 19 + 1$ | * |

Seberry and Zhang [89] have constructed amicable, disjoint $W(4mn, 2mn)$ U and V from Hadamard matrices of orders $4m$ and $4n$. Thus, we have

Theorem 11.21 (Seberry-Zhang [88]). *Suppose that $4m$ and $4n$ are the orders of Hadamard matrices. Then $U + iV$ (U, V above) is a complex Hadamard matrix of order $4mn$.*

The strong Kronecker product is used to prove Theorem 3.4.

11.3. Orders for Which Complex Hadamard Matrices Exist

We noted in Theorem 11.5 that symmetric conference matrices N always give a complex Hadamard matrix $iI + N$. So in Table 11.2 of complex Hadamard matrices, ci refers to the construction in Appendix A.1 for conference matrices. The construction $x2$ refers to Turyn's theorem [104], as well as to that of Kharaghani and Seberry [42] that Williamson-type matrices of order w give complex Hadamard matrices of order $2w$.

APPENDIX

A.1. Hadamard Matrices

One of us (Seberry) has a table containing odd integers $q < 40,000$ for which Hadamard matrices orders $2^t q$ exist. In Appendix A.3, we give this table for $q \leq 3000$. The key for the methods of construction follows: Note that not all construction methods appear, only those that, in the opinion of the authors, enabled us to compile the tables efficiently.

Amicable Hadamard Matrices

| <u>Key</u> | <u>Method</u> | <u>Explanation</u> |
|------------|---------------|---|
| a1 | $p^r + 1$ | $p^r \equiv 3 \pmod{4}$ is a prime power [110] |
| a2 | $2(q + 1)$ | $2q + 1$ is a prime power; $q \equiv 1 \pmod{4}$ is a prime [114] |
| a5 | nh | n, h , are amicable Hadamard matrices [110] |

Skew Hadamard Matrices

| <u>Key</u> | <u>Method</u> | <u>Explanation</u> |
|------------|-----------------|--|
| s1 | $2^t \prod k_i$ | t all positive integers; $k_i - 1 \equiv 3 \pmod{4}$ a prime power [66] |
| s2 | $(p - 1)^u + 1$ | p is a skew Hadamard matrix; $u > 0$ is an odd integer [105] |
| s3 | $2(q + 1)$ | $q \equiv 5 \pmod{8}$ is a prime power [98] |

TABLE 11.2 Complex Hadamard Matrices

| q | How | q | How | q | How | q | How | q | How |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 | | 89 | x2 | 177 | c1 | 265 | c1 | 353 | x2 |
| 3 | c1 | 91 | c1 | 179 | | 267 | | 355 | c1 |
| 5 | c1 | 93 | x2 | 181 | c1 | 269 | | 357 | |
| 7 | c1 | 95 | x2 | 183 | | 271 | c1 | 359 | |
| 9 | c1 | 97 | c1 | 185 | | 273 | | 361 | x2 |
| 11 | x2 | 99 | c1 | 187 | c1 | 275 | | 363 | x2 |
| 13 | c1 | 101 | x2 | 189 | x2 | 277 | x2 | 365 | c1 |
| 15 | c1 | 103 | | 191 | | 279 | c1 | 367 | c1 |
| 17 | x2 | 105 | x2 | 193 | x2 | 281 | x2 | 369 | |
| 19 | c1 | 107 | | 195 | c1 | 283 | | 371 | |
| 21 | c1 | 109 | x2 | 197 | x2 | 285 | c1 | 373 | x2 |
| 23 | c3 | 111 | | 199 | c1 | 287 | | 375 | x2 |
| 25 | c1 | 113 | c2 | 201 | c1 | 289 | c1 | 377 | |
| 27 | c1 | 115 | c1 | 203 | | 291 | | 379 | c1 |
| 29 | x2 | 117 | c1 | 205 | c1 | 293 | x2 | 381 | c1 |
| 31 | c1 | 119 | | 207 | | 295 | | 383 | |
| 33 | x2 | 121 | c1 | 209 | | 297 | c1 | 385 | c1 |
| 35 | | 123 | c3 | 211 | c1 | 299 | | 387 | c1 |
| 37 | c1 | 125 | x2 | 213 | | 301 | c1 | 389 | x2 |
| 39 | x2 | 127 | | 215 | | 303 | | 391 | |
| 41 | c1 | 129 | c1 | 217 | c1 | 305 | | 393 | |
| 43 | x2 | 131 | | 219 | | 307 | c1 | 395 | |
| 45 | c1 | 133 | | 221 | | 309 | c1 | 397 | x2 |
| 47 | | 135 | c1 | 223 | | 311 | | 399 | c1 |
| 49 | c1 | 137 | x2 | 225 | c1 | 313 | c1 | 401 | x2 |
| 51 | c1 | 139 | c1 | 227 | | 315 | x2 | 403 | |
| 53 | x2 | 141 | c1 | 229 | c1 | 317 | x2 | 405 | c1 |
| 55 | c1 | 143 | | 231 | c1 | 319 | | 407 | |
| 57 | c1 | 145 | c1 | 233 | x2 | 321 | c1 | 409 | x2 |
| 59 | | 147 | c1 | 235 | | 323 | | 411 | c1 |
| 61 | c1 | 149 | x2 | 237 | | 325 | x2 | 413 | |
| 63 | c1 | 151 | | 239 | | 327 | c1 | 415 | c1 |
| 65 | | 153 | x2 | 241 | x2 | 329 | | 417 | |
| 67 | | 155 | | 243 | x2 | 331 | c1 | 419 | |
| 69 | c1 | 157 | c1 | 245 | | 333 | | 421 | c1 |
| 71 | | 159 | c1 | 247 | | 335 | | 423 | x2 |
| 73 | x2 | 161 | | 249 | | 337 | c1 | 425 | |
| 75 | c1 | 163 | | 251 | | 339 | c1 | 427 | c1 |
| 77 | | 165 | | 253 | | 341 | | 429 | c1 |
| 79 | c1 | 167 | | 255 | c1 | 343 | | 431 | |
| 81 | x2 | 169 | c1 | 257 | x2 | 345 | | 433 | x2 |
| 83 | x2 | 171 | | 259 | | 347 | | 435 | x2 |
| 85 | c1 | 173 | x2 | 261 | c1 | 349 | x2 | 437 | |
| 87 | c1 | 175 | c1 | 263 | | 351 | c1 | 439 | c1 |

TABLE 11.2 Complex Hadamard Matrices (continued)

| <i>q</i> | How | <i>q</i> | How | <i>q</i> | How | <i>q</i> | How | <i>q</i> | How |
|----------|-----|----------|-----|----------|-----|----------|-----|----------|-----|
| 441 | c1 | 529 | x2 | 617 | x2 | 705 | c1 | 793 | |
| 443 | | 531 | c1 | 619 | c1 | 707 | | 795 | |
| 445 | | 533 | | 621 | | 709 | x2 | 797 | x2 |
| 447 | | 535 | c1 | 623 | | 711 | | 799 | c1 |
| 449 | x2 | 537 | | 625 | c1 | 713 | | 801 | c1 |
| 451 | x2 | 539 | | 627 | x2 | 715 | c1 | 803 | |
| 453 | | 541 | x2 | 629 | | 717 | c1 | 805 | c1 |
| 455 | | 543 | x2 | 631 | | 719 | | 807 | c1 |
| 457 | x2 | 545 | c3 | 633 | | 721 | | 809 | x2 |
| 459 | x2 | 547 | c1 | 635 | | 723 | | 811 | c1 |
| 461 | x2 | 549 | c1 | 637 | | 725 | | 813 | |
| 463 | | 551 | | 639 | c1 | 727 | c1 | 815 | |
| 465 | c1 | 553 | | 641 | x2 | 729 | x2 | 817 | |
| 467 | | 555 | c1 | 643 | | 731 | | 819 | c1 |
| 469 | c1 | 557 | x2 | 645 | c1 | 733 | | 821 | x2 |
| 471 | c1 | 559 | c1 | 647 | | 735 | x2 | 823 | |
| 473 | x2 | 561 | x6 | 649 | c1 | 737 | | 825 | |
| 475 | | 563 | | 651 | c1 | 739 | | 827 | |
| 477 | c1 | 565 | c1 | 653 | | 741 | c1 | 829 | c1 |
| 479 | | 567 | | 655 | | 743 | | 831 | |
| 481 | c1 | 569 | x2 | 657 | | 745 | c1 | 833 | |
| 483 | | 571 | | 659 | | 747 | c1 | 835 | c1 |
| 485 | | 573 | | 661 | c1 | 749 | | 837 | |
| 487 | | 575 | | 663 | x2 | 751 | | 839 | |
| 489 | c1 | 577 | c1 | 665 | | 753 | | 841 | c1 |
| 491 | | 579 | x2 | 667 | | 755 | | 843 | x2 |
| 493 | | 581 | | 669 | | 757 | x2 | 845 | |
| 495 | | 583 | | 671 | | 759 | x2 | 847 | c1 |
| 497 | | 585 | | 673 | x2 | 761 | c2 | 849 | c1 |
| 499 | c1 | 587 | | 675 | x2 | 763 | | 851 | |
| 501 | | 589 | | 677 | x2 | 765 | | 853 | |
| 503 | | 591 | c1 | 679 | | 767 | | 855 | c1 |
| 505 | c1 | 593 | x2 | 681 | c1 | 769 | x2 | 857 | |
| 507 | c1 | 595 | | 683 | | 771 | | 859 | |
| 509 | | 597 | c1 | 685 | c1 | 773 | x2 | 861 | c1 |
| 511 | c1 | 599 | | 687 | c1 | 775 | c1 | 863 | |
| 513 | | 601 | c1 | 689 | | 777 | c1 | 865 | |
| 515 | | 603 | | 691 | c1 | 779 | | 867 | c1 |
| 517 | c1 | 605 | | 693 | | 781 | | 869 | |
| 519 | | 607 | c1 | 695 | | 783 | | 871 | c1 |
| 521 | x2 | 609 | c1 | 697 | | 785 | | 873 | |
| 523 | | 611 | | 699 | | 787 | | 875 | |
| 525 | c1 | 613 | c2 | 701 | x2 | 789 | | 877 | c1 |
| 527 | | 615 | c1 | 703 | x2 | 791 | | 879 | x2 |

TABLE 11.2 Complex Hadamard Matrices (continued)

| q | How | q | How | q | How | q | How | q | How |
|-----|------|-----|------|-----|------|-----|------|-----|------|
| 881 | $x2$ | 905 | | 929 | $x2$ | 953 | $x2$ | 977 | $x2$ |
| 883 | | 907 | | 931 | $c1$ | 955 | | 979 | |
| 885 | $x2$ | 909 | | 933 | | 957 | $c1$ | 981 | |
| 887 | | 911 | | 935 | | 959 | | 983 | |
| 889 | $c1$ | 913 | | 937 | $c1$ | 961 | $x2$ | 985 | |
| 891 | | 915 | | 939 | $c1$ | 963 | | 987 | $c1$ |
| 893 | | 917 | | 941 | | 965 | | 989 | |
| 895 | $c1$ | 919 | | 943 | | 967 | $c1$ | 991 | |
| 897 | | 921 | | 945 | $c1$ | 969 | | 993 | |
| 899 | | 923 | | 947 | | 971 | | 995 | |
| 901 | $c1$ | 925 | $c1$ | 949 | | 973 | | 997 | $c1$ |
| 903 | | 927 | | 951 | $c1$ | 975 | $c1$ | 999 | $c1$ |

Skew Hadamard Matrices (continued)

| | | |
|------|------------------|--|
| $s4$ | $2(q+1)$ | $q = p^t$ is a prime power where $p \equiv 5 \pmod{8}$ and $t \equiv 2 \pmod{4}$ [99, 125] |
| $s5$ | $4m$ | $3 \leq m \leq 33, 127$ [35, 100, 18a] $m \in \{37, 43, 67, 113, 127, 157, 163, 181, 241\}$ [17, 16] |
| $s6$ | $4(q+1)$ | $q \equiv 9 \pmod{16}$ is a prime power [113] |
| $s7$ | $(t +1)(q+1)$ | $q = s^2 + 4t^2 \equiv 5 \pmod{8}$ is a prime power; $ t +1$ is a skew Hadamard matrix [117] |
| $s8$ | $4(q^2 + q + 1)$ | q is a prime power, $q^2 + q + 1 \equiv 3, 5, 7 \pmod{8}$ a prime, or $2(q^2 + q + 1) + 1$ is a prime power [94] |
| $s0$ | hm | h is a skew Hadamard matrix; m is an amicable Hadamard matrix [114] |

Spence Hadamard Matrices

| <u>Key</u> | <u>Method</u> | <u>Explanation</u> |
|------------|------------------|---|
| $p1$ | $4(q^2 + q + 1)$ | $q^2 + q + 1 \equiv 1 \pmod{8}$ is a prime [94] |
| $p2$ | $4n$ or $8n$ | $n, n-2$ are prime powers; if $n \equiv 1 \pmod{4}$, there exists a Hadamard matrix of order $4n$; if $n \equiv 3 \pmod{4}$, there exists a Hadamard matrix of order $8n$ [93] |
| $p3$ | $4m$ | m is an odd prime power for which an integer $s \geq 0$ such that $(m - (2^{s+1} + 1))/2^{s+1}$ is an odd prime power [93] |

Conference Matrices That Give Symmetric Hadamard Matrices The following methods give symmetric Hadamard matrices of order $2n$ and conference

matrices of order n with the exception of c6 which produces an Hadamard matrix. The order of the Hadamard matrix is given in the column headed "Method."

| <u>Key</u> | <u>Method</u> | <u>Explanation</u> |
|------------|----------------------------|---|
| c1 | $2(p^r + 1)$ | $p^r \equiv 1 \pmod{4}$ is a prime power [66, 25] |
| c2 | $2((h-1)^2 + 1)$ | h is a skew Hadamard matrix [7] |
| c3 | $2(q^2(q-2) + 1)$ | $q \equiv 3 \pmod{4}$ is a prime power $q-2$ is a prime power [63] |
| c4 | $2(5 \cdot 9^{2^t+1} + 1)$ | $t \geq 0$ [85] |
| c5 | $2((n-1)^s + 1)$ | n is a conference matrix $s \geq 2$ [105] |
| c6 | nh | n is a conference matrix h is a Hadamard matrix [25] |

Note: A conference matrix of order $n \equiv 2 \pmod{4}$ exists only if $n-1$ is the sum of two squares.

Hadamard Matrices Obtained from Williamson Matrices If a Williamson matrix of order $2^t q$ exists, then there is a Hadamard matrix of order $2^{t+2} q$, the same key as in the Index of Williamson Matrices in Appendix A.2 is used to index the Hadamard matrices produced from them.

OD Hadamard Matrices

| <u>Key</u> | <u>Method</u> | <u>Explanation</u> |
|------------|---------------|--|
| o1 | $2^{t+2} q$ | If a T -matrix of order $2^t q$ exists, then there is a Hadamard matrix of order $2^{t+2} q$ [12, 108] |
| o2 | ow | o is an OD-Hadamard matrix; w is a Williamson matrix [6, 12, 115] |
| o3 | $8pw$ | an $OD(8p; p, p, p, p, p, p, p, p)$ exists for $p = 1, 3$; there exist 8-Williamson matrices of order w [67] |

Yamada Hadamard Matrices

| <u>Key</u> | <u>Method</u> | <u>Explanation</u> |
|------------|---------------|--|
| y1 | $4q$ | $q \equiv 1 \pmod{8}$ is a prime power; $(q-1)/2$ is a Hadamard matrix [132] |
| y2 | $4(q+2)$ | $q \equiv 5 \pmod{8}$ is a prime power; $(q+3)/2$ is a skew Hadamard matrix [132] |
| y3 | $4(q+2)$ | $q \equiv 1 \pmod{8}$ is a prime power; $(q+3)/2$ is a conference matrix [132] |

Miyamoto Hadamard Matrices

| <u>Key</u> | <u>Method</u> | <u>Explanation</u> |
|------------|---------------|--|
| <i>m1</i> | $4q$ | $q \equiv 1 \pmod{4}$ is a prime power; $q - 1$ is a Hadamard matrix [64] |
| <i>m2</i> | $8q$ | $q \equiv 3 \pmod{4}$ is a prime power; $2q - 3$ is a prime power [64] |

Koukouvinos and Kounias

| <u>Key</u> | <u>Method</u> | <u>Explanation</u> |
|------------|---------------|---|
| <i>k1</i> | $2^t q$ | $2^t q = g_1 + g_2$, where g_1 and g_2 are the lengths of Golay sequences [53] |

Agayan Multiplication

| <u>Key</u> | <u>Method</u> | <u>Explanation</u> |
|------------|----------------|---|
| <i>d1</i> | $2^{t+s-1} pq$ | $2^t p$ and $2^s q$ are the orders of Hadamard matrices [1] |

Seberry

| <u>Key</u> | <u>Method</u> | <u>Explanation</u> |
|------------|---------------|--|
| <i>se</i> | $2^t q$ | t is the smallest integer such that for given odd q , $a(q + 1) + b(q - 3) = 2^t$ has a solution for a, b nonnegative integers [121] |

Craigen-Seberry-Zhang

| <u>Key</u> | <u>Method</u> | <u>Explanation</u> |
|------------|-----------------|---|
| <i>cz</i> | $2^{t+s+u+w-4}$ | $2^t a, 2^s b, 2^u c, 2^w d$ are the orders of Hadamard matrices [14] |

A.2. Index of Williamson Matrices

One of us (Seberry) has a list on the computer of odd integers $q < 40,000$ for which Williamson or Williamson type matrices exist. The following legend gives a list of constructions for these matrices, the method used, and the discoverer—with apologies to anyone excluded:

| <u>Key</u> | <u>Method</u> | <u>Explanation</u> |
|------------|------------------------------|---|
| w1 | {1, ..., 33, 37, 39, 41, 43} | [52, 18, 130] |
| w2 | $(p + 1)/2$ | $p \equiv 1 \pmod{4}$ a prime power [26, 106, 126] |
| w3 | 3^d | d a natural number [65, 109] |
| w4 | $[p(p + 1)]/2$ | $p \equiv 1 \pmod{4}$ a prime power [112, 127] |
| w5 | $s(4s + 3), s(4s - 1)$ | $s \in \{1, 3, 5, \dots, 31\}$ [120] |
| w6 | 93 | [120] |
| w7 | $[(f - 1)(4f + 1)]/4$ | $p = 4f + 1$, f odd, is a prime power of the form $1 + 4t^2$; $(f - 1)/8$ is the order of a good matrix [118] |
| w8 | $[(f + 1)(4f + 1)]/4$ | $p = 4f + 1$, f odd, is a prime power of the form $25 + 4t^2$; $(f + 1)/8$ is the order of a good matrix [118] |
| w9 | $[p(p - 1)]/2$ | $p = 4f + 1$ is a prime power; $(p - 1)/4$ is the order of a good matrix [118] |
| w0 | $(p + 2)(p + 1)$ | $p \equiv 1 \pmod{4}$ a prime power; $p + 3$ is the order of a symmetric Hadamard matrix [118] |
| wa | $[(f + 1)(4f + 1)]/2$ | $p = 4f + 1$, f odd, is a prime power of the form $9 + 4t^2$; $(f - 1)/2 \equiv 1 \pmod{4}$ a prime power [118] |
| wb | $[(f - 1)(4f + 1)]/2$ | $p = 4f + 1$, f odd, is a prime power of the form $49 + 4t^2$; $(f - 3)/2 \equiv 1 \pmod{4}$ a prime power [118] |
| wc | $2p + 1$ | $q = 2p - 1$ is a prime power, p is a prime [64, 87] |
| wd | $7 \cdot 3^i$ | $i \geq 0$ [65] |
| w#e | $7^{i+1}, 11 \cdot 7^i$ | $i \geq 0$ (gives 8-Williamson matrices) [78] |
| wf | $q^d(q + 1)/2$ | $q \equiv 1 \pmod{4}$ is a prime power, $d \geq 2$ [65, 95a] |
| wg | $p^2(p + 1)/2$ | $p \equiv 1 \pmod{4}$ is a prime power [80] |
| wh | $p^2(p + 1)/4$ | $p \equiv 3 \pmod{4}$ is a prime power; $(p + 1)/4$ is the order of a Williamson-type matrix [80] |
| wi | $q + 2$ | $q \equiv 1 \pmod{4}$ is a prime power; $(q + 1)/2$ is a prime power [64] |
| wj | $q + 2$ | $q \equiv 1 \pmod{4}$ is a prime power; $(q + 3)/2$ is the order of a symmetric conference matrix [64] |

| | | |
|--------|-------------------|---|
| wk | q | $q \equiv 1 \pmod{4}$ is a prime power; $(q-1)/2$ is the order of a symmetric conference matrix or the order of a symmetric Hadamard matrix [64] |
| wl | q | $q \equiv 1 \pmod{4}$ is a prime power; $(q-1)/4$ is the order of a Williamson-type matrix [64] |
| wm | q | $q \equiv 1 \pmod{4}$ is a prime power; $(q-1)/2$ is the order of a Hadamard matrix [87] |
| wn | wn | w is the order of a Williamson-type matrix; n is the order of a symmetric conference matrix |
| wo | $2wu$ | w and u are the orders of Williamson-type matrices [87] |
| $w\#p$ | $2q+1$ | $q+1$ is the order of an amicable Hadamard matrix; q is the order of a Williamson type matrix [87] |
| $w\#q$ | q | q is a prime power; and $(q-1)/2$ is the order of a Williamson-type matrix [87] |
| $w\#r$ | $2q+1$ | $q+1$ is the order of a symmetric conference matrix; q is the order of a Williamson-type matrix [87] |
| $w\#s$ | $2 \cdot 9^t + 1$ | $t > 0$ [87] |

$S = \{1, \dots, 31\}$ is the set of good matrices.

Note: The fact that if there is a Williamson matrix of order n , then there is a Williamson matrix of order $2n$, is used in the calculation of wh .

We now give in Table A.1 known Williamson-type matrices of orders < 2000 . The order in which the algorithms were applied was $w_1, w_2, w_3, w_4, w_5, w_6, wi, wj, wk, wl, wn, w\#p, w\#q, w\#r$, and then others if it appeared they might give a new order. To interpret the results in the table, we note that if there is an Hadamard matrix of order $4q$, then it can be a Williamson-type matrix, but this was not included. A notation $w\#x$ means that 8-Williamson matrices are known, but not four, so an $OD(8s; s, s, s, s, s, s, s, s)$ is needed to get an Hadamard matrix. The notation $47, 3, w\#p$ means that there are 8-Williamson matrices of order 47, and thus an Hadamard matrix of order $8 \cdot 47$. A notation with wn of 3 indicates that there are four Williamson-type matrices but they are of even order. The notation $35, 3, wn$ means that there are four Williamson-type matrices of order 70 and an Hadamard matrix of order 280.

TABLE A.1 Williamson and Williamson-Type Matrices

| <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How |
|----------|----------|-----|----------|----------|-----|----------|----------|-----|----------|----------|-----|----------|----------|-----|
| 1 | | w1 | 85 | | w2 | 169 | | w2 | 253 | 3 | wn | 337 | | w2 |
| 3 | | w1 | 87 | | w2 | 171 | 3 | wn | 255 | | w2 | 339 | | w2 |
| 5 | | w1 | 89 | | w1 | 173 | | w1 | 257 | | w1 | 341 | 3 | wn |
| 7 | | w1 | 91 | | w2 | 175 | | w2 | 259 | 3 | wn | 343 | 3 | wn |
| 9 | | w1 | 93 | | w5 | 177 | | w2 | 261 | | w2 | 345 | 3 | wn |
| 11 | | w1 | 95 | | w6 | 179 | 3 | w#q | 263 | | | 347 | 3 | w#q |
| 13 | | w1 | 97 | | w2 | 181 | | w2 | 265 | | w2 | 349 | | wk |
| 15 | | w1 | 99 | | w2 | 183 | 3 | wn | 267 | 3 | wn | 351 | | w2 |
| 17 | | w1 | 101 | | wk | 185 | 3 | wn | 269 | | | 353 | | wl |
| 19 | | w1 | 103 | 3 | w#q | 187 | | w2 | 271 | | w2 | 355 | | w2 |
| 21 | | w1 | 105 | 3 | wn | 189 | | w5 | 273 | 3 | wn | 357 | 3 | wn |
| 23 | | w1 | 107 | 3 | w#q | 191 | 3 | w#p | 275 | 3 | wn | 359 | | |
| 25 | | w1 | 109 | | wk | 193 | | wk | 277 | | wk | 361 | | wk |
| 27 | | w1 | 111 | 3 | wn | 195 | | w2 | 279 | | w2 | 363 | | wi |
| 29 | | w1 | 113 | | wk | 197 | | wk | 281 | | wl | 365 | | w2 |
| 31 | | w1 | 115 | | w2 | 199 | | w2 | 283 | 3 | w#q | 367 | | w2 |
| 33 | | w1 | 117 | | w2 | 201 | | w2 | 285 | | w2 | 369 | 3 | wn |
| 35 | 3 | wn | 119 | 3 | wn | 203 | 3 | w9 | 287 | 3 | wn | 371 | 3 | wn |
| 37 | | w1 | 121 | | w2 | 205 | | w2 | 289 | | w2 | 373 | | wl |
| 39 | | w1 | 123 | | wi | 207 | 3 | wn | 291 | 3 | wn | 375 | | wf |
| 41 | | w1 | 125 | | wk | 209 | 3 | wn | 293 | | wl | 377 | 3 | wn |
| 43 | | w1 | 127 | 3 | w#p | 211 | | w2 | 295 | | | 379 | | w2 |
| 45 | | w2 | 129 | | w2 | 213 | | | 297 | | w2 | 381 | | w2 |
| 47 | 3 | w#p | 131 | | | 215 | 3 | wn | 299 | 3 | wn | 383 | | |
| 49 | | w2 | 133 | 3 | wn | 217 | | w2 | 301 | | w2 | 385 | | w2 |
| 51 | | w2 | 135 | | w2 | 219 | 3 | wn | 303 | 3 | w7 | 387 | | w2 |
| 53 | | wk | 137 | | w1 | 221 | 3 | wn | 305 | 3 | wn | 389 | | wk |
| 55 | | w2 | 139 | | w2 | 223 | | | 307 | | w2 | 391 | 3 | wn |
| 57 | | w2 | 141 | | w2 | 225 | | w2 | 309 | | w2 | 393 | | |
| 59 | 3 | w#q | 143 | 3 | wn | 227 | 3 | w#q | 311 | | | 395 | 3 | wn |
| 61 | | w2 | 145 | | w2 | 229 | | w2 | 313 | | w2 | 397 | | wk |
| 63 | | w2 | 147 | | w2 | 231 | | w2 | 315 | | w5 | 399 | | w2 |
| 65 | 3 | wn | 149 | | wk | 233 | | w1 | 317 | | wk | 401 | | wk |
| 67 | 3 | w#q | 151 | 3 | w#q | 235 | | | 319 | 3 | w0 | 403 | 3 | wn |
| 69 | | w2 | 153 | | w4 | 237 | 3 | wn | 321 | | w2 | 405 | | w2 |
| 71 | | | 155 | 3 | wn | 239 | | | 323 | 3 | wn | 407 | 3 | wn |
| 73 | | wk | 157 | | w2 | 241 | | wk | 325 | | w4 | 409 | | wk |
| 75 | | w2 | 159 | | w2 | 243 | | wj | 327 | | w2 | 411 | | w2 |
| 77 | 3 | wn | 161 | 3 | wn | 245 | 3 | wn | 329 | | | 413 | | |
| 79 | | w2 | 163 | 3 | w#q | 247 | 3 | wn | 331 | | w2 | 415 | | w2 |
| 81 | | w3 | 165 | 3 | wn | 249 | 3 | wn | 333 | 3 | w9 | 417 | 3 | wn |
| 83 | | wi | 167 | 3 | w#p | 251 | 3 | w#q | 335 | | | 419 | | |

TABLE A.1 Williamson and Williamson-Type Matrices (continued)

| <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How |
|----------|----------|-----|----------|----------|-----|----------|----------|-----|----------|----------|-----|----------|----------|-----|
| 421 | | w2 | 513 | 3 | wn | 605 | 3 | wn | 697 | 3 | wn | 789 | | |
| 423 | | wi | 515 | 3 | w#r | 607 | | w2 | 699 | 3 | wn | 791 | 3 | wn |
| 425 | 3 | wn | 517 | | w2 | 609 | | w2 | 701 | | wk | 793 | 3 | wn |
| 427 | | w2 | 519 | 3 | wn | 611 | | | 703 | | w4 | 795 | 3 | wn |
| 429 | | w2 | 521 | | wl | 613 | | wl | 705 | | w2 | 797 | | wk |
| 431 | | | 523 | 3 | w#q | 615 | | w2 | 707 | 3 | wn | 799 | | w2 |
| 433 | | wk | 525 | | w2 | 617 | | wl | 709 | | wk | 801 | | w2 |
| 435 | | w4 | 527 | 3 | wn | 619 | | w2 | 711 | 3 | wn | 803 | 3 | wo |
| 437 | 3 | wn | 529 | | wl | 621 | 3 | wn | 713 | 3 | wn | 805 | | w2 |
| 439 | | w2 | 531 | | w2 | 623 | 3 | wn | 715 | | w2 | 807 | | w2 |
| 441 | | w2 | 533 | 3 | wn | 625 | | w2 | 717 | | w2 | 809 | | wl |
| 443 | | | 535 | | w2 | 627 | | wi | 719 | | | 811 | | w2 |
| 445 | 3 | wn | 537 | | | 629 | 3 | wn | 721 | | | 813 | 3 | wn |
| 447 | 3 | wn | 539 | 3 | wn | 631 | 3 | w#q | 723 | 3 | wn | 815 | | |
| 449 | | wk | 541 | | wk | 633 | 3 | wn | 725 | 3 | wn | 817 | 3 | wn |
| 451 | | wj | 543 | | wi | 635 | 3 | w#r | 727 | | w2 | 819 | | w2 |
| 453 | | | 545 | 3 | wn | 637 | 3 | wn | 729 | | w3 | 821 | | wk |
| 455 | 3 | wn | 547 | | w2 | 639 | | w2 | 731 | 3 | wo | 823 | 3 | w#q |
| 457 | | wk | 549 | | w2 | 641 | | wk | 733 | 3 | w#q | 825 | 3 | wn |
| 459 | | wi | 551 | 3 | wn | 643 | 3 | w#q | 735 | | wi | 827 | | |
| 461 | | wk | 553 | 3 | wn | 645 | | w2 | 737 | | | 829 | | w2 |
| 463 | 3 | w#q | 555 | | w2 | 647 | | | 739 | | | 831 | 3 | wn |
| 465 | | w2 | 557 | | wk | 649 | | w2 | 741 | | w2 | 833 | 3 | wn |
| 467 | 3 | w#q | 559 | | w2 | 651 | | w2 | 743 | | | 835 | | w2 |
| 469 | | w2 | 561 | 3 | wn | 653 | | | 745 | | w2 | 837 | 3 | wn |
| 471 | | w2 | 563 | 3 | w#q | 655 | 3 | w#p | 747 | | w2 | 839 | | |
| 473 | | w5 | 565 | | w2 | 657 | 3 | wn | 749 | | | 841 | | w2 |
| 475 | 3 | wn | 567 | 3 | wn | 659 | | | 751 | 3 | w#q | 843 | | wi |
| 477 | | w2 | 569 | | wm | 661 | | w2 | 753 | | | 845 | 3 | wn |
| 479 | | | 571 | 3 | w#q | 663 | | w5 | 755 | | | 847 | | w2 |
| 481 | | w2 | 573 | | | 665 | 3 | wn | 757 | | wl | 849 | | w2 |
| 483 | 3 | wn | 575 | 3 | wn | 667 | 3 | wn | 759 | | wi | 851 | 3 | wn |
| 485 | 3 | wn | 577 | | w2 | 669 | | | 761 | | wl | 853 | | |
| 487 | 3 | w#p | 579 | | wj | 671 | 3 | wn | 763 | | | 855 | | w2 |
| 489 | | w2 | 581 | 3 | wn | 673 | | wk | 765 | 3 | wn | 857 | | |
| 491 | | | 583 | 3 | wo | 675 | | wi | 767 | | | 859 | 3 | w#q |
| 493 | 3 | wo | 585 | 3 | wn | 677 | | wk | 769 | | wk | 861 | | w2 |
| 495 | 3 | wn | 587 | 3 | w#q | 679 | 3 | wn | 771 | 3 | wn | 863 | | |
| 497 | | | 589 | 3 | wn | 681 | | w2 | 773 | | wl | 865 | 3 | wn |
| 499 | | w2 | 591 | | w2 | 683 | | | 775 | | w2 | 867 | | w2 |
| 501 | | | 593 | | wk | 685 | | w2 | 777 | | w2 | 869 | 3 | wn |
| 503 | | | 595 | 3 | wn | 687 | | w2 | 779 | 3 | wn | 871 | | w2 |
| 505 | | w2 | 597 | | w2 | 689 | 3 | w9 | 781 | | | 873 | 3 | wn |
| 507 | | w2 | 599 | | | 691 | | w2 | 783 | 3 | wn | 875 | 3 | wn |
| 509 | | | 601 | | w2 | 693 | 3 | wn | 785 | 3 | wn | 877 | | w2 |
| 511 | | w2 | 603 | 3 | wn | 695 | 3 | wn | 787 | | | 879 | | wi |

TABLE A.1 Williamson and Williamson-Type Matrices (continued)

| <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How |
|----------|----------|------------|----------|----------|------------|----------|----------|------------|----------|----------|------------|----------|----------|------------|
| 881 | | <i>wk</i> | 973 | 3 | <i>wn</i> | 1065 | | <i>w2</i> | 1157 | 3 | <i>wn</i> | 1249 | | <i>wl</i> |
| 883 | 3 | <i>w#q</i> | 975 | | <i>w2</i> | 1067 | 3 | <i>wn</i> | 1159 | 3 | <i>wn</i> | 1251 | | <i>wi</i> |
| 885 | | <i>w5</i> | 977 | | <i>wk</i> | 1069 | | <i>w2</i> | 1161 | 3 | <i>wn</i> | 1253 | | |
| 887 | | | 979 | 3 | <i>wo</i> | 1071 | | <i>w2</i> | 1163 | | | 1255 | | |
| 889 | | <i>w2</i> | 981 | 3 | <i>wn</i> | 1073 | 3 | <i>wn</i> | 1165 | 3 | <i>wn</i> | 1257 | | |
| 891 | 3 | <i>wn</i> | 983 | | | 1075 | 3 | <i>wn</i> | 1167 | | <i>w2</i> | 1259 | | |
| 893 | | | 985 | 3 | <i>wn</i> | 1077 | | <i>w2</i> | 1169 | | | 1261 | | <i>w2</i> |
| 895 | | <i>w2</i> | 987 | | <i>w2</i> | 1079 | 3 | <i>wn</i> | 1171 | | <i>w2</i> | 1263 | 3 | <i>wn</i> |
| 897 | 3 | <i>wn</i> | 989 | 3 | <i>wn</i> | 1081 | | <i>w2</i> | 1173 | 3 | <i>wn</i> | 1265 | 3 | <i>wn</i> |
| 899 | 3 | <i>wn</i> | 991 | | | 1083 | 3 | <i>wn</i> | 1175 | | | 1267 | 3 | <i>wn</i> |
| 901 | | <i>w2</i> | 993 | 3 | <i>wn</i> | 1085 | 3 | <i>wn</i> | 1177 | | | 1269 | 3 | <i>wn</i> |
| 903 | 3 | <i>wn</i> | 995 | 3 | <i>wn</i> | 1087 | 3 | <i>w#p</i> | 1179 | | <i>w2</i> | 1271 | 3 | <i>wn</i> |
| 905 | 3 | <i>wn</i> | 997 | | <i>w2</i> | 1089 | 3 | <i>wn</i> | 1181 | | | 1273 | | |
| 907 | | | 999 | | <i>w2</i> | 1091 | | | 1183 | | <i>wf</i> | 1275 | | <i>w2</i> |
| 909 | 3 | <i>wn</i> | 1001 | 3 | <i>wn</i> | 1093 | 3 | <i>w#q</i> | 1185 | 3 | <i>wn</i> | 1277 | 3 | <i>w#q</i> |
| 911 | | | 1003 | | | 1095 | | <i>wi</i> | 1187 | 3 | <i>w#q</i> | 1279 | | <i>w2</i> |
| 913 | 3 | <i>wo</i> | 1005 | 3 | <i>wn</i> | 1097 | | <i>wl</i> | 1189 | | <i>w2</i> | 1281 | 3 | <i>wn</i> |
| 915 | 3 | <i>w9</i> | 1007 | 3 | <i>wn</i> | 1099 | | <i>w2</i> | 1191 | | <i>w2</i> | 1283 | 3 | <i>w#q</i> |
| 917 | | | 1009 | | <i>w2</i> | 1101 | 3 | <i>wn</i> | 1193 | | <i>wl</i> | 1285 | 3 | <i>wn</i> |
| 919 | 3 | <i>w#q</i> | 1011 | 3 | <i>wn</i> | 1103 | | | 1195 | | <i>w2</i> | 1287 | 3 | <i>wn</i> |
| 921 | 3 | <i>wn</i> | 1013 | 3 | <i>wn</i> | 1105 | | <i>w2</i> | 1197 | | <i>w2</i> | 1289 | | <i>wl</i> |
| 923 | 3 | <i>w#r</i> | 1015 | | <i>w2</i> | 1107 | | <i>w2</i> | 1199 | 3 | <i>wo</i> | 1291 | 3 | <i>w#q</i> |
| 925 | | <i>w2</i> | 1017 | 3 | <i>wn</i> | 1109 | | <i>wl</i> | 1201 | | <i>w2</i> | 1293 | | |
| 927 | 3 | <i>wn</i> | 1019 | 3 | <i>w#r</i> | 1111 | | <i>w2</i> | 1203 | | <i>wi</i> | 1295 | 3 | <i>wn</i> |
| 929 | | <i>wl</i> | 1021 | | <i>wk</i> | 1113 | 3 | <i>wn</i> | 1205 | 3 | <i>wn</i> | 1297 | | <i>w2</i> |
| 931 | | <i>w2</i> | 1023 | 3 | <i>wn</i> | 1115 | 3 | <i>w#r</i> | 1207 | | <i>w5</i> | 1299 | 3 | <i>wn</i> |
| 933 | | | 1025 | 3 | <i>wn</i> | 1117 | | <i>wk</i> | 1209 | | <i>w2</i> | 1301 | | <i>wl</i> |
| 935 | 3 | <i>wn</i> | 1027 | | <i>w2</i> | 1119 | | <i>w2</i> | 1211 | 3 | <i>wn</i> | 1303 | 3 | <i>w#q</i> |
| 937 | | <i>w2</i> | 1029 | 3 | <i>wn</i> | 1121 | | | 1213 | 3 | <i>w#q</i> | 1305 | | <i>w2</i> |
| 939 | | <i>w2</i> | 1031 | | | 1123 | | | 1215 | | <i>wi</i> | 1307 | 3 | <i>w#r</i> |
| 941 | | | 1033 | | <i>wk</i> | 1125 | 3 | <i>wn</i> | 1217 | | <i>wk</i> | 1309 | | <i>w2</i> |
| 943 | 3 | <i>wn</i> | 1035 | | <i>w2</i> | 1127 | 3 | <i>wn</i> | 1219 | | <i>w2</i> | 1311 | | <i>w2</i> |
| 945 | | <i>w2</i> | 1037 | 3 | <i>wn</i> | 1129 | | <i>wk</i> | 1221 | | <i>w2</i> | 1313 | 3 | <i>wn</i> |
| 947 | 3 | <i>w#q</i> | 1039 | | | 1131 | 3 | <i>wn</i> | 1223 | | | 1315 | | |
| 949 | 3 | <i>wn</i> | 1041 | | <i>w2</i> | 1133 | | | 1225 | | <i>w4</i> | 1317 | | <i>w2</i> |
| 951 | | <i>w2</i> | 1043 | 3 | <i>wn</i> | 1135 | | <i>w2</i> | 1227 | 3 | <i>wn</i> | 1319 | | |
| 953 | | <i>wl</i> | 1045 | | <i>w2</i> | 1137 | | <i>w2</i> | 1229 | | <i>wk</i> | 1321 | | <i>wl</i> |
| 955 | | | 1047 | 3 | <i>wn</i> | 1139 | | <i>w5</i> | 1231 | 3 | <i>w#p</i> | 1323 | | <i>wi</i> |
| 957 | | <i>w2</i> | 1049 | | <i>wm</i> | 1141 | | <i>w2</i> | 1233 | 3 | <i>wn</i> | 1325 | 3 | <i>wn</i> |
| 959 | 3 | <i>wn</i> | 1051 | 3 | <i>w#q</i> | 1143 | 3 | <i>wn</i> | 1235 | 3 | <i>wn</i> | 1327 | 3 | <i>w#p</i> |
| 961 | | <i>wk</i> | 1053 | 3 | <i>wn</i> | 1145 | 3 | <i>wn</i> | 1237 | | <i>w2</i> | 1329 | | <i>w2</i> |
| 963 | 3 | <i>wn</i> | 1055 | 3 | <i>wn</i> | 1147 | | <i>w2</i> | 1239 | | <i>w2</i> | 1331 | 3 | <i>wn</i> |
| 965 | 3 | <i>wn</i> | 1057 | | <i>w2</i> | 1149 | | <i>w2</i> | 1241 | 3 | <i>wo</i> | 1333 | 3 | <i>wn</i> |
| 967 | | <i>w2</i> | 1059 | 3 | <i>wn</i> | 1151 | | | 1243 | 3 | <i>wn</i> | 1335 | 3 | <i>wn</i> |
| 969 | 3 | <i>wn</i> | 1061 | | <i>wk</i> | 1153 | | <i>wk</i> | 1245 | 3 | <i>wn</i> | 1337 | | |
| 971 | | | 1063 | 3 | <i>w#q</i> | 1155 | | <i>w2</i> | 1247 | 3 | <i>wo</i> | 1339 | | <i>w2</i> |

TABLE A.1 Williamson and Williamson-Type Matrices (continued)

| <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How |
|----------|----------|------------|----------|----------|------------|----------|----------|------------|----------|----------|------------|----------|----------|------------|
| 1341 | | <i>wb</i> | 1433 | | | 1525 | | <i>w2</i> | 1617 | 3 | <i>wn</i> | 1709 | | <i>wk</i> |
| 1343 | 3 | <i>wn</i> | 1435 | 3 | <i>wn</i> | 1527 | | | 1619 | 3 | <i>w#q</i> | 1711 | | |
| 1345 | | <i>w2</i> | 1437 | | | 1529 | 3 | <i>wn</i> | 1621 | | <i>wk</i> | 1713 | | |
| 1347 | | <i>w2</i> | 1439 | | | 1531 | | <i>w2</i> | 1623 | | <i>wi</i> | 1715 | 3 | <i>w#r</i> |
| 1349 | | | 1441 | | | 1533 | 3 | <i>wn</i> | 1625 | 3 | <i>wn</i> | 1717 | | <i>w2</i> |
| 1351 | 3 | <i>wn</i> | 1443 | 3 | <i>wn</i> | 1535 | 3 | <i>wn</i> | 1627 | | <i>w2</i> | 1719 | | |
| 1353 | 3 | <i>wn</i> | 1445 | 3 | <i>wn</i> | 1537 | 3 | <i>wo</i> | 1629 | | <i>w2</i> | 1721 | | <i>wl</i> |
| 1355 | 3 | <i>wn</i> | 1447 | | | 1539 | 3 | <i>wn</i> | 1631 | 3 | <i>wn</i> | 1723 | 3 | <i>w#q</i> |
| 1357 | | <i>w2</i> | 1449 | | <i>w2</i> | 1541 | | | 1633 | | | 1725 | | <i>w2</i> |
| 1359 | | | 1451 | | | 1543 | | | 1635 | 3 | <i>wn</i> | 1727 | 3 | <i>wn</i> |
| 1361 | | <i>wk</i> | 1453 | | <i>wl</i> | 1545 | | <i>w2</i> | 1637 | | <i>wl</i> | 1729 | | <i>w2</i> |
| 1363 | | | 1455 | | <i>w2</i> | 1547 | 3 | <i>wn</i> | 1639 | 3 | <i>wo</i> | 1731 | | <i>w2</i> |
| 1365 | | <i>w2</i> | 1457 | | | 1549 | | <i>wk</i> | 1641 | 3 | <i>wn</i> | 1733 | | <i>wl</i> |
| 1367 | | | 1459 | | <i>w2</i> | 1551 | 3 | <i>wn</i> | 1643 | 3 | <i>wn</i> | 1735 | | <i>w2</i> |
| 1369 | | <i>wl</i> | 1461 | | | 1553 | | <i>wk</i> | 1645 | | | 1737 | 3 | <i>wn</i> |
| 1371 | | <i>w2</i> | 1463 | 3 | <i>wn</i> | 1555 | | <i>w2</i> | 1647 | 3 | <i>wn</i> | 1739 | | |
| 1373 | 3 | <i>w#q</i> | 1465 | 3 | <i>wn</i> | 1557 | 3 | <i>wn</i> | 1649 | 3 | <i>wn</i> | 1741 | | <i>w2</i> |
| 1375 | | <i>w2</i> | 1467 | 3 | <i>wn</i> | 1559 | | | 1651 | | <i>w2</i> | 1743 | | <i>w5</i> |
| 1377 | | <i>w2</i> | 1469 | 3 | <i>wn</i> | 1561 | | <i>w2</i> | 1653 | 3 | <i>wn</i> | 1745 | 3 | <i>wn</i> |
| 1379 | 3 | <i>wn</i> | 1471 | 3 | <i>w#p</i> | 1563 | | <i>w2</i> | 1655 | 3 | <i>wn</i> | 1747 | | |
| 1381 | 3 | <i>w#q</i> | 1473 | | | 1565 | 3 | <i>wn</i> | 1657 | | <i>w2</i> | 1749 | 3 | <i>wn</i> |
| 1383 | | <i>wi</i> | 1475 | | | 1567 | | | 1659 | | <i>wi</i> | 1751 | | |
| 1385 | 3 | <i>wn</i> | 1477 | | <i>w2</i> | 1569 | | <i>w2</i> | 1661 | | | 1753 | | <i>wl</i> |
| 1387 | 3 | <i>wn</i> | 1479 | | <i>w2</i> | 1571 | | | 1663 | | | 1755 | | <i>wi</i> |
| 1389 | | <i>w2</i> | 1481 | | <i>wl</i> | 1573 | 3 | <i>wn</i> | 1665 | | <i>w2</i> | 1757 | | |
| 1391 | | | 1483 | 3 | <i>w#q</i> | 1575 | 3 | <i>wn</i> | 1667 | | | 1759 | | <i>w2</i> |
| 1393 | 3 | <i>wn</i> | 1485 | | <i>w2</i> | 1577 | 3 | <i>wn</i> | 1669 | 3 | <i>w#q</i> | 1761 | | |
| 1395 | | <i>w2</i> | 1487 | | | 1579 | | | 1671 | 3 | <i>wn</i> | 1763 | 3 | <i>wn</i> |
| 1397 | | | 1489 | | <i>wl</i> | 1581 | 3 | <i>wn</i> | 1673 | | | 1765 | | <i>w2</i> |
| 1399 | | <i>w2</i> | 1491 | | | 1583 | | | 1675 | | | 1767 | | <i>w2</i> |
| 1401 | | <i>w2</i> | 1493 | | <i>wl</i> | 1585 | | <i>w2</i> | 1677 | 3 | <i>wn</i> | 1769 | 3 | <i>wn</i> |
| 1403 | 3 | <i>wn</i> | 1495 | 3 | <i>wn</i> | 1587 | | <i>wh</i> | 1679 | 3 | <i>wn</i> | 1771 | | <i>w2</i> |
| 1405 | | <i>w2</i> | 1497 | 3 | <i>wn</i> | 1589 | | | 1681 | | <i>w2</i> | 1773 | 3 | <i>wn</i> |
| 1407 | 3 | <i>wn</i> | 1499 | | | 1591 | | <i>w2</i> | 1683 | | <i>wj</i> | 1775 | 3 | <i>wn</i> |
| 1409 | | <i>wl</i> | 1501 | | <i>w2</i> | 1593 | 3 | <i>wn</i> | 1685 | 3 | <i>wn</i> | 1777 | | <i>wl</i> |
| 1411 | 3 | <i>wo</i> | 1503 | | | 1595 | 3 | <i>wn</i> | 1687 | | <i>w2</i> | 1779 | | <i>w2</i> |
| 1413 | 3 | <i>wn</i> | 1505 | 3 | <i>wn</i> | 1597 | | <i>wk</i> | 1689 | | | 1781 | 3 | <i>wn</i> |
| 1415 | | | 1507 | 3 | <i>wo</i> | 1599 | 3 | <i>wn</i> | 1691 | 3 | <i>wn</i> | 1783 | | |
| 1417 | | <i>w2</i> | 1509 | | | 1601 | | <i>wk</i> | 1693 | | <i>wl</i> | 1785 | 3 | <i>wn</i> |
| 1419 | | <i>w2</i> | 1511 | | | 1603 | 3 | <i>wn</i> | 1695 | | <i>w2</i> | 1787 | | |
| 1421 | 3 | <i>wn</i> | 1513 | 3 | <i>wn</i> | 1605 | | <i>w2</i> | 1697 | | <i>wl</i> | 1789 | 3 | <i>w#q</i> |
| 1423 | | | 1515 | 3 | <i>wn</i> | 1607 | | | 1699 | 3 | <i>w#q</i> | 1791 | | <i>w2</i> |
| 1425 | | <i>w5</i> | 1517 | 3 | <i>wn</i> | 1609 | | <i>w2</i> | 1701 | 3 | <i>wn</i> | 1793 | | |
| 1427 | | | 1519 | | <i>w2</i> | 1611 | | <i>w2</i> | 1703 | | | 1795 | | |
| 1429 | | <i>w2</i> | 1521 | | <i>w2</i> | 1613 | 3 | <i>w#q</i> | 1705 | 3 | <i>wn</i> | 1797 | | <i>w2</i> |
| 1431 | | <i>w2</i> | 1523 | 3 | <i>w#q</i> | 1615 | | <i>w2</i> | 1707 | | <i>w2</i> | 1799 | 3 | <i>wn</i> |

TABLE A.1 Williamson and Williamson-Type Matrices (continued)

| <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How |
|----------|----------|-----------|----------|----------|------------|----------|----------|------------|----------|----------|------------|----------|----------|------------|
| 1801 | | <i>wk</i> | 1841 | | | 1881 | | <i>w2</i> | 1921 | 3 | <i>wn</i> | 1961 | 3 | <i>wn</i> |
| 1803 | 3 | <i>wn</i> | 1843 | 3 | <i>wn</i> | 1883 | 3 | <i>w#r</i> | 1923 | 3 | <i>wn</i> | 1963 | | |
| 1805 | | <i>wh</i> | 1845 | 3 | <i>wn</i> | 1885 | | <i>w2</i> | 1925 | 3 | <i>wn</i> | 1965 | | <i>w2</i> |
| 1807 | | <i>w2</i> | 1847 | | | 1887 | 3 | <i>wn</i> | 1927 | | <i>w2</i> | 1967 | 3 | <i>wn</i> |
| 1809 | | <i>w2</i> | 1849 | | <i>w2</i> | 1889 | | <i>wm</i> | 1929 | | | 1969 | | |
| 1811 | | | 1851 | | <i>w2</i> | 1891 | | <i>w4</i> | 1931 | | | 1971 | 3 | <i>wn</i> |
| 1813 | 3 | <i>wn</i> | 1853 | 3 | <i>wo</i> | 1893 | | | 1933 | 3 | <i>w#q</i> | 1973 | 3 | <i>w#q</i> |
| 1815 | 3 | <i>wn</i> | 1855 | | <i>w2</i> | 1895 | 3 | <i>wn</i> | 1935 | | <i>wi</i> | 1975 | 3 | <i>wn</i> |
| 1817 | 3 | <i>wn</i> | 1857 | 3 | <i>wn</i> | 1897 | | <i>w2</i> | 1937 | 3 | <i>wn</i> | 1977 | | |
| 1819 | | <i>w2</i> | 1859 | 3 | <i>wn</i> | 1899 | | <i>w2</i> | 1939 | | <i>w2</i> | 1979 | | |
| 1821 | 3 | <i>wn</i> | 1861 | | <i>w2</i> | 1901 | 3 | <i>w#q</i> | 1941 | | <i>w2</i> | 1981 | | |
| 1823 | 3 | <i>wn</i> | 1863 | 3 | <i>wn</i> | 1903 | 3 | <i>wo</i> | 1943 | | | 1983 | 3 | <i>wn</i> |
| 1825 | 3 | <i>wn</i> | 1865 | 3 | <i>wn</i> | 1905 | 3 | <i>wn</i> | 1945 | | <i>w2</i> | 1985 | 3 | <i>wn</i> |
| 1827 | | <i>w5</i> | 1867 | | <i>w2</i> | 1907 | 3 | <i>w#q</i> | 1947 | 3 | <i>wn</i> | 1987 | | |
| 1829 | | | 1869 | 3 | <i>wn</i> | 1909 | 3 | <i>wn</i> | 1949 | | | 1989 | 3 | <i>wn</i> |
| 1831 | | | 1871 | | | 1911 | | <i>w2</i> | 1951 | 3 | <i>w#p</i> | 1991 | 3 | <i>wn</i> |
| 1833 | 3 | <i>wn</i> | 1873 | | <i>wk</i> | 1913 | | | 1953 | 3 | <i>wn</i> | 1993 | | <i>wl</i> |
| 1835 | 3 | <i>wn</i> | 1875 | | <i>wf</i> | 1915 | | | 1955 | 3 | <i>wn</i> | 1995 | | <i>w2</i> |
| 1837 | | <i>w2</i> | 1877 | | <i>wk</i> | 1917 | | <i>w2</i> | 1957 | | | 1997 | | <i>wk</i> |
| 1839 | | <i>w2</i> | 1879 | 3 | <i>w#q</i> | 1919 | 3 | <i>wn</i> | 1959 | | <i>w2</i> | 1999 | | <i>w#p</i> |

A.3. Tables of Hadamard matrices

Table A.2 gives the orders of known Hadamard matrices. The table gives the odd part q of an order, the smallest power of two, t , for which the Hadamard matrix is known and a construction method. If there is no entry in the t column the power is two. Thus, there are Hadamard matrices known of orders $2^2 \cdot 105$ and $2^3 \cdot 107$. We see at a glance, therefore, that the smallest order for which an Hadamard matrix is not yet known is $4 \cdot 107$. Since the theorem of Seberry ensures that a t exists for every q , there is either a t entry for each q , or $t = 2$ is implied.

With the exception of order $4 \cdot 163$, marked dj , which was announced recently [16], the method of construction used is indicated. The order in which the algorithms were applied reflects the fact that other tables were being constructed at the same time. Hence, the "Amicable Hadamard," "Skew Hadamard," "Conference Matrix," "Williamson Matrix," direct "Complex Hadamard" were implemented first (in that order). The tables reflect this and not the priority in time of a construction or its discoverer.

Next the "Spence," "Miyamoto," and "Yamada" direct constructions were applied because they were noticed to fill places in the table. The methods $o1$ and of Koukouvinos and Kounias were now applied as lists of ODs were constructed. These were then used to "plug in" the Williamson-type matrices implementing methods $o2$ and $o3$.

Finally, the multiplication theorems of Agayan, Seberry, and Zhang were applied. The Craigen, Seberry, and Zhang theorem was applied to the table that one of us (Seberry), had in the computer. The method and order of application was by personal choice to improve the efficiency of implementation. This means that some authors, for example, Baumert, Hall, Turyn, and Whiteman, who have priority of construction are not mentioned by name in the final table.

TABLE A.2 Orders of Known Hadamard Matrices

| <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How |
|----------|----------|-----------|----------|----------|------------|----------|----------|------------|----------|----------|------------|----------|----------|------------|
| 1 | | <i>a1</i> | 69 | | <i>c1</i> | 137 | | <i>a1</i> | 205 | | <i>c1</i> | 273 | | <i>a1</i> |
| 3 | | <i>a1</i> | 71 | | <i>a1</i> | 139 | | <i>c1</i> | 207 | | <i>a1</i> | 275 | | <i>o1</i> |
| 5 | | <i>a1</i> | 73 | | <i>wk</i> | 141 | | <i>a1</i> | 209 | | <i>o1</i> | 277 | | <i>wk</i> |
| 7 | | <i>c1</i> | 75 | | <i>c1</i> | 143 | | <i>a1</i> | 211 | | <i>c1</i> | 279 | | <i>c1</i> |
| 9 | | <i>c1</i> | 77 | | <i>a1</i> | 145 | | <i>c1</i> | 213 | | <i>o2</i> | 281 | | <i>a1</i> |
| 11 | | <i>a1</i> | 79 | | <i>c1</i> | 147 | | <i>a1</i> | 215 | | <i>a1</i> | 283 | 3 | <i>w#q</i> |
| 13 | | <i>c1</i> | 81 | | <i>w3</i> | 149 | | <i>wk</i> | 217 | | <i>c1</i> | 285 | | <i>c1</i> |
| 15 | | <i>a1</i> | 83 | | <i>a1</i> | 151 | | <i>y2</i> | 219 | | <i>o2</i> | 287 | | <i>o1</i> |
| 17 | | <i>a1</i> | 85 | | <i>c1</i> | 153 | | <i>w4</i> | 221 | | <i>a1</i> | 289 | | <i>c1</i> |
| 19 | | <i>c1</i> | 87 | | <i>a1</i> | 155 | | <i>a1</i> | 223 | 3 | <i>a1</i> | 291 | | <i>a1</i> |
| 21 | | <i>a1</i> | 89 | | <i>a2</i> | 157 | | <i>c1</i> | 225 | | <i>c1</i> | 293 | | <i>a1</i> |
| 23 | | <i>w1</i> | 91 | | <i>c1</i> | 159 | | <i>c1</i> | 227 | | <i>a1</i> | 295 | | <i>o1</i> |
| 25 | | <i>c1</i> | 93 | | <i>w5</i> | 161 | | <i>a1</i> | 229 | | <i>c1</i> | 297 | | <i>a1</i> |
| 27 | | <i>a1</i> | 95 | | <i>a1</i> | 163 | | <i>dj</i> | 231 | | <i>c1</i> | 299 | | <i>o1</i> |
| 29 | | <i>a2</i> | 97 | | <i>c1</i> | 165 | | <i>a1</i> | 233 | | <i>a2</i> | 301 | | <i>c1</i> |
| 31 | | <i>c1</i> | 99 | | <i>c1</i> | 167 | 3 | <i>w#p</i> | 235 | | <i>o1</i> | 303 | | <i>w7</i> |
| 33 | | <i>a1</i> | 101 | | <i>wk</i> | 169 | | <i>c1</i> | 237 | | <i>a1</i> | 305 | | <i>o1</i> |
| 35 | | <i>a1</i> | 103 | | <i>y2</i> | 171 | | <i>a1</i> | 239 | 4 | <i>a1</i> | 307 | | <i>c1</i> |
| 37 | | <i>c1</i> | 105 | | <i>a1</i> | 173 | | <i>a1</i> | 241 | | <i>wk</i> | 309 | | <i>c1</i> |
| 39 | | <i>wi</i> | 107 | 3 | <i>w#q</i> | 175 | | <i>c1</i> | 243 | | <i>a1</i> | 311 | 3 | <i>m3</i> |
| 41 | | <i>a1</i> | 109 | | <i>wk</i> | 177 | | <i>c1</i> | 245 | | <i>o1</i> | 313 | | <i>c1</i> |
| 43 | | <i>w1</i> | 111 | | <i>a1</i> | 179 | 3 | <i>w#q</i> | 247 | | <i>o1</i> | 315 | | <i>a1</i> |
| 45 | | <i>a1</i> | 113 | | <i>a2</i> | 181 | | <i>c1</i> | 249 | | <i>o2</i> | 317 | | <i>wk</i> |
| 47 | | <i>o1</i> | 115 | | <i>c1</i> | 183 | | <i>o2</i> | 251 | 3 | <i>w#q</i> | 319 | | <i>o1</i> |
| 49 | | <i>c1</i> | 117 | | <i>a1</i> | 185 | | <i>a1</i> | 253 | | <i>o1</i> | 321 | | <i>a1</i> |
| 51 | | <i>c1</i> | 119 | | <i>o1</i> | 187 | | <i>c1</i> | 255 | | <i>a1</i> | 323 | | <i>a1</i> |
| 53 | | <i>a1</i> | 121 | | <i>c1</i> | 189 | | <i>w5</i> | 257 | | <i>w1</i> | 325 | | <i>w4</i> |
| 55 | | <i>c1</i> | 123 | | <i>a1</i> | 191 | 3 | <i>w#p</i> | 259 | | <i>o1</i> | 327 | | <i>a1</i> |
| 57 | | <i>a1</i> | 125 | | <i>a1</i> | 193 | | <i>wk</i> | 261 | | <i>c1</i> | 329 | | <i>o1</i> |
| 59 | | <i>o1</i> | 127 | | <i>y2</i> | 195 | | <i>c1</i> | 263 | | <i>a1</i> | 331 | | <i>c1</i> |
| 61 | | <i>c1</i> | 129 | | <i>c1</i> | 197 | | <i>a1</i> | 265 | | <i>c1</i> | 333 | | <i>w9</i> |
| 63 | | <i>a1</i> | 131 | | <i>a1</i> | 199 | | <i>c1</i> | 267 | | <i>o2</i> | 335 | | <i>o1</i> |
| 65 | | <i>o1</i> | 133 | | <i>o1</i> | 201 | | <i>c1</i> | 269 | | <i>m2</i> | 337 | | <i>c1</i> |
| 67 | | <i>o1</i> | 135 | | <i>c1</i> | 203 | | <i>a1</i> | 271 | | <i>c1</i> | 339 | | <i>c1</i> |

TABLE A.2 Orders of Known Hadamard Matrices (continued)

| <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How |
|----------|----------|------------|----------|----------|------------|----------|----------|-----------|----------|----------|------------|----------|----------|-----------|
| 341 | | <i>o1</i> | 433 | | <i>wk</i> | 525 | | <i>a1</i> | 617 | | <i>a1</i> | 709 | | <i>wk</i> |
| 343 | | <i>o2</i> | 435 | | <i>w4</i> | 527 | | <i>o1</i> | 619 | | <i>c1</i> | 711 | | <i>a1</i> |
| 345 | | <i>o1</i> | 437 | | <i>a1</i> | 529 | | <i>wl</i> | 621 | | <i>o1</i> | 713 | | <i>a1</i> |
| 347 | 3 | <i>w#q</i> | 439 | | <i>c1</i> | 531 | | <i>c1</i> | 623 | | <i>o2</i> | 715 | | <i>c1</i> |
| 349 | | <i>wk</i> | 441 | | <i>c1</i> | 533 | | <i>a1</i> | 625 | | <i>c1</i> | 717 | | <i>c1</i> |
| 351 | | <i>c1</i> | 443 | 3 | <i>m3</i> | 535 | | <i>c1</i> | 627 | | <i>wi</i> | 719 | 4 | <i>a1</i> |
| 353 | | <i>wl</i> | 445 | | <i>o2</i> | 537 | 3 | <i>o3</i> | 629 | | <i>o1</i> | 721 | 3 | <i>d1</i> |
| 355 | | <i>c1</i> | 447 | | <i>a1</i> | 539 | | <i>o2</i> | 631 | 3 | <i>w#q</i> | 723 | | <i>o2</i> |
| 357 | | <i>a1</i> | 449 | | <i>wk</i> | 541 | | <i>wk</i> | 633 | | <i>a1</i> | 725 | | <i>o1</i> |
| 359 | 4 | <i>a1</i> | 451 | | <i>wj</i> | 543 | | <i>wi</i> | 635 | | <i>a1</i> | 727 | | <i>c1</i> |
| 361 | | <i>wk</i> | 453 | | <i>a1</i> | 545 | | <i>a1</i> | 637 | | <i>o2</i> | 729 | | <i>w3</i> |
| 363 | | <i>a1</i> | 455 | | <i>o1</i> | 547 | | <i>c1</i> | 639 | | <i>c1</i> | 731 | | <i>o2</i> |
| 365 | | <i>a1</i> | 457 | | <i>wk</i> | 549 | | <i>c1</i> | 641 | | <i>a2</i> | 733 | | <i>m2</i> |
| 367 | | <i>c1</i> | 459 | | <i>wi</i> | 551 | | <i>a1</i> | 643 | 3 | <i>w#q</i> | 735 | | <i>a1</i> |
| 369 | | <i>o1</i> | 461 | | <i>wk</i> | 553 | | <i>o2</i> | 645 | | <i>a1</i> | 737 | | <i>o2</i> |
| 371 | | <i>a1</i> | 463 | 3 | <i>w#q</i> | 555 | | <i>c1</i> | 647 | 3 | <i>m3</i> | 739 | 16 | <i>se</i> |
| 373 | | <i>wl</i> | 465 | | <i>c1</i> | 557 | | <i>wk</i> | 649 | | <i>c1</i> | 741 | | <i>a1</i> |
| 375 | | <i>a1</i> | 467 | | <i>a1</i> | 559 | | <i>c1</i> | 651 | | <i>c1</i> | 743 | | <i>a1</i> |
| 377 | | <i>o1</i> | 469 | | <i>c1</i> | 561 | | <i>a1</i> | 653 | | <i>a2</i> | 745 | | <i>c1</i> |
| 379 | | <i>c1</i> | 471 | | <i>c1</i> | 563 | | <i>a1</i> | 655 | | <i>y2</i> | 747 | | <i>c1</i> |
| 381 | | <i>a1</i> | 473 | | <i>w5</i> | 565 | | <i>c1</i> | 657 | | <i>o2</i> | 749 | 4 | <i>d1</i> |
| 383 | | <i>a1</i> | 475 | | <i>o1</i> | 567 | | <i>a1</i> | 659 | 17 | <i>se</i> | 751 | 3 | <i>a1</i> |
| 385 | | <i>c1</i> | 477 | | <i>a1</i> | 569 | | <i>wm</i> | 661 | | <i>c1</i> | 753 | | <i>a1</i> |
| 387 | | <i>c1</i> | 479 | 16 | <i>se</i> | 571 | 3 | <i>a1</i> | 663 | | <i>w5</i> | 755 | | <i>a1</i> |
| 389 | | <i>wk</i> | 481 | | <i>c1</i> | 573 | 3 | <i>a1</i> | 665 | | <i>a1</i> | 757 | | <i>wl</i> |
| 391 | | <i>o1</i> | 483 | | <i>a1</i> | 575 | | <i>o1</i> | 667 | | <i>o2</i> | 759 | | <i>wi</i> |
| 393 | | <i>a1</i> | 485 | | <i>o2</i> | 577 | | <i>c1</i> | 669 | 3 | <i>a1</i> | 761 | | <i>a2</i> |
| 395 | | <i>a1</i> | 487 | 3 | <i>w#p</i> | 579 | | <i>wj</i> | 671 | | <i>a1</i> | 763 | | <i>o2</i> |
| 397 | | <i>wk</i> | 489 | | <i>c1</i> | 581 | | <i>o2</i> | 673 | | <i>wk</i> | 765 | | <i>o1</i> |
| 399 | | <i>c1</i> | 491 | 15 | <i>se</i> | 583 | | <i>o1</i> | 675 | | <i>a1</i> | 767 | | <i>a1</i> |
| 401 | | <i>wk</i> | 493 | | <i>o1</i> | 585 | | <i>a1</i> | 677 | | <i>a1</i> | 769 | | <i>wk</i> |
| 403 | | <i>o1</i> | 495 | | <i>a1</i> | 587 | | <i>a1</i> | 679 | | <i>o2</i> | 771 | | <i>a1</i> |
| 405 | | <i>a1</i> | 497 | | <i>a1</i> | 589 | | <i>o2</i> | 681 | | <i>c1</i> | 773 | | <i>wl</i> |
| 407 | | <i>a1</i> | 499 | | <i>c1</i> | 591 | | <i>c1</i> | 683 | | <i>a1</i> | 775 | | <i>c1</i> |
| 409 | | <i>wk</i> | 501 | | <i>a1</i> | 593 | | <i>a1</i> | 685 | | <i>c1</i> | 777 | | <i>c1</i> |
| 411 | | <i>c1</i> | 503 | | <i>a1</i> | 595 | | <i>o1</i> | 687 | | <i>c1</i> | 779 | | <i>o1</i> |
| 413 | | <i>o1</i> | 505 | | <i>c1</i> | 597 | | <i>c1</i> | 689 | | <i>w9</i> | 781 | | <i>o2</i> |
| 415 | | <i>c1</i> | 507 | | <i>a1</i> | 599 | 8 | <i>a1</i> | 691 | | <i>c1</i> | 783 | | <i>o1</i> |
| 417 | | <i>a1</i> | 509 | | <i>a2</i> | 601 | | <i>c1</i> | 693 | | <i>o1</i> | 785 | | <i>o2</i> |
| 419 | 4 | <i>a1</i> | 511 | | <i>c1</i> | 603 | | <i>a1</i> | 695 | | <i>o2</i> | 787 | 3 | <i>m3</i> |
| 421 | | <i>c1</i> | 513 | | <i>o1</i> | 605 | | <i>o2</i> | 697 | | <i>o1</i> | 789 | 3 | <i>a1</i> |
| 423 | | <i>wi</i> | 515 | 3 | <i>w#r</i> | 607 | | <i>c1</i> | 699 | | <i>o2</i> | 791 | | <i>a1</i> |
| 425 | | <i>a1</i> | 517 | | <i>c1</i> | 609 | | <i>c1</i> | 701 | | <i>a1</i> | 793 | | <i>o2</i> |
| 427 | | <i>c1</i> | 519 | | <i>o2</i> | 611 | | <i>o1</i> | 703 | | <i>w4</i> | 795 | | <i>o1</i> |
| 429 | | <i>c1</i> | 521 | | <i>a1</i> | 613 | | <i>wl</i> | 705 | | <i>a1</i> | 797 | | <i>a1</i> |
| 431 | | <i>a1</i> | 523 | 3 | <i>w#q</i> | 615 | | <i>a1</i> | 707 | | <i>o2</i> | 799 | | <i>c1</i> |

TABLE A.2 Orders of Known Hadamard Matrices (continued)

| <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How |
|----------|----------|------------|----------|----------|------------|----------|----------|------------|----------|----------|------------|----------|----------|------------|
| 801 | | <i>a1</i> | 893 | | <i>a1</i> | 985 | | <i>o2</i> | 1077 | | <i>c1</i> | 1169 | 4 | <i>k1</i> |
| 803 | | <i>o2</i> | 895 | | <i>c1</i> | 987 | | <i>a1</i> | 1079 | | <i>o2</i> | 1171 | | <i>c1</i> |
| 805 | | <i>c1</i> | 897 | | <i>o2</i> | 989 | | <i>o2</i> | 1081 | | <i>c1</i> | 1173 | | <i>a1</i> |
| 807 | | <i>c1</i> | 899 | | <i>o2</i> | 991 | 3 | <i>a1</i> | 1083 | | <i>o2</i> | 1175 | | <i>o1</i> |
| 809 | | <i>a2</i> | 901 | | <i>c1</i> | 993 | | <i>o2</i> | 1085 | | <i>a1</i> | 1177 | 4 | <i>d1</i> |
| 811 | | <i>c1</i> | 903 | | <i>o1</i> | 995 | | <i>o2</i> | 1087 | 3 | <i>w#p</i> | 1179 | | <i>c1</i> |
| 813 | | <i>a1</i> | 905 | | <i>o2</i> | 997 | | <i>c1</i> | 1089 | | <i>o1</i> | 1181 | | <i>a1</i> |
| 815 | | <i>a1</i> | 907 | 3 | <i>m3</i> | 999 | | <i>c1</i> | 1091 | | <i>a1</i> | 1183 | | <i>wf</i> |
| 817 | | <i>o2</i> | 909 | | <i>o1</i> | 1001 | | <i>a1</i> | 1093 | 3 | <i>w#q</i> | 1185 | | <i>o2</i> |
| 819 | | <i>c1</i> | 911 | | <i>a1</i> | 1003 | | <i>o1</i> | 1095 | | <i>wi</i> | 1187 | 3 | <i>w#q</i> |
| 821 | | <i>wk</i> | 913 | | <i>o2</i> | 1005 | | <i>a1</i> | 1097 | | <i>wl</i> | 1189 | | <i>c1</i> |
| 823 | 3 | <i>w#q</i> | 915 | | <i>a1</i> | 1007 | | <i>a1</i> | 1099 | | <i>w2</i> | 1191 | | <i>c1</i> |
| 825 | | <i>a1</i> | 917 | 3 | <i>o2</i> | 1009 | | <i>c1</i> | 1101 | | <i>o2</i> | 1193 | | <i>wl</i> |
| 827 | | <i>a1</i> | 919 | 3 | <i>a1</i> | 1011 | | <i>o2</i> | 1103 | 3 | <i>m3</i> | 1195 | | <i>c1</i> |
| 829 | | <i>c1</i> | 921 | | <i>o2</i> | 1013 | | <i>a1</i> | 1105 | | <i>c1</i> | 1197 | | <i>a1</i> |
| 831 | | <i>a1</i> | 923 | | <i>a1</i> | 1015 | | <i>c1</i> | 1107 | | <i>c1</i> | 1199 | | <i>o2</i> |
| 833 | | <i>a1</i> | 925 | | <i>c1</i> | 1017 | | <i>o2</i> | 1109 | | <i>wl</i> | 1201 | | <i>c1</i> |
| 835 | | <i>c1</i> | 927 | | <i>o2</i> | 1019 | 3 | <i>w#r</i> | 1111 | | <i>c1</i> | 1203 | | <i>wi</i> |
| 837 | | <i>a1</i> | 929 | | <i>wl</i> | 1021 | | <i>wk</i> | 1113 | | <i>a1</i> | 1205 | | <i>o2</i> |
| 839 | 8 | <i>a1</i> | 931 | | <i>c1</i> | 1023 | | <i>a1</i> | 1115 | 3 | <i>w#r</i> | 1207 | | <i>w5</i> |
| 841 | | <i>c1</i> | 933 | 4 | <i>d1</i> | 1025 | | <i>a1</i> | 1117 | | <i>wk</i> | 1209 | | <i>c1</i> |
| 843 | | <i>a1</i> | 935 | | <i>a1</i> | 1027 | | <i>c1</i> | 1119 | | <i>c1</i> | 1211 | | <i>o2</i> |
| 845 | | <i>o1</i> | 937 | | <i>c1</i> | 1029 | | <i>o2</i> | 1121 | | <i>a1</i> | 1213 | | <i>m2</i> |
| 847 | | <i>c1</i> | 939 | | <i>c1</i> | 1031 | 6 | <i>a1</i> | 1123 | 3 | <i>m3</i> | 1215 | | <i>wi</i> |
| 849 | | <i>c1</i> | 941 | | <i>m2</i> | 1033 | | <i>wk</i> | 1125 | | <i>o1</i> | 1217 | | <i>wk</i> |
| 851 | | <i>o2</i> | 943 | | <i>o1</i> | 1035 | | <i>a1</i> | 1127 | | <i>a1</i> | 1219 | | <i>c1</i> |
| 853 | 3 | <i>a1</i> | 945 | | <i>a1</i> | 1037 | | <i>o2</i> | 1129 | | <i>wk</i> | 1221 | | <i>c1</i> |
| 855 | | <i>c1</i> | 947 | 3 | <i>w#q</i> | 1039 | 3 | <i>a1</i> | 1131 | | <i>a1</i> | 1223 | 8 | <i>a1</i> |
| 857 | | <i>m2</i> | 949 | | <i>o2</i> | 1041 | | <i>c1</i> | 1133 | 3 | <i>d1</i> | 1225 | | <i>w4</i> |
| 859 | 3 | <i>a1</i> | 951 | | <i>a1</i> | 1043 | | <i>o2</i> | 1135 | | <i>c1</i> | 1227 | | <i>o2</i> |
| 861 | | <i>c1</i> | 953 | | <i>a2</i> | 1045 | | <i>c1</i> | 1137 | | <i>a1</i> | 1229 | | <i>a2</i> |
| 863 | 3 | <i>m3</i> | 955 | 3 | <i>a1</i> | 1047 | | <i>o2</i> | 1139 | | <i>w5</i> | 1231 | | <i>y2</i> |
| 865 | | <i>o2</i> | 957 | | <i>c1</i> | 1049 | | <i>a2</i> | 1141 | | <i>c1</i> | 1233 | | <i>a1</i> |
| 867 | | <i>a1</i> | 959 | | <i>o2</i> | 1051 | 3 | <i>w#q</i> | 1143 | | <i>o2</i> | 1235 | | <i>o1</i> |
| 869 | | <i>o2</i> | 961 | | <i>wk</i> | 1053 | | <i>a1</i> | 1145 | | <i>o2</i> | 1237 | | <i>c1</i> |
| 871 | | <i>c1</i> | 963 | | <i>a1</i> | 1055 | | <i>a1</i> | 1147 | | <i>c1</i> | 1239 | | <i>c1</i> |
| 873 | | <i>a1</i> | 965 | | <i>o2</i> | 1057 | | <i>c1</i> | 1149 | | <i>c1</i> | 1241 | | <i>o2</i> |
| 875 | | <i>a1</i> | 967 | | <i>c1</i> | 1059 | | <i>o2</i> | 1151 | | <i>a1</i> | 1243 | | <i>o2</i> |
| 877 | | <i>c1</i> | 969 | | <i>o2</i> | 1061 | | <i>a1</i> | 1153 | | <i>wk</i> | 1245 | | <i>o2</i> |
| 879 | | <i>wi</i> | 971 | 6 | <i>a1</i> | 1063 | 3 | <i>w#q</i> | 1155 | | <i>c1</i> | 1247 | | <i>a1</i> |
| 881 | | <i>wk</i> | 973 | | <i>o2</i> | 1065 | | <i>a1</i> | 1157 | | <i>o2</i> | 1249 | | <i>wl</i> |
| 883 | 3 | <i>w#q</i> | 975 | | <i>c1</i> | 1067 | | <i>o2</i> | 1159 | | <i>o2</i> | 1251 | | <i>a1</i> |
| 885 | | <i>a1</i> | 977 | | <i>a1</i> | 1069 | | <i>c1</i> | 1161 | | <i>a1</i> | 1253 | | <i>a1</i> |
| 887 | | <i>a1</i> | 979 | | <i>o2</i> | 1071 | | <i>a1</i> | 1163 | | <i>a1</i> | 1255 | 3 | <i>a1</i> |
| 889 | | <i>c1</i> | 981 | | <i>a1</i> | 1073 | | <i>o2</i> | 1165 | | <i>o2</i> | 1257 | 4 | <i>o2</i> |
| 891 | | <i>o1</i> | 983 | | <i>a1</i> | 1075 | | <i>o2</i> | 1167 | | <i>c1</i> | 1259 | 4 | <i>a1</i> |

TABLE A.2 Orders of Known Hadamard Matrices (continued)

| <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How |
|----------|----------|------------|----------|----------|-----------|----------|----------|------------|----------|----------|------------|----------|----------|------------|
| 1261 | | <i>c1</i> | 1353 | | <i>o1</i> | 1445 | | <i>a1</i> | 1537 | | <i>o1</i> | 1629 | | <i>c1</i> |
| 1263 | | <i>a1</i> | 1355 | | <i>a1</i> | 1447 | 19 | <i>se</i> | 1539 | | <i>o1</i> | 1631 | | <i>o2</i> |
| 1265 | | <i>a1</i> | 1357 | | <i>c1</i> | 1449 | | <i>c1</i> | 1541 | | <i>a1</i> | 1633 | | <i>o2</i> |
| 1267 | | <i>o2</i> | 1359 | 3 | <i>d1</i> | 1451 | 6 | <i>a1</i> | 1543 | 3 | <i>a1</i> | 1635 | | <i>o2</i> |
| 1269 | | <i>o1</i> | 1361 | | <i>a1</i> | 1453 | | <i>wl</i> | 1545 | | <i>c1</i> | 1637 | | <i>a1</i> |
| 1271 | | <i>o1</i> | 1363 | | <i>o2</i> | 1455 | | <i>c1</i> | 1547 | | <i>o2</i> | 1639 | | <i>o2</i> |
| 1273 | | <i>o2</i> | 1365 | | <i>c1</i> | 1457 | | <i>a1</i> | 1549 | | <i>wk</i> | 1641 | | <i>a1</i> |
| 1275 | | <i>a1</i> | 1367 | 3 | <i>m3</i> | 1459 | | <i>c1</i> | 1551 | | <i>a1</i> | 1643 | | <i>a1</i> |
| 1277 | | <i>a1</i> | 1369 | | <i>wl</i> | 1461 | | <i>a1</i> | 1553 | | <i>a1</i> | 1645 | | <i>o1</i> |
| 1279 | | <i>c1</i> | 1371 | | <i>a1</i> | 1463 | | <i>a1</i> | 1555 | | <i>c1</i> | 1647 | | <i>o2</i> |
| 1281 | | <i>o1</i> | 1373 | | <i>m2</i> | 1465 | | <i>o2</i> | 1557 | | <i>o2</i> | 1649 | | <i>o2</i> |
| 1283 | 3 | <i>w#q</i> | 1375 | | <i>c1</i> | 1467 | | <i>a1</i> | 1559 | 4 | <i>a1</i> | 1651 | | <i>c1</i> |
| 1285 | | <i>o1</i> | 1377 | | <i>a1</i> | 1469 | | <i>o2</i> | 1561 | | <i>c1</i> | 1653 | | <i>o2</i> |
| 1287 | | <i>a1</i> | 1379 | | <i>o2</i> | 1471 | 3 | <i>w#p</i> | 1563 | | <i>w2</i> | 1655 | | <i>a1</i> |
| 1289 | | <i>a2</i> | 1381 | | <i>m2</i> | 1473 | 3 | <i>a1</i> | 1565 | | <i>o2</i> | 1657 | | <i>c1</i> |
| 1291 | 3 | <i>w#q</i> | 1383 | | <i>a1</i> | 1475 | | <i>o1</i> | 1567 | 19 | <i>se</i> | 1659 | | <i>wi</i> |
| 1293 | | <i>a1</i> | 1385 | | <i>o2</i> | 1477 | | <i>c1</i> | 1569 | | <i>c1</i> | 1661 | 3 | <i>d1</i> |
| 1295 | | <i>a1</i> | 1387 | | <i>o2</i> | 1479 | | <i>c1</i> | 1571 | 18 | <i>se</i> | 1663 | 3 | <i>m3</i> |
| 1297 | | <i>c1</i> | 1389 | | <i>c1</i> | 1481 | | <i>a1</i> | 1573 | | <i>o2</i> | 1665 | | <i>a1</i> |
| 1299 | | <i>o2</i> | 1391 | | <i>a1</i> | 1483 | 3 | <i>a1</i> | 1575 | | <i>a1</i> | 1667 | 3 | <i>m3</i> |
| 1301 | | <i>wl</i> | 1393 | | <i>o2</i> | 1485 | | <i>a1</i> | 1577 | | <i>o2</i> | 1669 | 3 | <i>w#q</i> |
| 1303 | 3 | <i>w#q</i> | 1395 | | <i>c1</i> | 1487 | 3 | <i>m3</i> | 1579 | 5 | <i>a1</i> | 1671 | | <i>o2</i> |
| 1305 | | <i>c1</i> | 1397 | 3 | <i>d1</i> | 1489 | | <i>wl</i> | 1581 | | <i>a1</i> | 1673 | | <i>a1</i> |
| 1307 | | <i>a1</i> | 1399 | | <i>c1</i> | 1491 | | <i>o2</i> | 1583 | 3 | <i>m3</i> | 1675 | | <i>o2</i> |
| 1309 | | <i>c1</i> | 1401 | | <i>c1</i> | 1493 | | <i>wl</i> | 1585 | | <i>c1</i> | 1677 | | <i>o1</i> |
| 1311 | | <i>c1</i> | 1403 | | <i>o2</i> | 1495 | | <i>o1</i> | 1587 | | <i>wh</i> | 1679 | | <i>o2</i> |
| 1313 | | <i>o2</i> | 1405 | | <i>c1</i> | 1497 | | <i>a1</i> | 1589 | 3 | <i>d1</i> | 1681 | | <i>c1</i> |
| 1315 | 3 | <i>d1</i> | 1407 | | <i>o1</i> | 1499 | 18 | <i>se</i> | 1591 | | <i>c1</i> | 1683 | | <i>wj</i> |
| 1317 | | <i>c1</i> | 1409 | | <i>a2</i> | 1501 | | <i>c1</i> | 1593 | | <i>o1</i> | 1685 | | <i>o2</i> |
| 1319 | 18 | <i>se</i> | 1411 | | <i>o2</i> | 1503 | | <i>a1</i> | 1595 | | <i>a1</i> | 1687 | | <i>c1</i> |
| 1321 | | <i>wl</i> | 1413 | | <i>a1</i> | 1505 | | <i>o2</i> | 1597 | | <i>wk</i> | 1689 | 3 | <i>o3</i> |
| 1323 | | <i>wi</i> | 1415 | | <i>a1</i> | 1507 | | <i>o2</i> | 1599 | | <i>o2</i> | 1691 | | <i>a1</i> |
| 1325 | | <i>o1</i> | 1417 | | <i>c1</i> | 1509 | 3 | <i>a1</i> | 1601 | | <i>a2</i> | 1693 | | <i>wl</i> |
| 1327 | 3 | <i>w#p</i> | 1419 | | <i>c1</i> | 1511 | | <i>a1</i> | 1603 | | <i>o2</i> | 1695 | | <i>a1</i> |
| 1329 | | <i>c1</i> | 1421 | | <i>a1</i> | 1513 | | <i>o2</i> | 1605 | | <i>c1</i> | 1697 | | <i>wl</i> |
| 1331 | | <i>a1</i> | 1423 | 3 | <i>a1</i> | 1515 | | <i>o2</i> | 1607 | | <i>a1</i> | 1699 | 3 | <i>a1</i> |
| 1333 | | <i>o2</i> | 1425 | | <i>w5</i> | 1517 | | <i>a1</i> | 1609 | | <i>c1</i> | 1701 | | <i>a1</i> |
| 1335 | | <i>o2</i> | 1427 | 3 | <i>m3</i> | 1519 | | <i>c1</i> | 1611 | | <i>c1</i> | 1703 | 3 | <i>o2</i> |
| 1337 | | <i>a1</i> | 1429 | | <i>c1</i> | 1521 | | <i>c1</i> | 1613 | | <i>a1</i> | 1705 | | <i>o2</i> |
| 1339 | | <i>c1</i> | 1431 | | <i>c1</i> | 1523 | | <i>a1</i> | 1615 | | <i>c1</i> | 1707 | | <i>a1</i> |
| 1341 | | <i>wb</i> | 1433 | | <i>m2</i> | 1525 | | <i>c1</i> | 1617 | | <i>o2</i> | 1709 | | <i>wk</i> |
| 1343 | | <i>o2</i> | 1435 | | <i>o1</i> | 1527 | 3 | <i>d1</i> | 1619 | 3 | <i>w#q</i> | 1711 | | <i>o2</i> |
| 1345 | | <i>c1</i> | 1437 | 6 | <i>a1</i> | 1529 | | <i>o2</i> | 1621 | | <i>wk</i> | 1713 | 3 | <i>o3</i> |
| 1347 | | <i>a1</i> | 1439 | 19 | <i>se</i> | 1531 | | <i>c1</i> | 1623 | | <i>a1</i> | 1715 | | <i>o2</i> |
| 1349 | | <i>o2</i> | 1441 | 3 | <i>a1</i> | 1533 | | <i>a1</i> | 1625 | | <i>o1</i> | 1717 | | <i>c1</i> |
| 1351 | | <i>o2</i> | 1443 | | <i>o2</i> | 1535 | | <i>o2</i> | 1627 | | <i>c1</i> | 1719 | 3 | <i>a1</i> |

TABLE A.2 Orders of Known Hadamard Matrices (continued)

| <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How |
|----------|----------|------------|----------|----------|------------|----------|----------|------------|----------|----------|------------|----------|----------|------------|
| 1721 | | <i>a1</i> | 1813 | | <i>o2</i> | 1905 | | <i>o2</i> | 1997 | | <i>wk</i> | 2089 | | <i>c1</i> |
| 1723 | 3 | <i>w#q</i> | 1815 | | <i>o1</i> | 1907 | 3 | <i>w#q</i> | 1999 | | <i>y2</i> | 2091 | | <i>a1</i> |
| 1725 | | <i>a1</i> | 1817 | | <i>o2</i> | 1909 | | <i>o2</i> | 2001 | | <i>c1</i> | 2093 | | <i>w5</i> |
| 1727 | | <i>a1</i> | 1819 | | <i>c1</i> | 1911 | | <i>a1</i> | 2003 | | <i>a1</i> | 2095 | 3 | <i>a1</i> |
| 1729 | | <i>c1</i> | 1821 | | <i>a1</i> | 1913 | 19 | <i>se</i> | 2005 | | <i>o1</i> | 2097 | | <i>a1</i> |
| 1731 | | <i>c1</i> | 1823 | 3 | <i>wn</i> | 1915 | 3 | <i>a1</i> | 2007 | | <i>c1</i> | 2099 | 3 | <i>w#r</i> |
| 1733 | | <i>a2</i> | 1825 | | <i>o2</i> | 1917 | | <i>c1</i> | 2009 | | <i>o2</i> | 2101 | | <i>c1</i> |
| 1735 | | <i>c1</i> | 1827 | | <i>a1</i> | 1919 | | <i>o2</i> | 2011 | | <i>c1</i> | 2103 | | <i>o2</i> |
| 1737 | | <i>a1</i> | 1829 | | <i>o2</i> | 1921 | | <i>o2</i> | 2013 | | <i>o2</i> | 2105 | | <i>a1</i> |
| 1739 | | <i>o2</i> | 1831 | 3 | <i>m3</i> | 1923 | | <i>a1</i> | 2015 | | <i>a1</i> | 2107 | | <i>o2</i> |
| 1741 | | <i>c1</i> | 1833 | | <i>a1</i> | 1925 | | <i>a1</i> | 2017 | | <i>wk</i> | 2109 | | <i>c1</i> |
| 1743 | | <i>a1</i> | 1835 | | <i>o2</i> | 1927 | | <i>c1</i> | 2019 | | <i>wi</i> | 2111 | | <i>a1</i> |
| 1745 | | <i>o2</i> | 1837 | | <i>c1</i> | 1929 | 3 | <i>o3</i> | 2021 | | <i>o2</i> | 2113 | | <i>wl</i> |
| 1747 | 3 | <i>m3</i> | 1839 | | <i>c1</i> | 1931 | | <i>a1</i> | 2023 | | <i>o2</i> | 2115 | | <i>c1</i> |
| 1749 | | <i>o1</i> | 1841 | 3 | <i>d1</i> | 1933 | 3 | <i>w#q</i> | 2025 | | <i>c1</i> | 2117 | | <i>a1</i> |
| 1751 | 3 | <i>d1</i> | 1843 | | <i>o2</i> | 1935 | | <i>wi</i> | 2027 | 3 | <i>w#r</i> | 2119 | 3 | <i>d1</i> |
| 1753 | | <i>wl</i> | 1845 | | <i>o1</i> | 1937 | | <i>o2</i> | 2029 | | <i>c1</i> | 2121 | | <i>c1</i> |
| 1755 | | <i>a1</i> | 1847 | 3 | <i>m3</i> | 1939 | | <i>c1</i> | 2031 | | <i>a1</i> | 2123 | | <i>o2</i> |
| 1757 | | <i>a1</i> | 1849 | | <i>c1</i> | 1941 | | <i>c1</i> | 2033 | 4 | <i>d1</i> | 2125 | | <i>o1</i> |
| 1759 | | <i>c1</i> | 1851 | | <i>c1</i> | 1943 | | <i>o2</i> | 2035 | | <i>o2</i> | 2127 | | <i>c1</i> |
| 1761 | | <i>a1</i> | 1853 | | <i>a1</i> | 1945 | | <i>c1</i> | 2037 | | <i>a1</i> | 2129 | | <i>a2</i> |
| 1763 | | <i>o2</i> | 1855 | | <i>c1</i> | 1947 | | <i>o1</i> | 2039 | 20 | <i>se</i> | 2131 | | <i>c1</i> |
| 1765 | | <i>c1</i> | 1857 | | <i>o2</i> | 1949 | 4 | <i>a1</i> | 2041 | | <i>o2</i> | 2133 | | <i>o2</i> |
| 1767 | | <i>c1</i> | 1859 | | <i>o2</i> | 1951 | | <i>y2</i> | 2043 | | <i>a1</i> | 2135 | | <i>a1</i> |
| 1769 | | <i>o2</i> | 1861 | | <i>c1</i> | 1953 | | <i>o1</i> | 2045 | | <i>a1</i> | 2137 | | <i>c1</i> |
| 1771 | | <i>c1</i> | 1863 | | <i>a1</i> | 1955 | | <i>o1</i> | 2047 | | <i>c1</i> | 2139 | | <i>wi</i> |
| 1773 | | <i>o2</i> | 1865 | | <i>a1</i> | 1957 | 3 | <i>d1</i> | 2049 | | <i>o1</i> | 2141 | | <i>a1</i> |
| 1775 | | <i>o2</i> | 1867 | | <i>c1</i> | 1959 | | <i>c1</i> | 2051 | | <i>o2</i> | 2143 | 3 | <i>w#q</i> |
| 1777 | | <i>wl</i> | 1869 | | <i>o2</i> | 1961 | | <i>o1</i> | 2053 | 3 | <i>w#q</i> | 2145 | | <i>c1</i> |
| 1779 | | <i>c1</i> | 1871 | 3 | <i>m3</i> | 1963 | 3 | <i>d1</i> | 2055 | | <i>a1</i> | 2147 | | <i>o2</i> |
| 1781 | | <i>o2</i> | 1873 | | <i>wk</i> | 1965 | | <i>c1</i> | 2057 | | <i>o2</i> | 2149 | | <i>c1</i> |
| 1783 | 7 | <i>a1</i> | 1875 | | <i>a1</i> | 1967 | | <i>a1</i> | 2059 | | <i>o2</i> | 2151 | | <i>o2</i> |
| 1785 | | <i>o1</i> | 1877 | | <i>a1</i> | 1969 | 4 | <i>o2</i> | 2061 | | <i>a1</i> | 2153 | | <i>wm</i> |
| 1787 | 3 | <i>m3</i> | 1879 | 3 | <i>a1</i> | 1971 | | <i>a1</i> | 2063 | 8 | <i>a1</i> | 2155 | 3 | <i>a1</i> |
| 1789 | 3 | <i>w#q</i> | 1881 | | <i>a1</i> | 1973 | | <i>a2</i> | 2065 | | <i>c1</i> | 2157 | | <i>a1</i> |
| 1791 | | <i>c1</i> | 1883 | 3 | <i>w#r</i> | 1975 | | <i>o2</i> | 2067 | | <i>c1</i> | 2159 | 3 | <i>d1</i> |
| 1793 | 4 | <i>a1</i> | 1885 | | <i>c1</i> | 1977 | | <i>a1</i> | 2069 | | <i>a2</i> | 2161 | | <i>wk</i> |
| 1795 | 5 | <i>d1</i> | 1887 | | <i>a1</i> | 1979 | 4 | <i>a1</i> | 2071 | | <i>o2</i> | 2163 | | <i>o2</i> |
| 1797 | | <i>a1</i> | 1889 | | <i>a2</i> | 1981 | 4 | <i>d1</i> | 2073 | | <i>a1</i> | 2165 | | <i>o2</i> |
| 1799 | | <i>o1</i> | 1891 | | <i>w4</i> | 1983 | | <i>o2</i> | 2075 | | <i>o2</i> | 2167 | | <i>o2</i> |
| 1801 | | <i>wk</i> | 1893 | 3 | <i>o3</i> | 1985 | | <i>o2</i> | 2077 | | <i>c1</i> | 2169 | | <i>c1</i> |
| 1803 | | <i>a1</i> | 1895 | | <i>o2</i> | 1987 | 16 | <i>se</i> | 2079 | | <i>c1</i> | 2171 | 4 | <i>d1</i> |
| 1805 | | <i>a1</i> | 1897 | | <i>c1</i> | 1989 | | <i>o1</i> | 2081 | | <i>wl</i> | 2173 | | <i>o1</i> |
| 1807 | | <i>c1</i> | 1899 | | <i>c1</i> | 1991 | | <i>a1</i> | 2083 | 3 | <i>w#q</i> | 2175 | | <i>a1</i> |
| 1809 | | <i>c1</i> | 1901 | | <i>a1</i> | 1993 | | <i>wl</i> | 2085 | | <i>o1</i> | 2177 | | <i>a1</i> |
| 1811 | | <i>a1</i> | 1903 | | <i>o2</i> | 1995 | | <i>c1</i> | 2087 | 4 | <i>a1</i> | 2179 | | <i>c1</i> |

TABLE A.2 Orders of Known Hadamard Matrices (continued)

| <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How | <i>q</i> | <i>t</i> | How |
|----------|----------|------------|----------|----------|------------|----------|----------|------------|----------|----------|------------|----------|----------|------------|
| 2181 | | <i>o2</i> | 2273 | | <i>a1</i> | 2365 | | <i>c1</i> | 2457 | | <i>w2</i> | 2549 | | <i>a2</i> |
| 2183 | | <i>a1</i> | 2275 | | <i>c1</i> | 2367 | | <i>a1</i> | 2459 | 3 | <i>w#q</i> | 2551 | | <i>c1</i> |
| 2185 | | <i>w5</i> | 2277 | | <i>o2</i> | 2369 | 3 | <i>d1</i> | 2461 | 3 | <i>a1</i> | 2553 | | <i>a1</i> |
| 2187 | | <i>a1</i> | 2279 | | <i>o2</i> | 2371 | 9 | <i>a1</i> | 2463 | | <i>a1</i> | 2555 | | <i>o2</i> |
| 2189 | | <i>o2</i> | 2281 | | <i>c1</i> | 2373 | | <i>a1</i> | 2465 | | <i>a1</i> | 2557 | | <i>c1</i> |
| 2191 | | <i>o2</i> | 2283 | | <i>o2</i> | 2375 | | <i>o2</i> | 2467 | | <i>c1</i> | 2559 | | <i>wi</i> |
| 2193 | | <i>o1</i> | 2285 | | <i>o2</i> | 2377 | | <i>wk</i> | 2469 | | <i>c1</i> | 2561 | | <i>a1</i> |
| 2195 | | <i>a1</i> | 2287 | 20 | <i>se</i> | 2379 | | <i>o2</i> | 2471 | | <i>a1</i> | 2563 | | <i>o2</i> |
| 2197 | | <i>wk</i> | 2289 | | <i>o2</i> | 2381 | | <i>m2</i> | 2473 | | <i>wk</i> | 2565 | | <i>a1</i> |
| 2199 | | <i>c1</i> | 2291 | | <i>o2</i> | 2383 | 3 | <i>w#q</i> | 2475 | | <i>w5</i> | 2567 | | <i>a1</i> |
| 2201 | | <i>a1</i> | 2293 | 22 | <i>se</i> | 2385 | | <i>a1</i> | 2477 | | <i>a1</i> | 2569 | | <i>o2</i> |
| 2203 | 3 | <i>a1</i> | 2295 | | <i>o1</i> | 2387 | | <i>a1</i> | 2479 | | <i>c1</i> | 2571 | 3 | <i>d1</i> |
| 2205 | | <i>a1</i> | 2297 | | <i>a1</i> | 2389 | | <i>wk</i> | 2481 | | <i>a1</i> | 2573 | | <i>o2</i> |
| 2207 | 4 | <i>a1</i> | 2299 | | <i>c1</i> | 2391 | | <i>o2</i> | 2483 | | <i>a1</i> | 2575 | | <i>w5</i> |
| 2209 | | <i>wk</i> | 2301 | | <i>a1</i> | 2393 | | <i>a2</i> | 2485 | | <i>c1</i> | 2577 | | <i>c1</i> |
| 2211 | | <i>c1</i> | 2303 | | <i>o2</i> | 2395 | | <i>c1</i> | 2487 | | <i>c1</i> | 2579 | 3 | <i>w#q</i> |
| 2213 | | <i>m2</i> | 2305 | | <i>o2</i> | 2397 | | <i>a1</i> | 2489 | 3 | <i>o2</i> | 2581 | | <i>o2</i> |
| 2215 | 4 | <i>d1</i> | 2307 | | <i>a1</i> | 2399 | 8 | <i>a1</i> | 2491 | | <i>o1</i> | 2583 | | <i>a1</i> |
| 2217 | | <i>a1</i> | 2309 | | <i>wk</i> | 2401 | | <i>c1</i> | 2493 | | <i>o2</i> | 2585 | | <i>o2</i> |
| 2219 | | <i>o2</i> | 2311 | | <i>c1</i> | 2403 | | <i>wi</i> | 2495 | | <i>o2</i> | 2587 | | <i>o2</i> |
| 2221 | | <i>c1</i> | 2313 | | <i>o1</i> | 2405 | | <i>a1</i> | 2497 | | <i>c1</i> | 2589 | 4 | <i>d1</i> |
| 2223 | | <i>o1</i> | 2315 | 4 | <i>a1</i> | 2407 | | <i>c1</i> | 2499 | | <i>o1</i> | 2591 | 3 | <i>m3</i> |
| 2225 | | <i>o2</i> | 2317 | | <i>o2</i> | 2409 | | <i>c1</i> | 2501 | | <i>o2</i> | 2593 | | <i>wl</i> |
| 2227 | 3 | <i>o2</i> | 2319 | | <i>c1</i> | 2411 | | <i>a1</i> | 2503 | 3 | <i>a1</i> | 2595 | | <i>c1</i> |
| 2229 | | <i>c1</i> | 2321 | | <i>a1</i> | 2413 | 3 | <i>d1</i> | 2505 | | <i>c1</i> | 2597 | | <i>o2</i> |
| 2231 | | <i>a1</i> | 2323 | | <i>o2</i> | 2415 | | <i>o1</i> | 2507 | | <i>o2</i> | 2599 | | <i>c1</i> |
| 2233 | | <i>o2</i> | 2325 | | <i>c1</i> | 2417 | | <i>wm</i> | 2509 | | <i>o2</i> | 2601 | | <i>wf</i> |
| 2235 | | <i>o2</i> | 2327 | 4 | <i>o2</i> | 2419 | | <i>o1</i> | 2511 | | <i>c1</i> | 2603 | | <i>o2</i> |
| 2237 | | <i>wk</i> | 2329 | | <i>c1</i> | 2421 | | <i>o2</i> | 2513 | 4 | <i>k1</i> | 2605 | | <i>c1</i> |
| 2239 | | <i>y2</i> | 2331 | | <i>a1</i> | 2423 | 4 | <i>a1</i> | 2515 | 3 | <i>d1</i> | 2607 | | <i>a1</i> |
| 2241 | | <i>a1</i> | 2333 | | <i>m2</i> | 2425 | | <i>o2</i> | 2517 | | <i>a1</i> | 2609 | | <i>wm</i> |
| 2243 | | <i>a1</i> | 2335 | 3 | <i>a1</i> | 2427 | | <i>o2</i> | 2519 | | <i>o2</i> | 2611 | | <i>o2</i> |
| 2245 | | <i>c1</i> | 2337 | | <i>c1</i> | 2429 | 4 | <i>d1</i> | 2521 | | <i>c1</i> | 2613 | | <i>o1</i> |
| 2247 | | <i>c1</i> | 2339 | 4 | <i>a1</i> | 2431 | | <i>c1</i> | 2523 | | <i>a1</i> | 2615 | | <i>a1</i> |
| 2249 | | <i>o2</i> | 2341 | 3 | <i>w#q</i> | 2433 | | <i>o2</i> | 2525 | | <i>a1</i> | 2617 | | <i>c1</i> |
| 2251 | 5 | <i>a1</i> | 2343 | | <i>a1</i> | 2435 | | <i>a1</i> | 2527 | | <i>o2</i> | 2619 | | <i>c1</i> |
| 2253 | | <i>a1</i> | 2345 | | <i>o2</i> | 2437 | | <i>wk</i> | 2529 | | <i>o2</i> | 2621 | | <i>m2</i> |
| 2255 | | <i>o1</i> | 2347 | 3 | <i>m3</i> | 2439 | | <i>c1</i> | 2531 | 3 | <i>m3</i> | 2623 | | <i>o2</i> |
| 2257 | | <i>c1</i> | 2349 | | <i>o1</i> | 2441 | | <i>wl</i> | 2533 | | <i>o2</i> | 2625 | | <i>a1</i> |
| 2259 | | <i>c1</i> | 2351 | | <i>a1</i> | 2443 | | <i>o2</i> | 2535 | | <i>a1</i> | 2627 | | <i>o2</i> |
| 2261 | | <i>a1</i> | 2353 | | <i>o2</i> | 2445 | | <i>c1</i> | 2537 | | <i>o2</i> | 2629 | 3 | <i>a1</i> |
| 2263 | | <i>o2</i> | 2355 | | <i>a1</i> | 2447 | | <i>a1</i> | 2539 | | <i>c1</i> | 2631 | | <i>c1</i> |
| 2265 | | <i>a1</i> | 2357 | | <i>m2</i> | 2449 | | <i>o2</i> | 2541 | | <i>a1</i> | 2633 | | <i>a1</i> |
| 2267 | | <i>a1</i> | 2359 | | <i>o2</i> | 2451 | | <i>a1</i> | 2543 | 6 | <i>a1</i> | 2635 | | <i>o1</i> |
| 2269 | 3 | <i>w#q</i> | 2361 | | <i>c1</i> | 2453 | | <i>a1</i> | 2545 | 3 | <i>a1</i> | 2637 | | <i>c1</i> |
| 2271 | | <i>o2</i> | 2363 | | <i>o2</i> | 2455 | | <i>c1</i> | 2547 | | <i>o2</i> | 2639 | | <i>o2</i> |

TABLE A.2 Orders of Known Hadamard Matrices (continued)

| q | t | How | q | t | How | q | t | How | q | t | How | q | t | How |
|------|-----|-----|------|-----|-----|------|-----|-----|------|-----|-----|------|-----|-----|
| 2641 | | c1 | 2713 | | wk | 2785 | | c1 | 2857 | | wl | 2929 | | c1 |
| 2643 | | o2 | 2715 | | a1 | 2787 | | c1 | 2859 | | c1 | 2931 | | c1 |
| 2645 | | o2 | 2717 | | a1 | 2789 | | m2 | 2861 | | a1 | 2933 | | a1 |
| 2647 | 3 | w#q | 2719 | | c1 | 2791 | | c1 | 2863 | | o2 | 2935 | | c1 |
| 2649 | | c1 | 2721 | | a1 | 2793 | | a1 | 2865 | 3 | d1 | 2937 | | o2 |
| 2651 | | o2 | 2723 | | a1 | 2795 | | o2 | 2867 | | a1 | 2939 | 8 | a1 |
| 2653 | | o2 | 2725 | | c1 | 2797 | | m2 | 2869 | | c1 | 2941 | | c1 |
| 2655 | | c1 | 2727 | | o2 | 2799 | | wi | 2871 | | a1 | 2943 | | o2 |
| 2657 | | a1 | 2729 | | wl | 2801 | | wk | 2873 | | a1 | 2945 | | a1 |
| 2659 | 3 | w#q | 2731 | 3 | w#q | 2803 | 3 | w#q | 2875 | | c1 | 2947 | | o2 |
| 2661 | 3 | d1 | 2733 | 3 | a1 | 2805 | | o1 | 2877 | | o2 | 2949 | | c1 |
| 2663 | | a1 | 2735 | | a1 | 2807 | | o1 | 2879 | 21 | se | 2951 | 3 | d1 |
| 2665 | | c1 | 2737 | | o1 | 2809 | | wk | 2881 | | o2 | 2953 | | wl |
| 2667 | | a1 | 2739 | | c1 | 2811 | | a1 | 2883 | | o2 | 2955 | | o2 |
| 2669 | | o2 | 2741 | | a2 | 2813 | | a1 | 2885 | | o2 | 2957 | | a1 |
| 2671 | 9 | a1 | 2743 | | o2 | 2815 | 3 | d1 | 2887 | 5 | a1 | 2959 | | y2 |
| 2673 | | a1 | 2745 | | a1 | 2817 | | o2 | 2889 | | w5 | 2961 | | o1 |
| 2675 | | o2 | 2747 | | a1 | 2819 | 3 | w#q | 2891 | | o2 | 2963 | 3 | w#q |
| 2677 | 9 | a1 | 2749 | | wk | 2821 | | c1 | 2893 | 3 | a1 | 2965 | | o2 |
| 2679 | | o2 | 2751 | | a1 | 2823 | 3 | d1 | 2895 | | a1 | 2967 | | a1 |
| 2681 | | a1 | 2753 | | a2 | 2825 | | a1 | 2897 | | a1 | 2969 | | a2 |
| 2683 | 7 | a1 | 2755 | | o2 | 2827 | | c1 | 2899 | 4 | d1 | 2971 | 3 | a1 |
| 2685 | | a1 | 2757 | | a1 | 2829 | | c1 | 2901 | | c1 | 2973 | 4 | d1 |
| 2687 | 21 | se | 2759 | | o2 | 2831 | | o2 | 2903 | 4 | a1 | 2975 | | o1 |
| 2689 | | wk | 2761 | | c1 | 2833 | | wk | 2905 | | o2 | 2977 | | c1 |
| 2691 | | c1 | 2763 | | o2 | 2835 | | c1 | 2907 | | c1 | 2979 | | o2 |
| 2693 | | a1 | 2765 | | a1 | 2837 | | wl | 2909 | | wk | 2981 | | a1 |
| 2695 | | o2 | 2767 | 3 | w#p | 2839 | | y2 | 2911 | | c1 | 2983 | | o2 |
| 2697 | | c1 | 2769 | | o2 | 2841 | 3 | a1 | 2913 | 7 | d1 | 2985 | | a1 |
| 2699 | 21 | se | 2771 | | a1 | 2843 | 3 | m3 | 2915 | | o1 | 2987 | 3 | w#r |
| 2701 | | w4 | 2773 | 3 | d1 | 2845 | | c1 | 2917 | | wl | 2989 | | o2 |
| 2703 | | o1 | 2775 | | o2 | 2847 | | c1 | 2919 | | wi | 2991 | | c1 |
| 2705 | | o1 | 2777 | | m2 | 2849 | | o2 | 2921 | 3 | d1 | 2993 | | a1 |
| 2707 | | c1 | 2779 | | c1 | 2851 | | c1 | 2923 | | o2 | 2995 | 9 | a1 |
| 2709 | | c1 | 2781 | | o2 | 2853 | | a1 | 2925 | | a1 | 2997 | | a1 |
| 2711 | 3 | m3 | 2783 | | a1 | 2855 | 4 | d1 | 2927 | 3 | m3 | 2999 | 22 | se |

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