

## Note

### Product of Four Hadamard Matrices

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*Communicated by V. Pless*

Received September 24, 1990

We prove that if there exist Hadamard matrices of order  $4m$ ,  $4n$ ,  $4p$ , and  $4q$  then there exists an Hadamard matrix of order  $16mpq$ . This improves and extends the known result of Agayan that there exists a Hadamard matrix of order  $8mn$  if there exist Hadamard matrices of order  $4m$  and  $4n$ . © 1992 Academic Press, Inc.

A *weighing matrix* [3] of order  $n$  with weight  $k$ , denoted  $W = W(n, k)$ , is a  $(0, \pm 1)$  matrix satisfying  $WW^T = kI_n$ . A  $W(n, n)$  is an Hadamard matrix.

Two matrices  $X$  and  $Y$  are said to be *amicable* if  $XY^t = YX^t$ . They are *disjoint* if  $X \cap Y = 0$  (here,  $\cap$  denotes the Hadamard, or entry-wise, product of matrices).

LEMMA 1. *If there exist Hadamard matrices of order  $4m$  and  $4n$  then there exist two  $(\pm 1)$  matrices,  $S$  and  $R$  of order  $4mn$ , satisfying*

$$(i) \quad SS^T + RR^T = 8mnI_{4mn},$$

$$(ii) \quad SR^T = RS^T = 0.$$

*Proof.* We write

$$H = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{pmatrix}, \quad K = \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{pmatrix},$$

where  $H$  and  $K$  are the given Hadamard matrices, each  $H_i$  being of size  $m \times 4m$  and each  $K_i$  being of size  $n \times 4n$ ,  $i = 1, 2, 3, 4$ . Now take

$$R = \frac{1}{2}(H_1 + H_2)^T \times K_1 + \frac{1}{2}(H_1 - H_2)^T \times K_2,$$

$$S = \frac{1}{2}(H_3 + H_4)^T \times K_3 + \frac{1}{2}(H_3 - H_4)^T \times K_4.$$

Clearly both  $R$  and  $S$  are square  $\pm 1$  matrices, and  $RR^T + SS^T = \frac{1}{2}(H_1^T H_1 + H_2^T H_2 + H_3^T H_3 + H_4^T H_4) \times 4nI$ . Since  $(H_1^T H_1 + H_2^T H_2 + H_3^T H_3 + H_4^T H_4) = H^T H$ , we have  $RR^T + SS^T = 8mnI_{4mn}$ . Since  $K_i K_j^T = 0$  for  $i \neq j$ ,  $R$  and  $S$  are as claimed.

This is Theorem 3 in [2], where  $S$  and  $R$  are called an *orthogonal pair*.

LEMMA 2. *If there exist Hadamard matrices of order  $4m$  and  $4n$  then there exist two disjoint, amicable  $W(4mn, 2mn)$ .*

*Proof.* Let  $R$  and  $S$  be the matrices constructed in Lemma 1. Let  $X = \frac{1}{2}(R + S)$  and  $Y = \frac{1}{2}(R - S)$ . We calculate

$$XX^T = YY^T = \frac{1}{4}(RR^T + SS^T) = 2mn I_{4mn}.$$

$X$  and  $Y$  are disjoint since  $R$  and  $S$  are  $\pm 1$  matrices. Therefore,  $X$  and  $Y$  are the desired weighing matrices.

This is Theorem 2 and Lemma 3 of [4], where it was obtained using *M-structures*. It may also be deduced from Theorem 3 and 7 of [2], which follows our method. The fact that the matrices are amicable is not needed for the theorem which follows. The two lemmas are clearly equivalent, for we may also write  $S = X + Y$ ,  $R = X - Y$ .

THEOREM 1. *If there exist Hadamard matrices of order  $4m$ ,  $4n$ ,  $4p$ ,  $4q$  then there exists an Hadamard matrix of order  $16mnpq$ .*

*Proof.* By Lemma 2, there exist two disjoint  $W(4mn, 2mn)$ ,  $X$  and  $Y$ . By Lemma 1, there exist two  $(\pm 1)$  matrices  $S$  and  $R$  of order  $4pq$  satisfying (i) and (ii).

Let  $H = X \times S + Y \times R$ . Then  $H$  is a  $(\pm 1)$  matrix and

$$\begin{aligned} HH^T &= XX^T \times SS^T + YY^T \times RR^T = 2mnI_{4mn} \times (SS^T + RR^T) \\ &= 2mnI_{4mn} \times 8pqI_{4pq} = 16mnpqI_{16mnpq}. \end{aligned}$$

Thus  $H$  is the required Hadamard matrix.

Theorem 1 gives an improvement and extension for the result of Agayan [1] that if there exist Hadamard matrices of order  $4m$  and  $4n$  then

there exists an Hadamard matrix of order  $8mn$ . Using the result of Agayan repeatedly on four Hadamard matrices of order  $4m$ ,  $4n$ ,  $4p$ ,  $4q$ , gives an Hadamard matrix of order  $32mpq$ .

#### REFERENCES

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