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Hadamard matrices of order $\equiv 8 \pmod{16}$ with maximal excess

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Abstract

Koukouvinos, C. and J. Seberry, Hadamard matrices of order $\equiv 8 \pmod{16}$ with maximal excess, Discrete Mathematics 92 (1991) 173–176.

Kounias and Farmakis, in 'On the excess of Hadamard matrices', Discrete Math. 68 (1988) 59–69, showed that the maximal excess (or sum of the elements) of an Hadamard matrix of order h, $\sigma(h)$ for $h = 4m(m-1)$ is given by

$$\sigma(4m(m-1)) \leq 4(m-1)^2(2m+1).$$

Kharaghani in 'An infinite class of Hadamard matrices of maximal excess' (to appear) showed this maximal excess can be attained if m is the order of a skew-Hadamard matrix.

We give another proof of Kharaghani's result, by generalizing an example of Farmakis and Kounias, 'The excess of Hadamard matrices and optimal designs', Discrete Math. 67 (1987) 165–176, and further show that the maximal excess of the bound is attained if $m \equiv 2 \pmod 4$ is the order of a conference matrix.

1. Introduction

An *Hadamard matrix*, H, of order h has elements ± 1 and satisfies $HH^{\mathrm{T}} = hI_h$. If $H = I + S$ where $S^{\mathrm{T}} = -S$ then H is called *skew-Hadamard*. The most recent results on skew-Hadamard matrices are given by Seberry [12].

A *conference matrix*, C, of order $n \equiv 2 \pmod 4$ has zero diagonal, other elements ± 1, and satisfies $CC^{\mathrm{T}} = (n-1)I_n$. Conference matrices are known for

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the orders:

(i) q^r+1, $q\equiv 1 \pmod 4$ is a prime power;

(ii) $(s-1)^{2t}+1$, s is the order of a skew-Hadamard matrix and t a positive integer;

(iii) $q^2(q+2)+1$, $q=4t-1$ a prime power, $q+3$ the order of a conference matrix, Mathon [9];

(iv) $5.9^{2t+1}+1$, $t\geqslant 0$ a nonnegative integer, Seberry and Whiteman [14];

(v) $(c-1)^t+1$, c the order of a conference matrix and t a nonnegative integer.

Conference matrices cannot exist unless $n-1$ is the sum of two squares: thus they cannot exist for orders 22, 34, 58, 70, 78, 94. The first few undecided orders are 66, 86, 118, 146, 154 and 186.

Let e denote the $1\times q$ matrix of ones, J the $q\times q$ matrix of ones and I the identity matrix. The skew-Hadamard and conference matrices of order $q+1$ can be denoted

$$\begin{bmatrix} 1 & e \\ -e^{\mathrm{T}} & I+Q \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & e \\ e^t & X \end{bmatrix},$$

where $Q^{\mathrm{T}}=-Q$, $X^{\mathrm{T}}=X$, $QQ^{\mathrm{T}}=qI-J$, $XX^{\mathrm{T}}=qI-J$, $eQ=0$, and $eX=0$. Q and X are called the *core* of the matrices from which they have been derived.

The *excess* of a Hadamard matrix, H, denoted $\sigma(H)$ is the sum of all its elements. $\sigma(n)$ is used to denote the *maximal excess* of all Hadamard matrices of order n. This concept has been studied by a number of authors (see [1–8, 10–11, 13, 15–16]) and some theoretical bounds of $\sigma(n)$ obtained and Hadamard matrices meeting that bound constructed. In this paper we give a new family of Hadamard matrices whose excess is maximal.

2. The constructions

Let $q\equiv 1 \pmod 4$ and $X^{\mathrm{T}}=X$, $eX=0$, $XX^{\mathrm{T}}=qI-J$, where X is the core of a conference matrix. Then consider the following matrix:

$$\left[\begin{array}{ccc|ccccc}
-A\times 1 & -A\times 1 & A\times 1 & -A\times 1 & B\times e & B\times e & C\times e & C\times e \\
A\times 1 & -A\times 1 & A\times 1 & A\times 1 & C\times e & C\times e & -B\times e & -B\times e \\
-A\times 1 & A\times 1 & A\times 1 & A\times 1 & B\times e & -B\times e & C\times e & -C\times e \\
A\times 1 & A\times 1 & A\times 1 & -A\times 1 & C\times e & -C\times e & -B\times e & B\times e \\
\hline
C\times e^{\mathrm{T}} & C\times e^{\mathrm{T}} & B\times e^{\mathrm{T}} & B\times e^{\mathrm{T}} & -A\times I-B\times X & A\times I-B\times X & A\times I+C\times X & A\times I-C\times X \\
B\times e^{\mathrm{T}} & B\times e^{\mathrm{T}} & -C\times e^{\mathrm{T}} & -C\times e^{\mathrm{T}} & A\times I-B\times X & A\times I+B\times X & A\times I-C\times X & -A\times I-C\times X \\
B\times e^{\mathrm{T}} & -B\times e^{\mathrm{T}} & -C\times e^{\mathrm{T}} & C\times e^{\mathrm{T}} & A\times I+C\times X & -A\times I+C\times X & A\times I+B\times X & A\times I-B\times X \\
-C\times e^{\mathrm{T}} & C\times e^{\mathrm{T}} & -B\times e^{\mathrm{T}} & B\times e^{\mathrm{T}} & A\times I-C\times X & A\times I+C\times X & -A\times I+B\times X & A\times I+B\times X
\end{array}\right]$$

(this is in fact an orthogonal design $OD(4(q+1); 4, 2q, 2q)$). We replace A by J_q, B by $I+X$ and C by $I-X$ to get the required Hadamard matrix of order $4q(q+1)$ and excess $4q^2(2q+3)$.

Let $q \equiv 3 \pmod 4$, $Q^{\mathrm{T}} = -Q$, $eQ = 0$, $QQ^{\mathrm{T}} = qI - J$, where Q is the core of a skew-Hadamard matrix. We generalize an example of Farmakis and Kounias [3] and consider the matrix:

$$\left[\begin{array}{cccc|cccc}
A\times 1 & -A\times 1 & -A\times 1 & -A\times 1 & B\times e & B\times e & B\times e & B\times e \\
A\times 1 & -A\times 1 & A\times 1 & A\times 1 & B\times e & B\times e & -B\times e & -B\times e \\
A\times 1 & A\times 1 & -A\times 1 & A\times 1 & B\times e & -B\times e & B\times e & -B\times e \\
A\times 1 & A\times 1 & A\times 1 & -A\times 1 & B\times e & -B\times e & -B\times e & B\times e \\
\hline
B\times e^{\mathrm{T}} & B\times e^{\mathrm{T}} & B\times e^{\mathrm{T}} & B\times e^{\mathrm{T}} & -A\times I+B\times Q & A\times I+B\times Q & A\times I-B\times Q & A\times I+B\times Q \\
-B\times e^{\mathrm{T}} & B\times e^{\mathrm{T}} & -B\times e^{\mathrm{T}} & B\times e^{\mathrm{T}} & A\times I-B\times Q & A\times I+B\times Q & -A\times I+B\times Q & A\times I+B\times Q \\
-B\times e^{\mathrm{T}} & -B\times e^{\mathrm{T}} & B\times e^{\mathrm{T}} & B\times e^{\mathrm{T}} & A\times I+B\times Q & -A\times I+B\times Q & A\times I+B\times Q & A\times I-B\times Q \\
-B\times e^{\mathrm{T}} & B\times e^{\mathrm{T}} & B\times e^{\mathrm{T}} & -B\times e^{\mathrm{T}} & A\times I+B\times Q & A\times I-B\times Q & A\times I+B\times Q & -A\times I+B\times Q
\end{array}\right]$$

(this is in fact an orthogonal design $\mathrm{OD}(4(q+1); 4, 4q)$). We replace A by J_q and B by $I+Q$ to get the required Hadamard matrix of order $4q(q+1)$ and excess $4q^2(2q+3)$. This construction is different from that of Kharaghani [6].

Thus we have shown the following theorem.

Theorem 1. *If there is a skew-Hadamard matrix of order* $m \equiv 0 \pmod 4$ *or a conference matrix of order* $m \equiv 2 \pmod 4$ *then there is a Hadamard matrix of order* $4m(m-1)$ ($\equiv 0 \pmod{16}$ *and* $\equiv 8 \pmod{16}$ *respectively*) *whose excess meets the Kounias—Farmakis bound, i.e.*

$$\sigma(4m(m-1)) = 4(m-1)^2(2m+1).$$

3. Numerical results

It can be seen by inspection that the matrices constructed above have the row-sum vector

$$(2qe_{q(3q+4)}, 2(q+2)e_{q^2}),$$

where q is odd.

The first few values obtained are given in the table, the values for $h=8$ and $h=48$ were known ([15, 10]), the values for $h \equiv 0 \pmod{16}$ arise from [6] but the remaining results are new.

q	$h = 4q(q+1)$	$\sigma(h) = 4q^2(2q+3)$	*Comment*
1	8	20	[15]
3	48	324	[10]
5	120	1300	New
7	224	3332	[6]
9	360	6804	New
11	528	12100	[6]
13	728	19604	New
15	960	29700	[6]
17	1224	42772	New
19	1520	59204	[6]
21			No conference matrix exists for order 22
23	2208	103684	[6]

References

[1] M.R. Best, The excess of a Hadamard matrix, Indag. Math. 39 (1977) 357–361.
[2] H. Enomoto and M. Miyamoto, On maximal weights of Hadamard matrices, J. Combin. Theory Ser. A 29 (1980) 94–100.
[3] N. Farmakis and S. Kounias, The excess of Hadamard matrices and optimal designs, Discrete Math. 67 (1987) 165–176.
[4] J. Hammer, R. Levingston and J. Seberry, A remark on the excess of Hadamard matrices and orthogonal designs, Ars Combin. 5 (1978) 237–254.
[5] B.A. Jenkins, C. Koukouvinos, S. Kounias, J. Seberry and R. Seberry, Some results on the excesses of Hadamard matrices, JCMCC 4 (1989) 155–185.
[6] H. Kharaghani, An infinite class of Hadamard matrices of maximal excess, Discrete Math., to appear.
[7] C. Koukouvinos and S. Kounias, Construction of some Hadamard matrices with maximum excess, to appear.
[8] S. Kounias and N. Farmakis, On the excess of Hadamard matrices, Discrete Math. 68 (1988) 59–69.
[9] R. Mathon, Symmetric conference matrices of order pq^2+1, Canad. J. Math. 30 (1978) 321–331.
[10] Y.S. Sathe and R.G. Shenoy, Construction of maximal weight Hadamard matrices of order 48 and 80, Ars Combin 19 (1985) 25–35.
[11] K.W. Schmidt and E.T.H. Wang, The weights of Hadamard matrices, J. Combin. Theory Ser. A 23 (1977) 257–263.
[12] J. Seberry, On skew Hadamard matices, Ars Combin. 6 (1978) 255–275.
[13] J. Seberry, SBIBD $(4k^2, 2k^2+k, k^2+k)$ and Hadamard matrices of order $4k^2$ with maximal excess are equivalent, Graphs Combin. 5 (1989) 373–383.
[14] J. Seberry and A.L. Whiteman, New Hadamard matrices and conference matrices obtained via Mathon's construction, Graphs Combin. 4 (1988) 355–377.
[15] W.D. Wallis, On the weights of Hadamard matrices, Ars Combin. 3 (1977) 287–292.
[16] M.Yamada, On a series of Hadamard matrices of order 2^t and the maximal excess of Hadamard matrices of order 2^{2t}, Graphs Combin. 4 (1988) 297–301.