Discrete Mathematics 88 (1991) 49–58
North-Holland

Supplementary difference sets and optimal designs

Christos Koukouvinos
Department of Mathematics, University of Thessaloniki, Thessaloniki, 54006, Greece

Stratis Kounias
Department of Statistics, University of Athens, Athens, 15784, Greece

Jennifer Seberry
Department of Computer Science, University College, University of New South Wales, Australian Defence Force Academy, A.C.T. 2600, Australia

Received 31 October 1988

Abstract

Koukouvinos, C., S. Kounias and J. Seberry, Supplementary difference sets and optimal designs, Discrete Mathematics 49–58.

D-optimal designs of order $n = 2v \equiv 2 \pmod 4$, where q is a prime power and $v = q^2 + q + 1$ are constructed using two methods, one with supplementary difference sets and the other using projective planes more directly.

An infinite family of Hadamard matrices of order $n = 4v$ with maximum excess $\sigma(n) = n\sqrt{n-3}$ where q is a prime power and $v = q^2 + q + 1$ is a prime, is also constructed.

1. Introduction

In [17–18] (Seberry) Wallis has given the following definition of *supplementary difference sets*:

If $B = \{b_1, b_2, \ldots, b_{k_1}\}$, $D = \{d_1, d_2, \ldots, d_{k_2}\}$ are two collections of k_1, k_2 residues mod v such that the congruence

$$b_i - b_j \equiv a \pmod v, \qquad d_i - d_j \equiv a \pmod v$$

has exactly λ solutions for any $a \not\equiv 0 \pmod v$ then B, D are called supplementary difference sets (abbreviated as SDS), denoted by $2\text{-}\{v; k_1, k_2; \lambda\}$.

In [5] Elliott and Butson have given the following definition of a *relative difference set*:

A set D of k elements in a group G of order vm is a difference set of G relative to a normal subgroup F of order $m \neq vm$ if the collection of differences

$r-s$, $r, s \in D$, $r \neq s$ contains only the elements of G which are not in F, and contains every such element exactly λ times. This relative difference set (abbreviated as RDS) will be denoted by $R(v, m, k, \lambda)$.

In this paper we consider the case $m=2$, i.e. $R(v, 2, k, \lambda)$. These RDS are called also *near difference sets* (see Ryser [13]). In [5] Elliott and Butson proved that if q is an odd prime power, then we can construct cyclic relative difference sets $R(v, 2, k, \lambda)$, where

$$n = 2v = 2(q^2+q+1), \qquad k = q^2, \qquad \lambda = \tfrac{1}{2}q(q-1) \tag{1}$$

Spence [16] showed that the construction of Elliott and Butson is also valid when q is a power of 2. For the construction of these $R(v, 2, k, \lambda)$ see also [11–12].

If $n \equiv 2 \pmod 4$, $v = n/2$ and R_1, R_2 are $v \times v$ commuting matrices, with elements ± 1, such that

$$R_1R_1^T + R_2R_2^T = (2v-2)I_v + qJ_v \tag{2}$$

then the $n \times n$ matrix

$$R = \begin{bmatrix} R_1 & R_2 \\ -R_2^T & R_1^T \end{bmatrix} \tag{3}$$

has the maximum determinant (Ehlich [4]) among all $n \times n \pm 1$ matrices.

Such matrices R are called D-*optimal designs* of order n and their construction is known for the following values of n: 2, 6, 10, 14, 18, 26, 30, 38, 42, 46, 50, 54, 62, 66, 82, 86 (Ehlich [4], Yang [20–24], Chadjipantelis and Kounias [2], Chadjipantelis, Kounias and Moyssiadis [3]).

If R_1, R_2 are circulant, then pre- and post-multiplying both sides of (2) by e^T and e respectively we obtain

$$(v-2k_1)^2 + (v-2k_2)^2 = 4v-2 \tag{4}$$

where e is the $v \times 1$ matrix of 1's and k_1, k_2 is the number of -1's in every row of R_1, R_2 respectively.

If R_1, R_2 satisfy (2) so do $\pm R_1, \pm R_2$, i.e. we can always take $1 \leq k_1 \leq k_2 \leq (v-1)/2$.

In [2] Chadjipantelis and Kounias proved that the existence of 2-$\{v; k_1, k_2; \lambda\}$ SDS, where k_1, k_2 satisfy (4) and $\lambda = k_1 + k_2 - (v-1)/2$ is equivalent to the existence of D-optimal designs of order $n = 2v \equiv 2 \pmod 4$. In this paper we construct D-optimal designs for $n \equiv 2 \pmod 4$ by using SDS.

Now we give some basic definitions.

An *Hadamard matrix*, called H-*matrix*, of order n is an $n \times n$ matrix H with elements $+1, -1$ satisfying

$$H^TH = HH^T = nI_n.$$

The sum of the elements of H, denoted by $\sigma(H)$, is called *excess* of H. The *maximum excess* of H, over all H-matrices of order n, is denoted by $\sigma(n)$, i.e.

$$\sigma(n) = \max \sigma(H) \text{ for all H-matrices of order } n \tag{5}$$

An equivalent notion is the *weight* $w(H)$ which is the number of 1's in H, then $\sigma(H) = 2w(H) - n^2$ and $\sigma(n) = 2w(n) - n^2$, see [9–10].

Kounias and Farmakis [10] proved that $\sigma(n) = n\sqrt{n}$ when $n = 4(2m+1)^2$ and a regular H-matrix exists thus satisfying the equality of Best's [1] inequality,

$$\sigma(n) \leqslant n\sqrt{n}.$$

Infinite families of H-matrices satisfying this bound have been found by Seberry [14] and Yamada [19].

Also, Kounias and Farmakis [10] proved that $\sigma(n) = n\sqrt{n-3}$ can be attained when $n = (2m+1)^2 + 3$ thus satisfying the equality of the Hammer–Levingston–Seberry [9] bound,

$$\sigma(n) \leqslant n\sqrt{n-3}$$

for this bound. This is discussed further in Section 3.

In this paper we also construct an infinite family of H-matrices of order $n = 4v$ with maximum excess $\sigma(n) = n\sqrt{n-3}$, where q is a prime power and $v = q^2 + q + 1$ is a prime.

2. On D-optimal designs of order $n \equiv 2 \pmod 4$

Spence [16] proved the following theorem.

Theorem 1 (Spence). *If there exists a cyclic projective plane of order q^2 then there exist two ± 1 matrices R_1, R_2, both circulant and of order $1 + q + q^2$, such that*

$$R_1R_1^{\mathrm{T}} + R_2R_2^{\mathrm{T}} = 2q(q+1)I + 2J \tag{6}$$

where I is the identity matrix of order $1 + q + q^2$ and J is the square matrix of order $1 + q + q^2$, all the entries of which are $+1$.

Now, by using the circulant matrices R_1, R_2 constructed by Spence in Theorem 1, and the matrix R in (3), we note the following theorem.

Theorem 2. *There exist* D-*optimal designs of order $n \equiv 2 \pmod 4$, where q is a prime power and*

$$n = 2v = 2(q^2 + q + 1).$$

Proof. Let $D = \{d_1, d_2, \ldots, d_k\}$ be a $R(v, 2, k, \lambda)$ as in (1) and $v = q^2 + q + 1$. The following two sets

$$\begin{aligned} D_1 &= \{(d+v)/2 \pmod v, \quad d \in D,\ d \text{ odd}\} \\ D_2 &= \{d/2 \pmod v, \quad d \in D,\ d \text{ even}\} \end{aligned} \tag{7}$$

constitute 2-$\{v, k_1, k_2; \lambda = k_1 + k_2 - (v-1)/2\}$ SDS, where

$$v = q^2 + q + 1, \qquad k_1 = \frac{q(q-1)}{2}, \qquad k_2 = \frac{q(q+1)}{2},$$
$$k_1 + k_2 = k = q^2, \qquad \lambda = k_1 + k_2 - \frac{v-1}{2} = k_1 \tag{8}$$

satisfying (4) (see Spence [16], Seberry Wallis and Whiteman [15]).

Since a $R(v, 2, k, \lambda)$ exists when q is a prime power, this completes the proof of Theorem 2. □

The matrices R_1, R_2 are the incidence circulant matrices of SDS described in (7) and are constructed by setting -1 in the positions indicated in D_1, D_2 respectively and $+1$ in the remaining positions. The following examples which are given in Table 1 illustrate the cases $q = 2, 3, 4, 5, 7$ of Theorem 2.

We give another proof of the above result which indicates possibilities for inequivalences and has less restrictions on the underlying structures.

First we note that a matrix, W, of order n with entries $0, +1, -1$, exactly k nonzero entries in each row and column and inner product of distinct rows zero is called a *weighing matrix* denoted $W = W(n, k)$. In fact

$$WW^{\mathrm{T}} = kI_n,$$

and a $W(n, n)$ is an Hadamard matrix.

Theorem 3. *Let Q and P be the incidence matrices of $(q^2+q+1, q+1, 1)$ difference sets. Further suppose QP has elements $0, 1, 2$. Then $W = QP - J$ is a weighing matrix of order q^2+q+1 and weight q^2 that is $WW^{\mathrm{T}} = q^2 I$ and W has entries $0, 1, -1$. Furthermore if $W = X - Y$, where X and Y have entries $0, 1$ then $R = J - X - Y$ satisfies $RR^{\mathrm{T}} = qI + J$, $RJ = (q+1)J$.*

Proof. Since P and Q are incidence matrices of $(q^2+q+1, q+1, 1)$ difference sets

$$PP^{\mathrm{T}} = QQ^{\mathrm{T}} = qI + J, \qquad PJ = QJ = (q+1)J$$

where P, Q, I, J are of order q^2+q+1. Now

$$\begin{aligned} WW^{\mathrm{T}} &= (QP - J)(P^{\mathrm{T}}Q^{\mathrm{T}} - J) = QPP^{\mathrm{T}}Q^{\mathrm{T}} - JP^{\mathrm{T}}Q^{\mathrm{T}} - QPJ + J^2 \\ &= Q(qI + J)Q^{\mathrm{T}} - 2(q+1)^2 + J^2 = qQQ^{\mathrm{T}} - (q+1)^2 J + J^2 \\ &= q^2 I + qJ - (q^2 + 2q + 1 - q^2 - q - 1)J = q^2 I. \end{aligned}$$

Since PQ had entries $0, 1, 2$ $PQ - J$ must have entries $0, 1, -1$.

Now $WJ = QPJ - J^2 = (q+1)^2 J - J^2 = qJ$. So $WJ = (X - Y)J = qJ$. $WW^{\mathrm{T}} = q^2 I$

Table 1
$R(v, 2, k, \lambda)$ where v, k, λ satisfy (1) and SDS 2-$\{v; k_1, k_2; \lambda\}$ where v, k_1, k_2, λ satisfy (8)

$n = 14$, $q = 2$, $v = 7$, $k = 4$, $k_1 = 1$, $k_2 = 3$; $\lambda = 1$

(i) $D = \{0, 1, 4, 6\}$
$D_1 = \{4\}$
$D_2 = \{0, 2, 3\}$

(ii) $D = \{0, 3, 5, 13\}$
$D_1 = \{3, 5, 6\}$
$D_2 = \{0\}$

$n = 26$, $q = 3$, $v = 13$, $k = 9$, $k_1 = 3$, $k_2 = 6$; $\lambda = 3$

(i) $D = \{0, 1, 6, 8, 10, 11, 12, 15, 18\}$
$D_1 = \{1, 7, 12\}$
$D_2 = \{0, 3, 4, 5, 6, 9\}$

(ii) $D = \{0, 1, 2, 8, 11, 18, 20, 22, 23\}$
$D_1 = \{5, 7, 12\}$
$D_2 = \{0, 1, 4, 9, 10, 11\}$

(iii) $D = \{4, 5, 7, 10, 11, 12, 15, 19, 21\}$
$D_1 = \{1, 3, 4, 9, 10, 12\}$
$D_2 = \{2, 5, 6\}$

(iv) $D = \{5, 8, 15, 17, 19, 20, 23, 24, 25\}$
$D_1 = \{1, 2, 3, 5, 6, 9\}$
$D_2 = \{4, 10, 12\}$

(v) $D = \{2, 4, 6, 7, 10, 11, 12, 18, 21\}$
$D_1 = \{4, 10, 12\}$
$D_2 = \{1, 2, 3, 5, 6, 9\}$

$n = 42$, $q = 4$, $v = 21$, $k = 16$, $k_1 = 6$, $k_2 = 10$; $\lambda = 6$

(i) $D = \{0, 1, 10, 11, 18, 20, 23, 25, 26, 29, 30, 34, 36, 37, 38, 40\}$
$D_1 = \{1, 2, 4, 8, 11, 16\}$
$D_2 = \{0, 5, 9, 10, 13, 15, 17, 18, 19, 20\}$

(ii) $D = \{0, 2, 4, 5, 6, 8, 12, 13, 16, 17, 19, 22, 24, 31, 32, 41\}$
$D_1 = \{5, 10, 13, 17, 19, 20\}$
$D_2 = \{0, 1, 2, 3, 4, 6, 8, 11, 12, 16\}$

$n = 62$, $q = 5$, $v = 31$, $k = 25$, $k_1 = 10$, $k_2 = 15$; $\lambda = 10$

(i) $D = \{0, 1, 2, 3, 5, 6, 7, 9, 10, 13, 15, 17, 23, 24, 25, 26, 30, 35, 39, 42, 45, 50, 51, 53, 58\}$
$D_1 = \{2, 4, 7, 10, 11, 16, 17, 18, 19, 20, 22, 23, 24, 27, 28\}$
$D_2 = \{0, 1, 3, 5, 12, 13, 15, 21, 25, 29\}$

(ii) $D = \{0, 1, 2, 5, 7, 9, 10, 21, 22, 25, 29, 34, 35, 37, 39, 43, 44, 45, 46, 48, 50, 51, 54, 57, 61\}$
$D_1 = \{2, 3, 4, 6, 7, 10, 13, 15, 16, 18, 19, 20, 26, 28, 30\}$
$D_2 = \{0, 1, 5, 11, 17, 22, 23, 24, 25, 27\}$

$n = 114$, $q = 7$, $v = 57$, $k = 49$, $k_1 = 21$, $k_2 = 28$; $\lambda = 21$

(i) $D = \{0, 8, 10, 12, 15, 18, 20, 22, 23, 25, 26, 32, 34, 39, 40, 41, 43, 45, 46, 47, 50, 51, 52, 55, 56, 59, 60, 61, 62, 68, 70, 71, 73, 74, 78, 81, 84, 85, 86, 87, 88, 90, 92, 93, 94, 101, 105, 110, 111\}$
$D_1 = \{1, 2, 7, 8, 12, 14, 15, 18, 22, 24, 27, 36, 40, 41, 48, 49, 50, 51, 52, 54, 56\}$
$D_2 = \{0, 4, 5, 6, 9, 10, 11, 13, 16, 17, 20, 23, 25, 26, 28, 30, 31, 34, 35, 37, 39, 42, 43, 44, 45, 46, 47, 55\}$

(ii) $D = \{0, 2, 3, 4, 8, 10, 11, 14, 21, 22, 23, 24, 27, 28, 31, 32, 33, 34, 36, 37, 39, 40, 43, 45, 47, 48, 50, 52, 54, 55, 56, 62, 69, 70, 72, 73, 74, 75, 77, 82, 83, 86, 87, 92, 98, 101, 103, 108, 110\}$
$D_1 = \{6, 8, 9, 10, 13, 15, 22, 23, 30, 34, 39, 40, 42, 44, 45, 47, 48, 50, 51, 52, 56\}$
$D_2 = \{0, 1, 2, 4, 5, 7, 11, 12, 14, 16, 17, 18, 20, 24, 25, 26, 27, 28, 31, 35, 36, 37, 41, 43, 46, 49, 54, 55\}$

says W has q^2 entries 1 or -1 in each row, say x ones and y minus ones. Then

$$x - y = q \qquad x + y = q^2$$

and thus

$$x = \tfrac{1}{2}q(q+1), \qquad y = \tfrac{1}{2}q(q-1).$$

Now any row of W has $x = \frac{1}{2}(q^2+q)$ ones, $y = \frac{1}{2}(q^2-q)$ minus ones and $q+1$ zeros.

Write any two rows of W as

$$\begin{array}{l} 1 \cdots\cdots\cdots\cdots\cdots\cdots 1 \quad - \cdots\cdots\cdots\cdots\cdots\cdots - \quad 0 \cdots\cdots\cdots\cdots\cdots\cdots 0 \\ \underbrace{1\cdots 1}_{a} \ \underbrace{-\cdots -}_{c} \ \underbrace{0\cdots 0}_{e} \ \underbrace{1\cdots 1}_{b} \ \underbrace{-\cdots -}_{d} \ \underbrace{0\cdots 0}_{f} \ \underbrace{1\cdots 1}_{x-a-b} \ \underbrace{-\cdots -}_{y-c-d} \ \underbrace{0\cdots 0}_{q+1-e-f} \end{array}$$

where there are, for example a columns $\binom{1}{1}$ and f columns $\binom{-1}{0}$.

Now the number of columns $\binom{0}{0}$ is $q+1-e-f$. Furthermore the inner product of each pair of rows is zero so $a+b-c-d=0$. Also

$$a + c + e = x \quad \text{(number of ones in first row)}$$
$$b + d + f = y \quad \text{(number of minus ones in first row)}.$$

Hence

$$\begin{aligned} q+1-e-f &= q+1+a+c-x+b+d-y = -q^2+q+1+a+c+b+d \\ &= -q^2+q+1+2c+2d \quad (\text{using } a+b-c-d=0) \\ &\leq -q^2+q+1+q^2-q \quad \text{(number of minus ones in second row)} \\ &\leq 1. \end{aligned}$$

Now $1 \geq q+1-e-f \geq 0$. Suppose $q+1-e-f=0$ then using

$$\begin{aligned} a+b+c+d+e+f &= q^2 \\ a+b-c-d \qquad &= 0 \\ e+f &= q+1 \end{aligned}$$

We have

$$2a + 2b = q^2 + q + 1.$$

But q^2+q+1 is always odd. So we have a contradiction and $q+1-e-f=1$. In other words each row of W has $q+1$ zeros and in each pair of rows of W exactly one zero is underneath a zero. Thus if $R = J - X - Y$ is the matrix with ones where W had zeros R is the incidence matrix of a $(q^2+1+1, q+1, 1)$ configuration. So

$$RR^{\mathrm{T}} = qI + J \quad \text{and} \quad RJ = (q+1)J.$$

Furthermore if P and q were defined on a cyclic (abelian) group, R is defined on the same group.

Theorem 4. *There exist two matrices A and B of order q^2+q+1 which satisfy*

$$AA^{\mathrm{T}}+BB^{\mathrm{T}}=2(q^2+q)I+2J.$$

Proof. Let $A=W+R$ and $B=W-R$ be defined as above. □

Corollary 5. *There is a* D*-optimal design of order* $2(q^2+q+1)$ *whenever there is a* $(q^2+q+1,\ q+1,\ 1)$ *difference set.*

Proof. Use

$$\begin{bmatrix} A & B \\ -B^{\mathrm{T}} & A^{\mathrm{T}} \end{bmatrix}$$

as before. □

Remark 1. This construction does not require the difference set to be defined on a cyclic group. Glynn [7], Geramita and Seberry [6, p. 152] have shown the conditions of the theorem can be met, for example if $P=Q$ in theorem.

Remark 2. We note that the sets D_1 and D_2 of $2-\{v;k_1,k_2;\lambda\}$ SDS described in (7) are disjoint.

For if

$$\frac{d_i+v}{2}\equiv\frac{d_j}{2}\ (\mathrm{mod}\ v)$$

then $d_i-d_j\equiv v\ (\mathrm{mod}\ 2v)$, $(d_i,\ d_j\in D)$ in violation of the definition of a RDS. (see Seberry Wallis and Whiteman [15]).

D-optimal designs have been constructed for $n=14$, $n=26$ by Ehlich [4] and Yang [22] and for $n=42$, $n=62$ by Yang [20, 23] and Chadjipantelis and Kounias [2]. All the other orders of D-optimal designs which are constructed by the above method are new.

3. The maximum excess of Hadamard matrices of order $n=4v$

First we show that the Hammer–Levingston–Seberry [9, p. 246] bound for $n=(2m+1)^2+3$ is the same as that found by Kounias and Farmakis [10, section 4].

Hammer, Levingston and Seberry [9, p. 217] show that for H-matrices of order n, writing x for the greatest even integer $<\sqrt{n}$, $t=x$ if $|n-x^2|<|(x+2)^2-n|$ and $t=x-2$ otherwise, i the integer part of $n((t+4)^2-n)/8(t+2)$, the excess of the H-matrices is bounded by

$$\sigma(n)=n(t+4)-4i.$$

Write $n = (2m+1)^2 + 3 = 4(m^2 + m + 1)$. Now x, even, is the greatest even integer $< \sqrt{n}$.

Let $x = 2a$, then $2a < \sqrt{n}$ and

$$4m^2 \leq 4a^2 < 4(m^2 + m + 1) < 4(m+1)^2$$

Hence $m \leq a < m + 1$.

Thus we can write

$$x = 2a = 2m, \qquad t = x - 2 = 2m - 2 \quad \text{and} \quad i = m^2 + m + 1.$$

Hence

$$\sigma(n) \leq (2m+2) - 4i = n(2m+2) - n = n(2m+1) = n\sqrt{n-3}$$

This was the result given in Kounias and Farmakis [10]. We summarize this as the following lemma.

Lemma 6. *The Hammer–Levingston–Seberry bound is equivalent to* $\sigma(n) \leq n(2m+1) = n\sqrt{n-3}$ *when* $n = (2m+1)^2 + 3$.

Kounias and Farmakis [10] proved that $\sigma(n) = n\sqrt{n-3}$ can be attained when $n = (2m+1)^2 + 3$ thus satisfying the equality of the above bound.

Spence [16] proved the following theorem.

Theorem 7 (Spence). *If there exists a cyclic projective plane of order* q^2 *and two supplementary difference sets in a cyclic group of order* $1 + q + q^2$, *then there exists a Hadamard matrix of the Goethals–Seidel type of order* $4(1 + q + q^2)$.

Now, from this theorem of Spence we note the following theorem.

Theorem 8. *There exist* H-*matrices of order* $n = (2q+1)^2 + 3$, *with maximum excess* $\sigma(n) = n\sqrt{n-3}$, *where* q *is a prime power and* $v = q^2 + q + 1$ *is a prime.*

Proof. It is easy to see (Spence [16], Seberry Wallis and Whiteman [15]) that if $v = q^2 + q + 1$ is a prime, then we can construct two sets D_3 and D_4 as

$$2 - \left\{ v; k_3, k_4; k_3 + k_4 - \frac{v+1}{2} \right\} \tag{9}$$

SDS, where D_3 is the set of quadratic residues of v, and D_4 is the set of quadratic nonresidues of v, $k_3 = k_4 = q(q+1)/2$, $\lambda = k_3 + k_4 - (v+1)/2 = q(q+1)/2 - 1$.

By using (7) and (9) SDS, we can construct a

$$4 - \left\{ v; k_1, k_2, k_3, k_4; \lambda = \sum_{i=1}^{4} k_i - v \right\}$$

which may be used to construct H-matrices (H_{4v}) of the Goethals–Seidel type.

Now, it is obvious that $n = 4v = 4(q^2 + q + 1) = (2q + 1)^2 + 3$, and from Lemma 3 and the result of Kounias and Farmakis [10], we note that these H-matrices have maximum excess $\sigma(n) = n\sqrt{n-3}$. □

If we construct the R_3, R_4 incidence circulant matrices of (9) SDS, we have

$$R_3R_3^T + R_4R_4^T = 2(q^2 + q + 2)I_v - 2J_v. \tag{10}$$

Hence from (6) and (10) we obtain:

$$R_1R_1^T + R_2R_2^T + R_3R_3^T + R_4R_4^T = 4(q^2 + q + 1)I_v = 4vI_v. \tag{11}$$

The following matrix G, whose construction is due to Goethals and Seidel [8], is an H-matrix of order $4(q^2 + q + 1)$:

$$G = \begin{bmatrix} R_1 & R_2W & R_3W & R_4W \\ -R_2W & R_1 & -R_4^TW & R_3^TW \\ -R_3W & R_4^TW & R_1 & -R_2^TW \\ -R_4W & R_3^TW & R_2^TW & R_1 \end{bmatrix} \tag{12}$$

where $W = [w_{ij}]$ is the permutation matrix of order $v = q^2 + q + 1$ defined by

$$w_{ij} = \begin{cases} 1, & \text{if } i + j \equiv 1 \pmod v, \\ 0, & \text{otherwise.} \end{cases}$$

The circulant $(1, -1)$ matrices R_1, R_2, R_3, R_4 of order v, have row sums $2q + 1, 1, 1, 1$ respectively, then G gives the row-sum vector $(2qe_{3n/4}^T, (2q + 4)e_{n/4}^T)$ where re_s^T denotes the $1 \times s$ vector $(r, r, \ldots, r)$.

Example. From Theorem 8 we obtain the following orders of H-matrices with maximum excess:

$n = 28$ $(q = 2, v = 7)$,
$n = 52$ $(q = 3, v = 13)$,
$n = 124$ $(q = 5, v = 31)$,
$n = 292$ $(q = 8, v = 73)$,
$n = 1228$ $(q = 17, v = 307)$,
$n = 3028$ $(q = 27, v = 757)$,
$n = 6892$ $(q = 41, v = 1723)$,
$n = 14164$ $(q = 59, v = 3541)$, etc.

H-matrices with maximum excess have been constructed for $n = 28$, $n = 52$, $n = 124$ from the results of Hammer, Levingston and Seberry [9] using Williamson-type matrices alone, or from the results of Kounias and Farmakis [10]. All the other orders of H-matrices with maximum excess are new.

References

[1] M.R. Best, The excess of a Hadamard matrix, Indag. Math. 39 (1977) 357–361.
[2] T. Chadjipantelis and S. Kounias, Supplementary difference sets and D-optimal designs for $n \equiv 2 \pmod 4$, Discrete Math. 57 (1985) 211–216.
[3] T. Chadjipantelis, S. Kounias and C. Moyssiadis, Construction of D-optimal designs for $n \equiv 2 \pmod 4$ using block circulant matrices, J. Combin. Theory Ser. A 40 (1985) 125–135.
[4] H. Ehlich, Determinantenabschätzungen für binäre matrizen, Math. Z. 83 (1964) 123–132.
[5] J.E.H. Elliott and A.T. Butson, Relative difference sets, Illinois J. Math. 10 (1966) 517–531.
[6] A.V. Geramita and J. Seberry, Orthogonal Designs: Quadratic Forms and Hadamard Matrices (Marcel Dekker, New York, 1979).
[7] D.G. Glynn, Finite projective planes and related combinatorial systems, Ph.D. Thesis, University of Adelaide, 1978.
[8] J.M. Goethals and J.J. Seidel, A skew-Hadamard matrix of order 36, J. Austral. Math. Soc. 11 (1970) 343–344.
[9] J. Hammer, R. Levingston and J. Seberry, A remark on the excess of Hadamard matrices and orthogonal designs, Ars Combin. 5 (1978) 237–254.
[10] S. Kounias and N. Farmakis, On the excess of Hadamard matrices, Discrete Math. 68 (1988) 59–69.
[11] C.W.H. Lam, On relative difference sets, Proc. Seventh Manitoba Conf. on Numerical Math. and Computing, Winnipeg 1977, Congr. Numer. 20 (1977) 445–474.
[12] P.A. Leonard, Cyclic relative difference sets, Amer. Math. Monthly 93 (1986) 106–111.
[13] H.J. Ryser, Variants of the cyclic difference sets, Proc. Amer. Math. Soc. 41 (1973) 45–50.
[14] J. Seberry, SBIBD $(4k^2, 2k^2+k, k^2+k)$ and Hadamard matrices of order $4k^2$ with maximal excess are equivalent, Graphs Combin. 5 (1989) 373–383.
[15] J. Seberry Wallis and A.L. Whiteman, Some results on weighing matrices, Bull. Austral. Math. Soc. 12 (1975) 433–447.
[16] E. Spence, Skew-Hadamard matrices of the Goethals–Seidel type, Canad. J. Math. 27 (1975) 555–560.
[17] J. (Seberry) Wallis, On supplementary difference sets, Aequationes Math. 8 (1972) 242–257.
[18] J. (Seberry) Wallis, A note on supplementary difference sets, Aequationes Math. 10 (1974) 46–49.
[19] M. Yamada, On a series of Hadamard matrices of order 2^t and the maximal excess of Hadamard matrices of order 2^{2t}, Graphs and Combinatorics 4 (1988) 297–301.
[20] C.H. Yang, Some designs of maximal (+1, −1)-determinant of order $n \equiv 2 \pmod 4$, Math. Comp. 20 (1966) 147–148.
[21] C.H. Yang, A construction for maximal (+1, −1)-matrix of order 54, Bull. Amer. Math. Soc. 72 (1966) 293.
[22] C.H. Yang, On designs of maximal (+1, −1)-matrices of order $n \equiv 2 \pmod 4$, Math. Comp. 22 (1968) 175–180.
[23] C.H. Yang, On designs of maximal (+1, −1)-matrices of order $n \equiv 2 \pmod 4$ II, Math. Comp. 23 (1969) 201–205.
[24] C.H. Yang, Maximal binary matrices and sum of two squares, Math. Comp. 30 (1976) 148–153.