Addendum to Further results on base sequences, disjoint complementary sequences, $OD(4t; t, t, t, t)$, and the excess of Hadamard matrices.

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Abstract

It is known that if there are base sequences of lengths $m+p$, $m+p$, m, m and y is a Yang number then there are T-sequences of length $(2m+p)y$.

Let $G = \{g : g = 2^a 10^b 26^c,\ a,\ b,\ c$ non negative integers$\}$.

We show that base sequences currently exist for $p = 1$ and $m \in \{1, \ldots, 18, 20, 21, 23, 25, 29\} \cup G$. Yang numbers currently exist for $y \in \{3, 5, \ldots, 33, 37, 41, 45, 51, 53, 59, 65, 81, \ldots$ and $2g+1 > 81$, $g \in G\}$.

This means T-sequences exist for

$$
\begin{aligned}
0_1 &= \{t : t \text{ odd } \le 59\} \cup \{s : 63 \le s \le 199, s \text{ not prime}, s \ne 183\} \\
0_2 &= \{ym : y \text{ a Yang number}, m \text{ a base sequence}\} \\
0_3 &= \{g + g' : g, g' \in G (\text{e.g. we may take } g' = 1)\} \\
E_1 &= \{2yp : y \text{ a Yang number and } p \in 0_1 \cup 0_2 \cup 0_3 \cup E_1\}.
\end{aligned}
$$

In particular we find a new SBIBD$(4k^2, 2k^2+k, k^2+k)$ for $k = 43$, and new Hadamard matrices with maximum known excess for $n = 860$ and 1204.

1 Introduction

We use the notation and definitions of Koukouvinos, Kounias and Seberry [5].

First we reformulate some results of Yang [14] (Table 1).

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Theorem 1 (C.H. Yang) *(restated) Q, R, S, T are T-sequences of lengths $(2m+p)(2n+1)$ where $0', 0$ are sequences of zeros of lengths $m+p$ and m respectively, $F = \{f_1, \ldots, f_n\}$, $G = \{g_1, \ldots, g_n\}$, $H = \{h_1, \ldots, h_n\}$, when*

$$Q=\{Af_n, Cg_1 + Dh_1 ;0',0;Af_{n-1}, Cg_2 + Dh_2 ;0',0;\ldots;Af_1, Cg_n + Dh_n;0',0;-B',0\}$$

$$R=\{Bf_n, Dg_n - Ch_n;0',0;Bf_{n-1}, Dg_{n-1} - Ch_{n-1};0',0;\ldots;Bf_1, Dg_1 - Ch_1 ;0',0; A',0\}$$

$$S=\{0',0;Ag_n - Bh_1, -Cf_n;0',0;Ag_{n-1} - Bh_2, -Cf_{n-1};\ldots;0',0;Ag_1 - Bh_n, -Cf_1;0',D'\}$$

$$T=\{0',0;Bg_1 + Ah_n, -Df_n;0',0;Bg_2 + Ah_{n-1}, -Df_{n-1};\ldots;0',0;Bg_n + Ah_1, -Df_1;0',C'\}$$

(i) A, B, C, D are suitable sequences with elements 0, +1 or -1 of lengths $m+p$, $m+p$, m, m, that is A and B are disjoint, C and D are disjoint and A, B, C, D have zero non-periodic autocorrelation function;

(ii) F, G, H are Yang quasi-symmetric sequences with elements 0, +1 or -1 of length n, that is F has elements ± 1, $G + H$ has elements ± 1 and $g_i = 0 \Leftrightarrow g_{n+1-i} = 0$, $h_i = 0 \Leftrightarrow h_{n+1-i} = 0$, $g_i = 0 \Leftrightarrow h_i \neq 0$, $g_i \neq 0 \Leftrightarrow h_i = 0$ and F, G, H have zero non periodic autocorrelation function.

Remark 1 Let f, g, h be the row sums of F, G, H then $f^2 + g^2 + h^2 = 2n$ (counting elements and using the zero autocorrelation function). This means Yang quasi symmetric sequences do not exist for $n = 2^{2a-1}(8b+7)$, a, b non-negative integers as the numbers $4^a(8b+7)$ can only be written as the sum of four squares. In particular $n \neq 14, 30, 46, 62, \ldots$

Yang has given these sequences for n = 3, 5, 7, 9, 11, 12, 13, 15, 25, 29 and he notes they exist for $n = 2^a 10^b 26^c$ (Golay numbers).

Therefore we know these sequences

(i) exist for $n \in \{1, \ldots, 5, 7, \ldots, 13, 15, 16, 20, 25, 26, 29, 32, \ldots\}$;

(ii) do not exist for $n \in \{6, 14, 30, \ldots\}$, 6 is decided by a complete hand search;

(iii) $n \in \{17, 18, 19, 21, \ldots, 24, 27, 28, 31, \ldots\}$ are undecided.

Corollary 2 (C.H.Yang) *(restated) Under the Conditions of the Theorem 1, Q, R, S, T are a set of 4-complementary sequences of length $(2m+p)n$, where A, B, C, D are base sequences of lengths $m+p$, $m+p$, m, m.*

$$Q=\{Af_n, Cg_1 + Dh_1 ; Af_{n-1}, Cg_2 + Dh_2 ;\ldots; Af_1, Cg_n + Dh_n\}$$

$$R=\{Bf_n, Dg_n - Ch_n ;Bf_{n-1}, Dg_{n-1} - Ch_{n-1};\ldots; Bf_1, Dg_1 - Ch_1\}$$

$$S=\{Ag_n - Bh_1, -Cf_n; Ag_{n-1} - Bh_2, -Cf_{n-1} ;\ldots; Ag_1 - Bh_n, Cf_1\}$$

$$T=\{Bg_1 + Ah_n, -Df_n; Bg_2 + Ah_{n-1}, -Df_{n-1} ;\ldots;Bg_n + Ah_1, -Df_1\}$$

Theorem 3 (C.H. Yang) *(restated) Q, R, S, T are T-sequences of lengths $b(4m+1) = (2s+p)(4m+1)$, where $0', 0$ are sequences of zeros of lengths $s+p$ and s respectively, $X = \{x_1, \ldots x_m\}$, $Y = \{y_1, \ldots, y_m\}$, $G = \{g_1, \ldots, g_{2m}\}$, $H = \{h_1, \ldots, h_{2m}\}$, when*

$$Q=\{Ag_{2m}-Bh_1,-Cy_1;Ag_{2m-1}-Bh_2,-D'x_1;\ldots;Ag_2-Bh_{2m-1},-Cy_m;$$
$$;Ag_1-Bh_{2m},-D'x_m;0',-D';0',0;0',0;\ldots;0',0;0',0;\}$$

$$R=\{Bg_1+Ah_{2m},-Dy_1;Bg_2+Ah_{2m-1},C'x_1;\ldots;Bg_{2m-1}+Ah_2,-Dy_m;$$
$$;Bg_{2m}+Ah_1,C'x_m;0',C';0',0;0',0;\ldots;0',0;0',0;\}$$

$$S=\{0',0;0',0;\ldots;0',0;0',0;-B,0;-Bx_m,C'g_{2m}+D'h_{2m};$$
$$;A'y_1,C'g_{2m-1}+D'h_{2m-1};\ldots;-Bx_1,C'g_2+D'h_2;A'y_m,C'g_1+D'h_1\}$$

$$T=\{0',0;0',0;\ldots;0',0;0',0;A,0;Ax_m,D'g_1-C'h_1;$$
$$;B'y_1,D'g_2-C'h_2;\ldots;Ax_1,D'g_{2m-1}-C'h_{2m-1};B'y_m,D'g_{2m}-C'h_{2m}\}$$

(i) *A,B,C,D are suitable sequences with elements 0, +1 or -1 of lengths $s+p$, $s+p$, s, s, that is A and B are disjoint, C and D are disjoint and A,B,C,D have zero non-periodic autocorrelation function;*

(ii) *X,Y,G,H are Yang quasi-symmetric sequences with elements 0, +1 or -1 of lengths m, m, $2m$, $2m$ respectively, that is X,Y have elements ±1, $G+H$ has elements ±1 and $g_i=0 \Leftrightarrow g_{2m+1-i}=0$, $h_i=0 \Leftrightarrow h_{2m+1-i}=0$, $g_i=0 \Leftrightarrow h_i \neq 0$, $g_i \neq 0 \Leftrightarrow h_i=0$ and the sequences $E=(X|0,1)$, $F=(Y|0)$, G, H have zero non-periodic autocorrelation function.*

Remark 2 Yang has given these sequences for $n=4m+1$, $m \leq 11$.

Remark 3 In Theorem 3 there are in fact five sequences used of length $4m+2$, where $G'+H$ and $G+H$ have elements ±1 and $\{G,H,Y,X+W\}$ have zero non-periodic autocorrelation function

$$\begin{array}{llllllll}
G'=\{g_{2m},0; & g_{2m-1},0; & g_{2m-2},0; & g_{2m-3},0;\ldots, & 0,g_3 & ;g_2,0 & ;g_1,0 & ;0,0\} \\
H=\{h_1,0 & ;h_2,0 & ;h_3,0 & ;\quad h_4,0;\ldots & h_{2m-2},0; & h_{2m-1},0; & h_{2m},0; & 0,0\} \\
Y=\{0,y_1 & ;0,0 & ;0,y_2 & ;\quad 0,0;\ldots & 0,y_m; & 0,0 & ;0,0 & ;0,0\} \\
X=\{0,0 & ;0,x_1 & ;0,0 & ;\quad 0,x_2;\ldots & 0,x_{m-1}; & 0,0 & ;0,x_m & ;0,0\} \\
W=\{0,0 & ;0,0 & ;0,0 & ;\quad 0,0;\ldots & 0,0; & 0,0 & ;0,0 & ;0,1\}
\end{array}$$

So that writing P' for the reverse of the sequence P

$$\begin{array}{lllllllll}
Q= & A\times G' & - & B\times H & - & C\times Y & - & D'\times X & - \; D'\times W \\
R= & B\times G & + & A\times H' & - & D\times Y & + & C'\times X & + \; C'\times W \\
S= & D'\times H' & + & C'\times G' & + & A'\times Y & - & B\times X' & - \; B\times W' \\
T= & -C'\times H & + & D'\times G & + & B'\times Y & + & A\times X' & + \; A\times W'
\end{array}$$

and the sequences used are $\{Q,0_{2mb}\}$, $\{R,0_{2mb}\}$, $\{0_{2mb},S\}$, $\{0_{2mb},T\}$ of length $(4m+1)b$ where $b=2s+p$ (A,B of length $s+p$, C, D of length s).

Theorem 4 (C.H. Yang) *(restated) Let A,B,C,D be base sequences of lengths $m+p$, $m+p$, m, m and F,G,H,E be base sequences of lengths $n+1$, $n+1$, n, n. Then the following Q,R,S,T*

$$Q=\{Af_{n+1}|Cg_1;-B'e_1|Dh_1;Af_n|Cg_2;-B'e_2|Dh_2;\ldots;$$
$$;Af_2|Cg_n;-B'e_n|Dh_n;Af_1|Cg_{n+1}\}$$

$$R=\{Bf_{n+1}|Dg_{n+1};Ae_1|-Ch_n;Bf_n|Dg_n;Ae_2|-Ch_{n-1};\ldots;$$
$$;Bf_2|Dg_2;Ae_n|-Ch_1;Bf_1|Dg_1\}$$

$$S=\{Ag_{n+1}|-Cf_1;-Bh_1|-D'e_1;Ag_n|-Cf_2;-Bh_2|-D'e_2;\ldots;$$
$$;Ag_2|-Cf_n;-Bh_n|-D'e_n;Ag_1|-Cf_{n+1}\}$$

$$T=\{Bg_1|-Df_1;Ah_n|C'e_1;Bg_2|-Df_2;Ah_{n-1}|C'e_2;\ldots;$$
$$;Bg_n|-Df_n;Ah_1|C'e_n;Bg_{n+1}|-Df_{n+1}\}$$

are 4-complementary (1,-1)-sequences of length $(2m+p)(2n+1)$.

2 Preliminary Results

Theorem 5 *If there exist 4-complementary sequences of length t and y is a Yang number then there exist 4-disjoint T-sequences of length 2yt.*

Example 1 Yang's construction allows us to multiply base sequences E, F, G, H of lengths $m+p$, $m+p$, m, m by 3 by first forming suitable sequences

$$A = \frac{1}{2}(E+F),\ B + \frac{1}{2}(E-F),\ C = \frac{1}{2}(G+H),\ D = \frac{1}{2}(G-H)$$

which are then used to form T-sequences (0_{m+1} and $0'_m$ are sequences of $m+1$ and m zeros respectively)

$$\begin{aligned} X &= (A, C; 0, 0', B^*, 0') \\ Y &= (B, D; 0, 0'; -A^*, 0') \\ Z &= (0, 0'; A, -C; 0, D^*) \\ W &= (0, 0'; B, -D; 0, -C^*) \end{aligned}$$

for example, of length $3(2m+p)$. X^* means the elements of the sequence X are written in the reverse order. Clearly if $p=0$ the construction can still be used to form sequences but of even length.

Remark 4 We also note that the construction can be used to multiply by any Yang number and can be used recursively because X, Y, Z, W just formed (or any T-sequence) can be used to form 4-complementary sequences of lengths $2ym$:

$$\begin{aligned} E' &= X+Y+Z+W & \text{or}\quad E' &= -X+Y+Z+W \\ F' &= -X+Y-Z+W & F' &= X-Y+Z+W \\ G' &= -X-Y+Z+W & G' &= X+Y-Z+W \\ H' &= -X+Y+Z-W & H' &= X+Y+Z-W. \end{aligned}$$

Then the method used again to form similar sequences and then longer T-sequences of lengths $4y_1y_2m$ where y_i is a Yang number.

Remark 5 Yang numbers exist for $y \in \{3, 5, \ldots, 33, 37, 41, 45, 51, 53, 59, 65, 81, \ldots$, and $2g+1$, $g = 2^a.10^b.26^c$, a, b, c are non-negative integers$\}$.

3 A new maximal excess and $SBIBD$

Using the base sequences of lengths $m+1$, $m+1$, m, m, $m=21$ as follows, which have been obtained by correcting a typographical error in Yang [14]: $2m+1 = 43 = 4^2+3^2+3^2+3^2$

1 1 −1 1 −1 1 −1 −1 −1 −1 −1 −1 −1 1 −1 −1 1 1 −1 1 −1 −1
1 1 −1 −1 −1 −1 1 −1 1 −1 1 1 −1 −1 −1 1 −1 1 1 1 −1 1
1 −1 1 1 −1 1 1 1 1 1 1 −1 −1 −1 1 1 −1 −1 1 1 1
1 1 −1 1 −1 1 1 1 1 −1 −1 1 −1 −1 1 1 −1 −1 −1 −1 1

we can construct the following T-sequences (T-matrices) of length 43.

$$
\begin{aligned}
T_1 &= \{1,4,-5,6,7,8,9,-13,-14,15,16,-17,-18,21\} \\
T_2 &= \{-2,3,10,11,-12,19,20\} \\
T_3 &= \{-22,-23,24,26,29,31,34,36,-39,-41,42\} \\
T_4 &= \{-25,-27,28,30,32,33,-35,37,-38,40,43\}
\end{aligned}
$$

Then using Lemma 1 in [6] we can construct an Hadamard matrix of order $n = 172$ with maximal excess $\sigma(n) = n\sqrt{n-3} = 2236$, using only the above T-matrices with row sums $4,3,3,3$.

$\underline{n = 25}$		
$25(2s+1)$	$=$	$(5d)^2 + (5c)^2 + (5b)^2 + (5a)^2$
$\underline{n = 25}$		
$25(2s+1)$	$=$	$(-2a - b + 4c + 2d)^2 + (a - 2b - 2c + 4d)^2 + (4a - 2b + 2c - d)^2 + (2a + 4b + c + 2d)^2$
$25(2s+1)$	$=$	$(-2b - a + 4c + 2d)^2 + (b - 2a - 2c + 4d)^2 + (4b - 2a + 2c - d)^2 + (2b + 4a + c + 2d)^2$
$25(2s+1)$	$=$	$(-2a - b + 4d + 2c)^2 + (a - 2b - 2d + 4c)^2 + (4a - 2b + 2d - c)^2 + (2a + 4b + d + 2c)^2$
$25(2s+1)$	$=$	$(-2b - a + 4d + 2c)^2 + (b - 2a - 2d + 4c)^2 + (4b - 2a + 2d - c)^2 + (2b + 4a + d + 2c)^2$
$\underline{n = 29}$		
$29(2s+1)$	$=$	$(2b + 3c + 4d)^2 + (2a + 4c - 3d)^2 + (3a - 4b + 2d)^2 + (4a + 3b - 2c)^2$
$29(2s+1)$	$=$	$(2a + 3c + 4d)^2 + (2b + 4c - 3d)^2 + (3b - 4a + 2d)^2 + (4b + 3a - 2c)^2$
$29(2s+1)$	$=$	$(2b + 3d + 4c)^2 + (2b + 4d - 3c)^2 + (3a - 4b + 2c)^2 + (4a + 3b - 2d)^2$
$29(2s+1)$	$=$	$(2a + 3d + 4c)^2 + (2b + 4d - 3c)^2 + (3b - 4a + 2c)^2 + (4b + 3a - 2d)^2$
$\underline{n = 37}$		
$37(2s+1)$	$=$	$(c + 6d)^2 + (6c - d)^2 + (a - 6b)^2 + (6a + b)^2$
$37(2s+1)$	$=$	$(c + 6d)^2 + (6c - d)^2 + (b - 6a)^2 + (6b + a)^2$
$37(2s+1)$	$=$	$(d + 6c)^2 + (6d - c)^2 + (a - 6b)^2 + (6a + b)^2$
$37(2s+1)$	$=$	$(d + 6c)^2 + (6d - c)^2 + (b - 6a)^2 + (6b + a)^2$
$\underline{n = 45}$		
$45(2s+1)$	$=$	$(2a - 4b - 3c - 4d)^2 + (4a + 2b + 4c - 3d)^2 + (3a - 4b + 2c + 4d)^2 + (4a + 3b - 4c + 2d)^2$
$45(2s+1)$	$=$	$(2b - 4a - 3c - 4d)^2 + (4b + 2a + 4c - 3d)^2 + (3b - 4a + 2c + 4d)^2 + (4b + 3a - 4c + 2d)^2$
$45(2s+1)$	$=$	$(2a - 4b - 3d - 4c)^2 + (4a + 2b + 4d - 3c)^2 + (3a - 4b + 2d + 4c)^2 + (4a + 3b - 4d + 2c)^2$
$45(2s+1)$	$=$	$(2b - 4a - 3d - 4c)^2 + (4b + 2a + 4d - 3c)^2 + (3b - 4a + 2d + 4c)^2 + (4b + 3a - 4d + 2c)^2$

Table 1. Decompositions arising from Yang's composition $n(2s+1)$ where $2s+1 = a^2 + b^2 + c^2 + d^2$ using theorem 3.

We can also extend the values of k in theorem 8 of Seberry, see [8], to include $k = 43$, by using the method which was developed in Koukouvinos, Kounias and Seberry [6]. For $k = 43$ we use T-sequences with row sums $(t_1, t_2, t_3, t_4) = (4, 3, -3, 3)$ to form the following circulant matrices of commuting variables which gives an $OD(4.43; 43, 43, 43, 43)$:

$$\begin{aligned} X &= AT_1 + BT_2 - CT_3 + DT_4 \\ Y &= -BT_1 + AT_2 - DT_3 - CT_4 \\ Z &= -CT_1 - DT_2 - AT_3 + BT_4 \\ W &= -DT_1 + CT_2 + BT_3 + AT_4 \end{aligned}$$

The variables are then replaced by Williamson matrices with row sums $(a, b, c, d) = (7, 7, 7, 5)$ see [9, p388-9] to form an Hadamard matrix of order $n = 7396 = 4.43^2$ with maximal excess.

Hence there exists an $SBIBD(4k^2, 2k^2 + k, k^2 + k)$ for $k = 43$.

New constructions for T-sequences are listed in Table 2.

t		Sum of squares	Comment	KF Upper Bound	Excess
129	3 × 43	$10^2 + 4^2 + 3^2 + 2^2$	$y = 3, m = 21$	11696	10320
215	5 × 43	$11^2 + 9^2 + 3^2 + 2^2$	$y = 5, m = 21$	25184	21500
215	5 × 43	$10^2 + 9^2 + 5^2 + 3^2$	$y = 5, m = 21$	25184	23220 †
225	25 × 9	15^2	$y = 25, m = 4$	27000	27000 * #
225	25 × 9	$14^2 + 5^2 + 2^2$	$y = 25, m = 4$	27000	25200
225	25 × 9	$12^2 + 8^2 + 4^2 + 1^2$	$y = 25, m = 4$	27000	22500
225	25 × 9	$12^2 + 6^2 + 6^2 + 3^2$	$y = 25, m = 4$	27000	24300
225	25 × 9	$10^2 + 10^2 + 4^2 + 3^2$	$y = 25, m = 4$	27000	24300
261	29 × 9	$12^2 + 9^2 + 6^2$	$y = 29, m = 4$	33708	28188
261	29 × 9	$12^2 + 8^2 + 7^2 + 2^2$	$y = 29, m = 4$	33708	30276
275	25 × 11	$12^2 + 11^2 + 3^2 + 1^2$	$y = 25, m = 5$	36432	29700
301	7 × 43	$16^2 + 5^2 + 4^2 + 2^2$	$y = 7, m = 21$	41736	38528
301	7 × 43	$11^2 + 10^2 + 8^2 + 4^2$	$y = 7, m = 21$	41736	39732 †
301	7 × 43	$10^2 + 10^2 + 10^2 + 1^2$	$y = 7, m = 21$	41736	37324

Table 2. New constructions for 4-disjoint T-sequences. (this table should be used in conjunction with [5])

* Hadamard matrix with maximal excess
†New maximum known
Previously known maximal excess [8]

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