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Graphs and Combinatorics 5, 373–383 (1989)

Graphs and Combinatorics

$SBIBD(4k^2, 2k^2 + k, k^2 + k)$ and Hadamard Matrices of Order $4k^2$ with Maximal Excess Are Equivalent

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Abstract. We show that an $SBIBD(4k^2, 2k^2 + k, k^2 + k)$ is equivalent to a regular Hadamard matrix of order $4k^2$ which is equivalent to an Hadamard matrix of order $4k^2$ with maximal excess.

We find many new $SBIBD(4k^2, 2k^2 + k, k^2 + k)$ including those for even k when there is an Hadamard matrix of order $2k$ (in particular all $2k \leq 210$) and $k \in \{1, 3, 5, \ldots, 29, 33, \ldots, 41, 45, 51, 53, 61, \ldots, 69, 75, 81, 83, 89, 95, 99, 625, 3^{2m}, 25 \cdot 3^{2m}, m \geq 0\}$.

1. Introduction

An *Hadamard matrix* of order n is an $n \times n$ matrix H with elements $+1$, -1, satisfying $H^T H = HH^T = nI_n$. The sum of the elements of H, denoted by $\sigma(H)$, is called *excess* of H. The maximum excess of H, over all Hadamard matrices of order n, is denoted by $\sigma(n)$, i.e.

$$\sigma(n) = \max \sigma(H) \qquad \text{for all Hadamard matrices of order } n \tag{1}$$

An equivalent notion is the *weight* $w(H)$ which is the number of 1's in H, then $\sigma(H) = 2w(H) - n^2$ and $\sigma(n) = 2w(n) - n^2$, see [5, 10, 16, 25].

Kounias and Farmakis [13] proved that $\sigma(n) = n\sqrt{n}$ when $n = 4(2m+1)^2$ thus satisfying the equality of Best's inequality:

$$\sigma(n) \leq n\sqrt{n}$$

A *regular Hadamard matrix* has constant row and column sum. These are discussed by Seberry Wallis [24, pp. 341–346].

A *symmetric balanced incomplete block design* or $SBIBD(v, k, \lambda)$ can be defined as a square matrix of order v with entries 0 or 1, with k 1's in row and column and the inner product of an pair of distinct rows is λ. For more detals see Street and Street [17].

An *orthogonal design* $D = x_1A_1 + x_2A_2 + \cdots + x_uA_u$ of order n and type $(s_1, \ldots, s_u)$, written $OD(n; s_1, s_2, \ldots, s_u)$, on the commuting variables $x_1, \ldots, x_u$ is a square matrix with entries 0, $\pm x_1, \ldots, \pm x_u$ where x_i or $-x_i$ occurs s_i times in each row and column and distinct rows are formally orthogonal. That is

$$DD^T = \sum_{j=1}^{u} s_j x_j^2.$$

Each A_j is a $(0, 1, -1)$-matrix satisfying $A_j A_j^T = s_j I_n$ and is called a *weighing matrix* of weight s_j. A weighing matrix of order n and weight n is called an *Hadamard matrix*.

We define the *excess of the orthogonal design D* as

$$\sigma(D) = \sigma(A_1) + \cdots + \sigma(A_u),$$

where $\sigma(A_i)$ is the sum of the entries of A_i, this is equivalent to putting all the variables equal to $+1$.

Suitable matrices are matrices with elements $+1$ and -1 which can be used to replace the variables of ODs to form Hadamard matrices. Of special interest are *Williamson type matrices*, which are 4 matrices, W_1, W_2, W_3, W_4 with elements $+1$ or -1 of order w which satisfy

$$\sum_{i=1}^{4} W_i W_i^T = 4wI_w$$

$$W_i W_j^T = W_j W_i^T$$

Our construction follows that of Hammer, Levingston and Seberry [8] who formed orthogonal designs $OD(4t; t, t, t, t)$ and then replaced the variables by suitable matrices.

This practice for constructing Hadamard matrices derived from extensions due to Baumert-Hall [1] who found the first $OD(12; 3, 3, 3, 3)$ and Cooper and (Seberry) Wallis [4] who first introduced T-matrices to form $OD(4t; t, t, t, t)$. The variables of these ODs are then replaced by Williamson type matrices of order w to form Hadamard matrices of order $4wt$. These are discussed extensively by Geramita and Seberry [7, pp. 120–125]. Cohen, Rubie, Koukouvinos, Seberry and Yamada [3] survey the most recent results. This method was also used by Koukouvinos and Kounias [12] to find Hadamard matrices with maximal excess.

2. The Equivalence Theorem

Theorem 1. *There is an Hadamard matrix of order $n = 4s^2$ and maximal excess $n\sqrt{n} = 8s^3$ if and only if there is an $SBIBD(4s^2, 2s^2 + s, s^2 + s)$.*

Proof. If there is an $SBIBD$, B, with parameters $(4s^2, 2s^2 + s, s^2 + s)$ then $A = 2B - J$ has elements $+1$ and -1. A has $2s^2 + s$ elements $+1$ in each row (and column) and $2s^2 - s$ elements -1 in each row (and column). Thus the row (column) sum of each row (column) of A is $2s^2 + s - (2s^2 - s) = 2s$. Thus the excess of $A = 4s^2 \times 2s = 8s^3$ = number of rows (columns) of A times the row (column) sum of A.

Further

$$\begin{aligned} AA^T &= (2B - J)(2B - J)^T \\ &= 4BB^T - 2JB^T - 2BJ + J^2 \\ &= (s^2 I + (s^2 + s)J) - 4(2s^2 + s)J + 4s^2 J \\ &= 4s^2 I \end{aligned}$$

Thus A is an Hadamard matrix.

Conversely, let A be an Hadamard matrix of order $n = 4s^2$ and maximal excess $8s^3$.

Let the column sum of the ith column of A be x_i. Then all $x_i \geq 0$, otherwise that entire column could be negated giving an Hadamard matrix with greater excess

$$\sum_{i=1}^{n} x_i = 8s^3, \qquad x_i \geq 0 \text{ all } i \tag{2}$$

since the sum of the column sums is the excess. Now let e be the $1 \times n$ matrix of ones. Since A is an Hadamard matrix we have

$$AA^T = 4s^2 I$$

$$eAA^Te^T = 4s^2ee^T = 16s^4 = (x_1x_2\ldots x_n)(x_1x_2\ldots x_n)^T.$$

So

$$\sum_{i=1}^{n} x_i^2 = 16s^4. \tag{3}$$

The only solution to (2) and (3) is

$$x_1 = x_2 = \cdots = x_n = 2s.$$

[Suppose $x_i = 2s + t_i$, then

$$\sum_{i=1}^{n} x_i = 8s^3 + \sum_{i=1}^{n} t_i = 8s^3. \qquad \text{So } \textstyle\sum_{i=1}^{n} t_i = 0.$$

Also

$$\sum_{i=1}^{n} x_i^2 = 16s^4 = 16s^4 + 4s\sum_{i=1}^{n} t_i + \sum_{i=1}^{n} t_i^2.$$

Thus

$$\sum_{i=1}^{n} t_i^2 = 0,$$

so $t_i = 0$ for all i.]

But this means each column of A has $2s^2 + s$ elements $+1$ and $2s^2 - s$ elements -1. Now, since A is an Hadamard matrix, the columns of A are orthogonal, so if two columns are written

$$\begin{matrix} 1\ldots1 & 1\ldots1 & -1\ldots-1 & -1\ldots-1 \\ \underbrace{1\ldots1}_{x} & \underbrace{-1\ldots-1}_{2s^2+s-x} & \underbrace{1\ldots1}_{2s^2+s-x} & \underbrace{-1\ldots-1}_{4s^2-2(2s^2+s-x)-x} \end{matrix}$$

where $x, 2s^2+s-x, 2s^2+s-x, -2s+x$ are the number of columns of each type. Now since the rows are orthogonal

$$x - (2s^2+s-x) - (2s^2+s-x) - 2s + x = 0$$

$$4s^2 + 4s = 4x$$

$$x = s^2 + s$$

Thus A has $2s^2 + s$ elements $+1$ in each column and $s^2 + s$ elements $+1$ in any column overlapping with elements $+1$ in every other column. A similar argument can be used for the rows. Thus $B = \frac{1}{2}(A + J)$ is an $SBIBD(4s^2, 2s^2 + s, s^2 + s)$. $\square$

In Seberry Wallis [24, p. 343] it is pointed out that Goethals and Seidel [9] and Shrikhande and Singh [19] have established:

Theorem 2. *If there exists a* $BIBD(2k^2 - k, 4k^2 - 1, 2k + 1, k, 1)$ *then there exists a symmetric Hadamard matrix with constant diagonal of order* $4k^2$.

Moreover Shrikhande [18], [21] has studied these designs and showed they exist for $k = 2^t$, $t \geq 1$. They are also known for $k = 3, 5, 6, 7$ [7].

In Seberry Wallis [24, pp. 344–346] it is established that symmetric Hadamard matrices with constant diagonal thus exist for 2^{2t}, $t \geq 1$, 36, 100, 144, 196 (after Theoem 5.14 of [24]) and using results of (Seberry) Wallis-Whiteman [23] and Szekeres [20] they are shown to exist with the extra property of regularity for $4 \cdot 5^2, 4 \cdot 13^2, 4 \cdot 29^2, 4 \cdot 51^2$, and $4\left(2\left(\frac{p-3}{4}\right) + 1\right)^2$, for $p \equiv 3 \pmod 4$ a prime power (after Theorem 5.15 of [24]).

Remark 1. Now a Theorem of Goethals and Seidel [9] (see Geramita and Seberry [8]) tells us that if there is an Hadamard matrix with constant diagonal of order $4k$ there is a regular symmetric Hadamard matrix with constant diagonal of order $4(2k)^2$. So an Hadamard matrix of order $4t$ gives a regular symmetric Hadamard matrix of order $4k^2$, $k = 2t$. In particular known results give these matrices for $2t \leq 210$.

Remark 2. Now combining these results, and noting that regular symmetric Hadamard matrices with constant diagonal of orders $4s^2$ and $4t^2$ give a regular symmetric Hadamard matrix with constant diagonal with order $4(2st)^2$, we have them for orders $4k^2$ for

(i) all even $k \leq 210$, all even $2t$ when there is an Hadamard matrix of order $4t$;
(ii) $k \in \{1, 3, 5, 9, 11, 13, 15, 21, 23, 25, 29, 33, 35, 39, 41, 45, 51, 53, 63, 65, 69, 75, 81, 83, 89, 95, 99, 105, 111, 113, 119, 125, 131, 135, 141, 153, 155, 165, 173, 179, 183, 189, 191, 209\}$.

We now wish to establish the existence of some of the remaining undecided cases.

We first note a theorem given by Seberry Wallis: [24, p. 280]

Theorem 3. *A regular Hadamard matrix* H *of order* $4k^2$ *with row sum* $\pm 2k$ *exists if and only if there exists an* $SBIBD(4k^2, 2k^2 \pm k, k^2 \pm k)$.

We observe that the stipulation that the row sum is $\pm 2k$ is unnecessary for if the matrix is regular it must have constant row sum, x, say.

Thus $eH^T = (x, \ldots, x)$ where e is the $1 \times 4k^2$ matrix of ones. Now $H^TH = 4k^2I$, so

$$16k^4 = 4k^2ee^T = eH^THe^T = (x, \ldots, x)(x, \ldots, x)^T = 4k^2x^2.$$

Thus $x = \pm 2k$. The matrix with constant row sum $-2k$ is the negative of the matrix with constant row sum $2k$. □

We can now combine the results obtained so far as

Theorem 4 (Equivalence Theorem). *The following are equivalent*:

(i) *there exists an Hadamard matrix of order* $4k^2$ *with maximal excess* $(8k^3)$;
(ii) *there exists a regular Hadamard matrix of order* $4k^2$;
(iii) *there is an* $SBIBD(4k^2, 2k^2 + k, k^2 + k)$ (*and its complement the* $SBIBD(4k^2, 2k^2 - k, k^2 + k)$).

This result was also observed by Best [1].

We now wish to consider the undecided cases. First we look at a known family of Williamson matrices.

3. Matrices of Order $4q^2$, $2q^2 - 1$ a Prime Power

We show that if $p \equiv 1 \pmod 4$ is a prime power, $p = 2q^2 - 1$, then the Hadamard matrix found as in Hammer, Levingston and Seberry [10, p. 244] with excess $2(p + 1)(x + y)$, $p = x^2 + y^2$ has

$$\sigma(2(p + 1)) > 2(p + 1)(x + y).$$

Since $p = x^2 + y^2 = 2q^2 - 1$ the excess is $4q^2(x + y)$ and the order is $4q^2$. But an Hadamard matrix of order $4q^2$ has maximal excess $8q^3$. So we consider $x + y$. Now $x = (2q^2 - 1 - y^2)^{1/2}$ so $E = x + y$ is maximal for $\frac{dE}{dx} = 0$ or $x = y$. But that means

$$x = y = (q^2 - 0.5)^{1/2}$$

As x is an integer this means $x = y < q$ so $x + y < 2q$ and the excess $2(p + 1)(x + y) < 8q^3$. So this method cannot give maximal excess for matrix orders $4q^2$.

In some cases the construction gives quite high excess. The results are tabulated in Table 1.

Table 1

q	$2q^2 - 1 = x^2 + y^2$ (prime)	Hadamard order $= 4q^2$	Maximal Excess $= 4q^2.2q$	Found Excess $= 4q^2(x + y)$
43	$3697 = 36^2 + 49^2$	4.43^2	$4.43^2.86$	$4.43^2.85$
49	$4801 = 24^2 + 65^2$	4.49^2	$4.49^2.98$	$4.49^2.89$
59	$6961 = 20^2 + 81^2$	4.59^2	$4.59^2.118$	$4.59^2.101$
69	$9521 = 40^2 + 89^2$	4.69^2	$4.69^2.138$	$4.69^2.129$
73	$10657 = 64^2 + 81^2$	4.73^2	$4.73^2.146$	$4.73^2.145$
85	$14449 = 7^2 + 120^2$	4.85^2	$4.85^2.170$	$4.85^2.127$
87	$15137 = 41^2 + 116^2$	4.87^2	$4.87^2.174$	$4.87^2.157$
91	$16561 = 81^2 + 100^2$	4.91^2	$4.91^2.182$	$4.91^2.181$

4. The Hammer – Levingston – Seberry Construction Revisited

Hammer, Levingston and Seberry [10] suggested (following Cooper and (Seberry) Wallis [4]) using 4 circulant (or type 1) matrices of order t, X_1, X_2, X_3, X_4, with entries 0, +1, −1 row sums x_1, x_2, x_3, x_4 respectively satisfying

$$\begin{cases} \text{(i)} & \sum_{i=1}^{4} X_i X_i^T = tI_t, \\ \text{(ii)} & X_i J = x_i J, \\ \text{(iii)} & X_i * X_j = 0,\ i \neq j. \\ \text{(iv)} & \sum_{i=1}^{4} X_i \text{ is a } (1,-1)\text{-matrix}, \\ \text{(v)} & x_1^2 + x_2^2 + x_3^2 + x_4^2 = t. \end{cases}$$

These matrices are called T-matrices.

This means $\sigma(X_i)$, the excess of X_i is tx_i, $i = 1, 2, 3, 4$, because each X_i is circulant (or type 1 = block circulant).

Let y_1, y_2, y_3, y_4 be commuting variables and

$$U = \begin{bmatrix} -y_1 & y_2 & y_3 & y_4 \\ y_2 & y_1 & y_4 & -y_3 \\ y_3 & -y_4 & y_1 & y_2 \\ y_4 & y_3 & -y_2 & y_1 \end{bmatrix} = (u_{ij}), \quad V = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & -y_1 & -y_4 & y_3 \\ y_3 & y_4 & -y_1 & -y_2 \\ y_4 & -y_3 & y_2 & -y_1 \end{bmatrix} = (v_{ij}).$$

Now we can form A_i by either choosing

$$A_i = \sum_{k=1}^{4} u_{ik} X_k, \qquad i = 1, 2, 3, 4,$$

or

$$A_i = \sum_{k=1}^{4} v_{ik} X_k, \qquad i = 1, 2, 3, 4.$$

A_i will be circulant (or type 1) according as X_i is circulant (or type 1).

Now the elements of A_i are variables, so the excess is a linear expression in x_i (constants) and y_i (variables). Depending on which coefficients are used (the u_{ik} or v_{ik}) the excesses of the A_i will be:

Case 1.

$$\begin{aligned} \sigma(A_1) &= (-y_1x_1 + y_2x_2 + y_3x_3 + y_4x_4)t \\ \sigma(A_2) &= (y_2x_1 + y_1x_2 + y_4x_3 - y_3x_4)t \\ \sigma(A_3) &= (y_3x_1 - y_4x_2 + y_1x_3 + y_2x_4)t \\ \sigma(A_4) &= (y_4x_1 + y_3x_2 - y_2x_3 + y_1x_4)t \end{aligned}$$

Case 2.

$$\begin{aligned} \sigma(A_1) &= (y_1x_1 + y_2x_2 + y_3x_3 + y_4x_4)t \\ \sigma(A_2) &= (y_2x_1 - y_1x_2 - y_4x_3 + y_3x_4)t \\ \sigma(A_3) &= (y_3x_1 + y_4x_2 - y_1x_3 - y_2x_4)t \\ \sigma(A_4) &= (y_4x_1 - y_3x_2 + y_2x_3 - y_1x_4)t \end{aligned}$$

Write

$$G = \begin{bmatrix} -A_1 & A_2R & A_3R & A_4R \\ A_2R & A_1 & A_4^TR & -A_3^TR \\ A_3R & -A_4^TR & A_1 & A_2^TR \\ A_4R & A_3^TR & -A_2^TR & A_1 \end{bmatrix},$$

$$H = \begin{bmatrix} A_1 & A_2R & A_3R & A_4R \\ -A_2R & A_1 & A_4^TR & -A_3^TR \\ -A_3R & -A_4^TR & A_1 & A_2^TR \\ -A_4R & A_3^TR & -A_2^TR & A_1 \end{bmatrix}$$

where R is the back diagonal matrix and A_1, A_2, A_3, A_4 are circulant matrices (or type 1).

Now if the matrices from Case 1 are used in G we get

$$\sigma(OD) = 2\sigma(A_1) + 2\sigma(A_2) + 2\sigma(A_3) + 2\sigma(A_4)$$

$$= 2t(y_1 y_2 y_3 y_4)\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Call this case $1G$. If the matrices from Case 1 are used in H we get

$$\sigma(OD) = 4\sigma(A_1) = 4t(y_1 y_2 y_3 y_4)\begin{bmatrix} -x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Call this case $1H$.

If the matrices of Case 2 are used in G we get

$$\sigma(OD) = 2t(y_1 y_2 y_3 y_4)\begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Call this Case $2G$. While if the matrices from Case 2 are used in H we get

$$\sigma(OD) = 4t(y_1 y_2 y_3 y_4)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Call this Case $2H$.

Case $1H$ is never used as for positive x_i and y_i (which can always be assumed as a row or matrix with negative row sum or excess can be just negated to get a row or matrix with positive row sum or excess).

If each of the variables y_i is replaced by 1 we get the excesses

$$4t(x_1+x_2+x_3+x_4),\quad 4t(-x_1+x_2+x_3+x_4),\quad 8tx_1,\quad 4t(x_1+x_2+x_3+x_4),$$

respectively. So the excess of the corresponding Hadamard matrix of order $4t$ is

$$\sigma(4t) \geq 4t \max(2x_1, x_1+x_2+x_3+x_4).$$

Where x_i is the row sum of the T-matrices.

Example 1. Suppose that X_1, X_2, X_3, X_4 are the circulant matrices of order 9 with first rows

$$(1\,1\,0\,1\,0\,0\,0\,0\,0), (0\,0\,1\,0\,-1\,0\,0\,0\,0), (0\,0\,0\,0\,0\,1\,0\,0\,-1), (0\,0\,0\,0\,0\,0\,1\,-1\,0)$$

Then $x_1 = 3$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ and

$$\sigma(36) \geq 36\max(6, 3+0+0+0) = 216 = 36\sqrt{36}$$

So we in fact have the matrix of order 36 with maximal excess.

Now instead of replacing y_1, y_2, y_3, y_4 by 1 we replace them by suitable matrices (for example Williamson matrices) W_1, W_2, W_3, W_4 of order w with row and column sums a, b, c, d respectively where

$$e\left(\sum_{i=1}^{4} W_iW_i^T\right)e^T = w(a^2+b^2+c^2+d^2) = 4wee^T = 4w^2$$

e being the $1 \times w$ matrix of 1s.

So

$$\sigma(W_1) = aw, \qquad \sigma(W_2) = bw, \qquad \sigma(W_3) = cw, \qquad \sigma(W_4) = dw$$

and

$$\sigma(4tw) = 2tw(a\ b\ c\ d)\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \text{(case } 1G\text{)}$$

$$\sigma(4tw) = 4tw(a\ b\ c\ d)\begin{bmatrix} -x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \text{(case } 1H\text{)}$$

$$\sigma(4tw) = 2tw(a\ b\ c\ d)\begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \text{(case } 2G\text{)}$$

$$\sigma(4tw) = 4tw(a\ b\ c\ d)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad \text{(case } 2H\text{)}$$

Example 2. Suppose that X_1, X_2, X_3, X_4 are as in Example 1. Then $x_1 = 3$, $x_2 = x_3 = x_4 = 0$. Thus

$$\sigma(36w) = 54w(-a + b + c + d) \qquad \text{(case } 1G\text{)}$$

$$\sigma(36w) = -108wa \qquad \text{(case } 1H\text{)}$$

$$\sigma(36w) = 54w(a + b + c + d) \qquad \text{(case } 2G\text{)}$$

$$\sigma(36w) = 108wa \qquad \text{(case } 2H\text{)}$$

We now observe that if

$$J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

then

$$W_1 = W_2 = W_3 = \begin{bmatrix} J & B & -B \\ -B & J & B \\ B & -B & J \end{bmatrix} \qquad W_4 = \begin{bmatrix} B & B & B \\ B & B & B \\ B & B & B \end{bmatrix}$$

where $\sum_{i=1}^{4} W_i W_i^T = 36I$, $W_i W_j^T = W_j W_i^T$, and the row sums are $a = b = c = d = 3$.

Thus $\sigma(36 \cdot 9) \geq \max(54 \cdot 9 \cdot 6, 54 \cdot 9 \cdot 12, 108 \cdot 9 \cdot 3)$ from cases $1G$, $2G$ and $2H$ respectively i.e.

$$\sigma(4 \cdot 9 \cdot 9) \geq 8 \cdot 3^6 = 9 \cdot 36\sqrt{9 \cdot 36}$$

So we have the Hadamard matrix with maximal excess.

This method is that used by Koukouvinos and Kounias [12] (but not quite in this form) to construct their maximal excess Hadamard matrices. For convenience we state these results as a theorem.

Theorem 5. *Suppose there are Williamson type matrices of order w and row sums a, b, c, d. Suppose there are T-matrices of order t and row sums* x_1, x_2, x_3, x_4 *then the excess of the Hadamard matrix of order* $4wt$ *formed from these matrices satisfies (writing* $\mathbf{A}$ *for* $(a\ \ b\ \ c\ \ d)$ *and* $\mathbf{X}$ *for* $(x_1 x_2 x_3 x_4)^T$.*)*

$$\sigma(4wt) \geq \max\left(4wt\mathbf{A}\mathbf{X}, 2wt\mathbf{A}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix}\mathbf{X}, 2wt\mathbf{A}\begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}\mathbf{X}\right)$$

5. Some Numerical Results

We have seen that we can extablish the existence of *SBIBD*s and regular Hadamard matrices by looking for Hadamard matrices with maximal excess.

First we note that Koukouvinos and Kounias [12] have shown:

Theorem 6. *Hadamard matrices of order* $4k^2$ *with maximal excess exist for* $k = 1$, $3, \ldots, 13, \ldots, 17, 19, 21, \ldots, 25, 33, 37$.

Combining the results of Remark 2 with this theorem and using the Equivalence Theorem we get many more matrices with maximal excess.

Yamada [26, Section 3] has shown there are Hadamard matrices

(i) of order $4 \cdot 3^{2m}$ and excess $8 \cdot 3^{3m}$
(ii) of order $2^{2t} \cdot 5^2$ and excess $2^{3t} \cdot 5^3$.

This means there are Hadamard matrices with maximal excess for orders $4 \cdot 27^2$ and $4 \cdot 81^2$.

In Geramita and Seberry [8, p. 175] the T-matrices to construct an $OD(4 \cdot 61; 61, 61, 61, 61)$ with row sums 6, 5, 0, 0 are given. This gives an excess of either $4 \cdot 61(6y_3 + 5y_4)$ or $2 \cdot 61(y_1 + 11y_2 + y_3 + 11y_4)$. Now $121 = 11^2 + 0^2$ is a prime power so there are Williamson matrices of order $n = \frac{121+1}{2} = 61$, with row sums 1, 11, 1, 11. Thus there is an Hadamard matrix of order $4 \cdot 61^2$ with excess $2 \cdot 61^2 \cdot 4 \cdot 61$.

Now the sequences $\{1\ 0_{11}\}$, $\{0\ 1\ 1\ 1\ 1\ 1\ -1\ -1\ 1\ -1\ 1\ -1\}$ can be used for A or B and $\{1\ 0\ 1\ 0\ -1\ 0\ 1\ 0\ -1\ 0\ 1\}$, $\{0\ 1\ 0\ 1\ 0\ -1\ 0\ 1\ 0\ 1\ 0\}$ can be used for C or D in Yang's construction [27] to form T-matrices of order 69. Depending on the order the matrices are used we can get T-matrices of row sums 6, 5, 2, 2 or 7,4 0,2 or 6, 1, 4, 4 and order 69. We use the T-matrices with row sums 6, 5, 2, 2.

For $t = 25$ use the T-matrices given in Geramita and Seberry [8, p. 175] which give an $OD(100; 25, 25, 25, 25)$ and which have row sums 5, 0, 0, 0. This gives an excess of $50(5y_1 + 5y_2 + 5y_3 + 5y_4) = 2 \cdot 5^3(y_1 + y_2 + y_3 + y_4)$. Now as Yamada remarks [26, section 3] there are Williamson matrices of order $n = 3^{2m}$ with row sums 3^m, 3^m, 3^m, 3^m. So there are Hadamard matrices of order $4 \cdot 5^2 \cdot 3^{2m}$ with maximal excess $8 \cdot 5^3 \cdot 3^{3m}$. There are also (see [24, p. 389] Williamson matrices of order $n = 25$ with row sums 5, 5, 5, 5. So there are Hadamard matrices of order $4 \cdot 5^4$ with maximal excess $8 \cdot 5^6$.

Lemma 7. *There are Hadamard matrices of order* $100 \cdot 3^{2m}$, $m \geq 0$ *and maximal excess* $1000 \cdot 3^{3m}$. *There are Hadamard matrices of order* $4 \cdot 5^4$ *and maximal excess* $8 \cdot 5^6$.

6. Summarizing

Theorem 8. *Hadamard matrices of order* $4k^2$ *with maximal excess exist for*

(i) k *even* $k \leq 210$, *or an Hadamard matrix of order* $2k$ *exists,*
(ii) $k \in \{1, 3, 5, \ldots, 29, 33, \ldots, 41, 45, 51, 53, 61, \ldots, 69, 75, 81, 83, 89, 95, 99, 625, 3^{2m}, 5^2 \cdot 3^{2m}, m \geq 0\}$.

This means that regular Hadamard matrices of order $4k^2$ *and* $SBIBD(4k^2, 2k^2 \pm k, k^2 \pm k)$ *also exist for these* k *values.*

Remark. Koukouvinos, Kounias and Seberry have subsequently, in "Further Hadamard matrices with maximal excess and new $SBIBD(4k^2, 2k^2 + k, k^2 + k)$" Utilitas Math. (to appear) extended (ii) to include

$$k \in \{31, 43, 49, 55, 57, 85, 87, 91, 93, 115, 117\}$$

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Received: March 18, 1988