A survey of base sequences, disjoint complementary sequences and $OD(4t; t, t, t, t)$

Gavin Cohen*
David Rubie*
Jennifer Seberry*
Christos Koukouvinos†
Stratis Kounias†
and
Mieko Yamada‡

Abstract

We survey the existence of base sequences, that is four sequences of lengths $m+p, m+p, m, m$, p odd with zero auto correlation function which can be used with Yang numbers and four disjoint complementary sequences (and matrices) with zero non-periodic (periodic) autocorrelation function to form longer sequences.

We survey their application to make orthogonal designs $OD(4t; t, t, t, t)$.

We give the method of construction of $OD(4t; t, t, t, t)$ for $t = 1, 3, \ldots,$ 41, 45, ..., 65, 67, 69, 75, 77, 81, 85, 87, 91, 93, 95, 99, 101, 105, 111, 115, 117, 119, 123, 125, 129, 133, 141, ..., 147, 153, 155, 159, 161, 165, 169, 171, 175, 177, 183, 185, 189, 195, 201, 203, 205, 209.

1 Definitions and Introduction

An *orthogonal design of order n and type* $(s_1, \ldots, s_u), s_i$ positive integers, is an $n \times n$ matrix X, with entries from $\{0, \pm x_1, \ldots, \pm x_u\}$ (the x_i commuting indeterminates) satisfying

$$XX^T = (\sum_{i=1}^{u} s_i x_i^2) I_n.$$

We write this as $OD(n; s_1, s_2, \ldots, s_u)$.

Alternatively, each X has s_i entries of the type $\pm x_i$ and the distinct rows are orthogonal under the Euclidean inner product.

We may view X as a matrix with entries in the field of fractions of the integral domain $Z[x_1, \ldots, x_u]$, (Z the rational integers), and then if we let $(\sum_{i=1}^{u} s_i x_i^2)$, X is an invertible matrix with inverse $\frac{1}{f} X^T$. Thus $XX^T = f I_n$, and so our alternative of the rows of X applies equally well to the columns of X.

An orthogonal design with no zeros and in which each of the entries is replaced by +1 or -1 is called an *Hadamard matrix*. Alternatively, an Hadamard matrix of order n, H, has entries +1 or -1 and the distinct rows are orthogonal so

$$HH^T = nI_n$$

.

Orthogonal designs and Hadamard matrices are extensively described in [4] and [14].

A special orthogonal design, the $OD(4t; t, t, t, t)$, is especially useful in constructing Hadamard matrices. An $OD(12; 3, 3, 3, 3)$ was first found by Baumert–Hall [2] and an $OD(20; 5, 5, 5, 5)$ by Welch. $OD(4t; t, t, t, t)$ are sometimes called Baumert–Hall arrays.

Early work of Golay [5,6] was concerned with two (1,-1) sequences, but Welti [27], Tseng [21] and Tseng and Liu [22] approached the subject from the point of view of two orthonormal vectors, each corresponding to one of two orthogonal waveforms. Later work, including Turyn's [23,24] used four or more sequences.

Since we are concerned with orthogonal designs, we shall consider sequences of commmuting variables.

Let $X = \{\{a_{11}, \ldots, a_{1n}\}, \{a_{21}, \ldots, a_{2n}\} \ldots \{a_{m1}, \ldots, a_{mn}\}\}$ be m sequences of commuting variables of length n.

The *nonperiodic auto-correlation function of the family of sequences* X (denoted N_X) is a function defined by

$$N_X(j) = \sum_{i=1}^{n-j}(a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \ldots + a_{m,i}a_{m,i+j}).$$

. Note that if the following collection of m matrices of order n is formed,

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ & a_{11} & & a_{1,n-1} \\ & & \ddots & \\ O & & & a_{11} \end{bmatrix}, \begin{bmatrix} a_{21} & a_{22} & \ldots & a_{2n} \\ & a_{21} & & a_{2,n-1} \\ & & \ddots & \\ O & & & a_{21} \end{bmatrix}, \ldots,$$

$$\begin{bmatrix} a_{m1} & a_{m2} & \ldots & a_{mn} \\ & a_{m1} & & a_{m,n-1} \\ & & \ddots & \\ O & & & a_{m1} \end{bmatrix}$$

then $N_X(j)$ is simply the sum of the inner products of rows 1 and $j+1$ of these matrices.

The *periodic auto-correlation function of the family of sequences* X (denoted P_X) is a function defined by

$$P_X(j) = \sum_{i=1}^{n}(a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \ldots + a_{m,i}a_{m,i+j}),$$

where we assume the second subscript is actually chosen from the complete set of residues $\pmod n$.

We can interpret the function P_X in the following way: form the m circulant matrices which have first rows, respectively,

$$[a_{11}a_{12}\ldots a_{1n}], [a_{21}a_{22}\ldots a_{2n}], \ldots, [a_{m1}a_{m2}\ldots a_{mn}];$$

then $P_X(j)$ is the sum of the inner products of rows 1 and $j+1$ of these matrices.

We say the *weight* of a set of sequences X is the number of nonzero entries in X.

If X is as above with $N_X(j) = 0$, $j = 1, 2, \ldots, n-1$, then we will call X *m–complementary sequences* of length n.

If $X = \{A_1, A_2, \ldots, A_m\}$ are m-complementary sequences of length n and weight $2k$ such that

$$Y = \{(A_1 + A_2)/2, (A_1 - A_2)/2, \ldots (A_{2i-1} + A_{2i})/2, (A_{2i-1} - A_{2i})/2, \ldots\}$$

are also m-complementary sequences (of weight k), then X will be said to be *m–complementary disjointable sequences* of length n. X will be said to be *m–complementary disjoint sequences* of length n if all $\binom{m}{2}$ pairs of sequences are disjoint, i.e., $A_i * A_j = 0$ for all i, j, where $*$ is the Hadamard product.

For example { 1 1 0 1 }, { 0 0 1 0 -1 }, { 0 0 0 0 0 1 0 0 -1 }, { 0 0 0 0 0 0 1 -1 } are disjoint as they have zero non-periodic autocorrelation function and precisely one $a_{ij} \neq 0$ for each j. (Here -1 means "minus 1".)

Notation: We sometimes use - for -1, and $\bar{x}$ for $-x$, and A^* to mean the order of the entries in the sequence A are reversed.

One more piece of notation is in order. If g_r denotes a sequence of integers of length r, then by xg_r we mean the sequence of integers of length r obtained from g_r by multiplying each member of g_r by x.

Proposition 1 *Let X be a family of sequences as above. Then*

$$P_X(j) = N_X(j) + N_X(n-j), \ j = 1, \ldots, n-1.$$

Corollary 2 *If $N_X(j) = 0$ for all $j = 1, \ldots, n-1$, then $P_X(j) = 0$ for all $j = 1, \ldots, n-1$.*

Note. $P_X(j)$ may equal 0 for all $j = 1, \ldots, n-1$, even though the $N_X(j)$ are not.

If $X = \{\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}\}$ are two sequences where $a_i, b_j \in \{1, -1\}$ and $N_X(j) = 0$ for $j = 1, \ldots, n-1$, then the sequences in X are called *Golay complementary sequences of length n.* e.g. (writing - for minus 1),

$n = 2$ 11 and 1−
$n = 10$ 1 − −1 − 1 − − − 1 and 1 − − − − − −11−
$n = 26$ 111 − −111 − 1 − − − − − 1 − 11 − −1 − − − −
− − −11 − − − 1 − 11 − 1 − 1 − 11 − −1 − − − −.

We note that if X is as above and A is the circulant matrix with first row $\{a_1, \ldots, a_n\}$ and B the circulant matrix with first row $\{b_1, \ldots, b_n\}$, then

$$AA^T + BB^T = \sum_{i=1}^{n}(a_i^2 + b_i^2)I_n.$$

Consequently, such matrices may be used to obtain Hadamard matrices constructed from two circulants.

We would like to use Golay sequences to construct other orthogonal designs, but first we consider some of their properties.

Lemma 3 *Let $X = \{\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}\}$ be Golay complementary sequences of length n. Suppose k_1 of the a_i are positive and k_2 of the b_i are positive. Then*

$$n = (k_1 + k_2 - n)^2 + (k_1 - k_2)^2,$$

and n is even.

Proof: Since $P_X(j) = 0$ for all j, we may consider the two sequences as $2 - \{n; k_1, k_2; \lambda\}$ supplementary difference sets with $\lambda = k_1 + k_2 - \frac{1}{2}n$. But the parameters (counting difference two ways) satisfy $\lambda(n-1) = k_1(k_1-1) + k_2(k_2-1)$. On substituting λ in this equation we obtain the result of the enunciation.

Geramita and Seberry [4, pp133-7], Andres [1] and James [9] have studied the smaller values of n, k_1, k_2 of the lemma showing the only orders for Golay sequences of order ≤ 68 which exist are 2, 4, 8, 10, 16, 20, 26, 32, 40, 52 and 64. Malcolm Griffin [7] has shown no Golay sequences can exist for lengths $n = 2.9^t$. The value $n = 18$ previously excluded by a complete search but is now theoretically excluded by Griffin's theorem and independently by a result of Kruskal [13] and C.H.Yang [29], [30], [31]. Recent work of Andres [1] and James [9] has led to a greatly improved computer algorithms for studying these sequences.

Summary of Golay Properties

Two sequences $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are called *Golay complementary sequences* of length n if all their entries are ± 1 and

$$\sum_{i=1}^{n-j}(x_i x_{i+j} + y_i y_{i+j}) = 0 \text{ for every } j \ne 0,\ j = 1, \ldots, n-1, \text{ i.e, } N_X = 0.$$

These sequences have the following properties:

1. $\sum_{i=1}^{n}(x_i x_{i+j} + y_i y_{i+j}) = 0$ for every $j \ne 0$, $j = 1, \ldots, n-1$ (where the subscripts are reduced modulo n), i.e. $P_X = 0$.

2. n is even and the sum of two squares.

3. $x_{n-i+1} = e_i x_i \Longleftrightarrow y_{n-i+1} = -e_i y_i$ where $e_i = \pm 1$.

4.

$$\left[\sum_{i\in S} x_i \mathrm{Re}(\zeta^{2i+1})\right]^2 + \left[\sum_{i\in D} x_i \mathrm{Im}(\zeta^{2i+1})\right]^2 + \left[\sum_{i\in S} y_i \mathrm{Im}(\zeta^{2i+1})\right]^2 +$$

$$\left[\sum_{i\in D} y_i \mathrm{Re}(\zeta^{2i+1})\right]^2 = \tfrac{1}{2} n,$$

where $S = \{i : 0 \le i < n, e_i = 1\}$, $D = \{i : 0 \le i < n, e_i = -1\}$ and ζ is a $2n$-th root of unity.

5. They exist for orders $2^a 10^b 26^c$, a, b, c nonnegative integers.

6. They do not exist for orders 2.9^c (c a positive integer), 34, 36, 50, 58 or 68.

7. They do not exist for orders 2.49^c (c a positive integer) (Kounias, Koukouvinos and Sotirakoglou).

We now discuss other sequences with zero auto-correlation function.

Other Sequences with Zero Auto-Correlation Function

Lemma 4 *Suppose $X = \{X_1, X_2, \ldots, X_m\}$ is a set of $(0,1,-1)$ sequences of length n for which $N_X = 0$ or $P_X = 0$. Further suppose the weight of X_i is x_i and the sum of the elements of x_i is a_i. Then*

$$\sum_{i=1}^{m} a_i^2 = \sum_{i=1}^{m} x_i.$$

Proof: Form circulant matrices Y_i for each X_i. Then

$$Y_i J = a_i J \text{ and } \sum_{i=1}^{m} Y_i Y_i^T = \sum_{i=1}^{m} x_i.$$

Now considering

$$\sum_{i=1}^{m} Y_i Y_i^T J = \sum_{i=1}^{m} a_i^2 J = \sum_{i=1}^{m} x_i J,$$

we have the result.

Example. Suppose X_1, X_2, X_3, X_4 have elements from +1 and -1 and lengths 19,19,18,18. The total weight of these sequences is 74 now the four rows sums must square to 74 so we could have

$$\begin{array}{ll} 3^2+1^2+8^2+0^2 & 1^2+1^2+6^2+6^2 \\ 7^2+5^2+0^2+0^2 & \quad\text{or} \\ 7^2+3^2+4^2+0^2 & 5^2+3^2+6^2+2^2 \end{array}$$

In algorithms to search for these squares the row sum of 8 and length 18 means there are 13 elements +1 and 5 elements -1 considerably shortening any search.

Now a few simple observations are in order, and for convenience we put them together as a lemma though more has been observed by Whitehead [28].

Lemma 5 *Let $X = \{A_1, A_2, \ldots, A_m\}$ be m-complementary sequences of length n. Then*

(i) $Y = \{A_1^*, A_2^*, \ldots, A_i^*, A_{i+1}, \ldots, A_m\}$ are m-complementary sequences of length n where A_i^* means reverse the elements of A_i;

(ii) $W = \{A_1, A_2, \ldots, A_i, -A_{i+1}, \ldots, -A_m\}$ are m-complementary sequences of length n;

(iii) $Z = \{\{A_1, A_2\}, \{A_1, -A_2\}, \ldots, \{A_{2i-1}, A_{2i}\}, \{A_{2i-1}, -A_{2i}\}, \ldots\}$ are $m-$ (or $m+1$ if m was odd when we let A_{m+1} be n zeros) complementary sequences of length $2n$;

(iv) $U = \{\{A_1/A_2\}, \{A_1/-A_2\}, \ldots, \{A_{2i-1}/A_{2i}\}, \{A_{2i-1}/-A_{2i}\}, \ldots\}$, where A_j/A_k means $a_{j1}, a_{k1}, a_{j2}, a_{k2}, \ldots a_{jn}, a_{kn}$, are $m-$ (or $m+1$ if m was odd when we let A_{m+1} be n zeros) complementary sequences of length $2n$.

(v) $V = \{A_1^*, A_2^*, \ldots, A_m^*\}$ where $A_i^* = \{a_{i1}, -a_{i2}, a_{i3}, -a_{i4}, \ldots\}$ are m complementary sequences of length n.

By a lengthy but straightforward calculation, it can be shown that:

Theorem 6 *Suppose $X = \{A_1, \ldots, A_{2m}\}$ are 2m-complementary sequences of length n and weight u and $Y = \{B_1, B_2\}$ are 2-complementary disjointable sequences of length t and weight 2k. Then there are 2m-complementary sequences of length nt and weight ku.*

The same result is true if X are 2m-complementary disjointable sequences of length n and weight 2u and Y are 2-complementary sequences of weight k.

Proof: Using an idea of R.J.Turyn, we consider

$$A_{2i-1} \times (B_1 + B_2)/2 + A_{2i} \times (B_1 - B_2)/2 \quad \text{and}$$
$$A_{2i-1} \times (B_1^* - B_2^*)/2 - A_{2i} \times (B_1^* + B_2^*)/2, \quad \text{and}$$

for $i = 1, \ldots, m$, which are the required sequences in the first case, and

$$(A_{2i-1} + A_{2i})/2 \times B_1 + (A_{2i-1} - A_{2i})/2 \times B_2^* \quad \text{and}$$
$$(A_{2i-1} + A_{2i})/2 \times B_2 - (A_{2i-1} - A_{2i})/2 \times B_1^* \quad \text{and}$$

for $i = 1, \ldots, m$, which are the required sequences for the second case.

The proof now follows by an exceptionally tedious but straightforward verification.

Corollary 7 *Since there are Golay sequences of lengths 2,10 and 26, there are Golay sequences of length $2^a 10^b 26^c$ for a, b, c non-negative integers.*

Corollary 8 *There are 2-complementary sequences of lengths $2^a 6^b 10^c 14^d 26^e$ of weights $2^a 5^b 10^c 13^d 26^e$, where a, b, c, d, e are non-negative integers.*

Proof: Use the sequences of Tables 5 and 6 of Appendix H of [4].

Base Sequences

Four sequences of elements +1,-1 of lengths $m+p, m+p, m, m$ where p is odd which have zero non-periodic autocorrelation function are called *base sequences*. In Table 1 base sequences are displayed for lengths $m+1, m+1, m, m$ for $m+1 \in \{ 2, 3, \ldots, 18, 21, 24, 30\}$. If X and Y are Golay sequences $\{1, X\}, \{1, -X\}, \{Y\}, \{Y\}$ are base sequences of lengths $m+1, m+1, m, m$. So base sequences exist for all $m = 2^a 10^b 26^c$, a, b, c non-negative integers, $p = 1$. The cases for $m = 17$, $p = 1$, were found by A.Sproul and J.Seberry and for $m = 23$, $p = 1$ by R.Turyn. These sequences are also discussed in Geramita and Seberry [4,pp129-148].

Base sequences are crucial to Yang's [29], [30], [31] constructions for finding longer T-sequences of odd length.

We know propose to use R.J.Turyn's idea of using m-complementary sequences to construct orthogonal designs.

Table 1: Base Sequences of Lengths m+1, m+1, m, m

Length	Sums of Squares	Sequences
m = 1	$2^2+0^2+1^2+1^2$	({1 1}, {1-1}, {1}, {1})
m = 2	$3^2+1^2+0^2+0^2$	({1 1 1}, {1 1-1}, {1-1}, {1-1})
m = 3	$2^2+0^2+3^2+1^2$	({1 1-1 1}, {1 1-1-1}, {1 1 1}, {1-1 1})
m = 4	$3^2+3^2+0^2+0^2$	({1 1-1 1 1}, {1 1 1 1-1}, {1 1-1-1}, {1-1 1-1})
m = 4	$3^2+1^2+2^2+2^2$	({1 1 1 1-1}, {-1 1 1 1-1}, {1 1-1 1}, {1 1-1 1})
m = 5	$2^2+0^2+3^2+3^2$	({1 1-1 1-1 1}, {1 1 1-1-1-1}, {1 1-1 1 1}, {1 1-1 1 1})
m = 6	$3^2+1^2+4^2+0^2$	({1 1-1 1-1 1 1}, {1 1-1-1-1 1-1}, {-1 1 1 1 1 1}, {-1-1 1 1-1 1 })
m = 6	$5^2+1^2+0^2+0^2$	({1 1 1-1 1 1 1}, {1 1-1-1-1 1-1}, {1 1-1 1-1-1}, {1 1-1 1-1-1 })
m = 7	$2^2+0^2+5^2+1^2$	({1 1 1 1-1-1-1 1}, {1 1-1 1-1 1-1-1}, {1 1 1-1 1 1 1}, {1-1-1 1-1-1 1})
m = 7	$4^2+2^2+3^2+1^2$	({-1 1 1 1 1 1-1 1}, {1 1 1-1-1 1-1 1}, {-1 1 1-1 1 1 1}, {1-1 1 1 1-1-1})
m = 8	$5^2+3^2+0^2+0^2$	({1 1 1 1-1 1 1-1 1}, {-1 1 1 1-1 1 1-1 1}, {1 1 1-1-1-1 1-1}, {1 1 1-1-1-1 1-1})
m = 9	$4^2+2^2+3^2+3^2$	({1 1 1 1 1-1-1 1-1 1}, {-1 1 1 1 1-1-1 1-1 1}, {1 1 1-1 1 1 1-1-1}, {1-1 1 1 1-1 1 1-1})
m = 10	$5^2+3^2+2^2+2^2$	({1 1-1-1 1 1 1 1 1 1-1}, {-1 1-1-1 1 1 1 1 1 1-1}, {-1 1 1 1-1 1-1 1 1-1}, {-1 1 1 1-1 1-1 1 1-1})

Length	Sums of Squares	Sequences
m = 10	$1^2 + 5^2 + 0^2 + 4^2$	{{1 1-1 1 1-1-1-1-1 1 1}, {-1 1 1-1-1 1 1 1 1 1 1} {-1-1 1 1-1 1-1 1-1 1}, {-1 1 1-1 1-1 1 1 1 1}}
m = 11	$6^2 + 0^2 + 3^2 + 1^2$	{{-1 1 1 1-1 1-1 1 1 1 1 1}, {-1 1 1-1 1 1-1-1-1 1 1-1}, {-1-1 1 1 1 1 1-1 1 1-1}, {-1 1 1 1 1-1 1-1 1-1-1}}
m = 11	$4^2 + 2^2 + 5^2 + 1^2$	{{1 1 1 1 1 1-1-1 1-1 1-1}, {1-1-1-1-1-1 1 1-1 1-1 1}, {1 1 1 1-1-1 1 1-1 1 1}, {1-1 1-1-1 1 1-1-1-1 1}}
m = 12	$7^2 + 1^2 + 0^2 + 0^2$	{{1 1 1 1-1 1-1 1-1 1 1 1 1}, {1 1 1-1-1 1-1 1-1-1 1 1-1}, {1 1 1-1 1 1-1-1 1-1-1-1}, {1 1 1-1-1 1-1 1 1-1-1-1}}
m = 12	$5^2 + 5^2 + 0^2 + 0^2$	{{1-1-1-1 1-1 1-1-1-1 1-1-1}, {1 1-1 1-1-1 1-1 1 1 1 1 1}, {1 1-1 1-1-1 1 1 1-1-1-1}, {1 1-1 1 1 1-1-1 1-1-1-1}}
m = 12	$3^2 + 1^2 + 6^2 + 2^2$	{{1 1 1 1-1 1-1-1 1 1-1-1 1}, {1 1 1 1-1 1-1-1 1 1-1-1-1}, {1 1 1 1 1-1 1-1 1-1 1 1}, {1 1 1-1-1 1-1 1-1-1 1 1}}
m = 13	$6^2 + 4^2 + 1^2 + 1^2$	{{1 1 1 1 1 1-1-1 1-1-1 1-1 1}, {1-1-1-1-1-1 1 1-1-1 1-1 1-1}, {1 1 1 1-1-1-1 1-1-1 1-1-1}, {1-1 1-1-1 1-1-1-1 1 1 1-1}}
m = 14	$7^2 + 3^2 + 0^2 + 0^2$	{{1 1-1 1 1 1-1 1-1 1 1 1-1 1 1}, {1 1 1-1 1 1-1-1-1 1 1-1 1 1-1}, {1 1 1 1-1-1 1-1 1 1-1-1-1-1}, {1-1-1-1-1 1-1 1-1 1 1 1 1-1}}
m = 15	$6^2 + 4^2 + 1^2 + 3^2$	{{1 1 1 1 1-1 1 1-1-1 1 1 1-1 1-1}, {1-1-1-1-1 1-1-1 1 1-1-1-1 1-1 1}, {1 1 1-1 1-1 1 1-1-1-1-1-1 1 1}, {1-1 1 1 1 1 1-1-1 1-1 1-1-1 1}}
m = 16	$5^2 + 3^2 + 4^2 + 4^2$	{{1 1 1 1 1-1-1 1-1 1-1 1-1-1 1 1 1}, {1-1-1 1 1-1 1-1 1-1 1 1-1-1-1-1-1}, {1 1 1 1 1-1-1 1 1-1 1-1 1 1-1-1}, {1 1 1 1 1-1-1 1 1-1 1-1 1 1-1-1}}
m = 17	$2^2 + 4^2 + 5^2 + 5^2$	{{1-1 1 1 1-1-1 1-1-1-1-1 1 1 1 1 1-1}, {1-1 1 1-1 1-1-1 1 1 1-1-1-1 1 1 1 1}, {1 1 1-1 1 1 1-1 1 1 1-1 1-1-1 1-1}, {1 1 1-1 1 1 1-1 1 1 1-1 1-1-1 1-1}}
m = 20	$3^2 + 1^2 + 6^2 + 6^2$	{{1 1-1-1 1 1 1 1 1 1-1 1-1-1 1-1 1-1-1-1 1}, {-1 1-1-1 1 1 1 1 1 1-1 1-1-1 1-1 1-1-1-1 1}, {1-1-1 1 1 1 1 1 1-1-1 1 1-1 1-1 1 1 1-1}, {1-1-1 1 1 1 1 1 1 1-1-1 1 1-1 1-1 1 1 1-1}}
m = 20	$7^2 + 5^2 + 2^2 + 2^2$	{{1 1-1-1 1 1 1 1 1 1-1-1 1 1 1-1 1-1 1 1-1}, {-1 1-1-1 1 1 1 1 1 1-1-1 1 1 1-1 1-1 1 1-1}, {1-1-1 1 1 1 1 1 1-1 1-1-1-1 1-1 1-1-1 1}, {1-1-1 1 1 1 1 1 1-1 1-1-1-1 1-1 1-1-1 1}}
m = 23	$2^2 + 8^2 + 5^2 + 1^2$	{{1-1-1-1 1-1 1-1-1-1-1 1 1 1 1 1 1-1-1 1-1-1-1 1}, {1-1-1 1-1-1 1-1 1 1 1-1-1-1-1-1 1-1-1-1 1-1-1-1}, {1 1 1-1-1-1 1 1-1-1 1-1-1-1-1 1-1-1-1-1 1-1 1}, {1 1-1-1 1-1 1 1-1 1-1 1 1 1-1 1-1-1 1-1-1-1 1}}
m = 29	$8^2 + 6^2 + 3^2 + 3^2$	{{1 1 1 1 1-1 1 1 1 1-1 1-1-1 1 1-1-1 1 1 1 1-1 1-1-1-1 1-1 1}, {1-1-1-1-1 1-1-1-1-1 1-1 1 1-1-1 1 1-1-1-1-1 1-1 1 1 1-1 1-1}, {1 1 1-1 1-1-1-1 1-1 1 1-1-1-1 1-1-1 1 1 1 1-1 1 1 1 1-1-1}, {1-1 1 1 1 1-1 1 1 1 1-1-1 1-1-1-1 1 1-1 1-1-1-1 1-1 1 1-1}}

Table 1(cont): Base Sequences of Lengths m+1, m+1, m, m

Four (1,-1) sequences $A = (X, U, Y, V)$, where

$$
\begin{aligned}
X &= \{x_1 = 1, x_2, x_3, \ldots, x_m, -x_m, \ldots, -x_3, -x_2, -x_1 = -1\}, \\
U &= \{u_1 = 1, u_2, u_3, \ldots, u_m, -u_m, \ldots, -u_3, -u_2, 1\}, \\
Y &= \{y_1, y_2, \ldots, y_{m-1}, y_m, y_{m-1}, \ldots, y_3, y_2, y_1\}, \\
V &= \{v_1, v_2, \ldots, v_{m-1}, v_m, v_{m-1}, \ldots, v_3, v_2, v_1\},
\end{aligned}
$$

which have $N_A = 0$ and $8m - 6$ is the sum of two squares.
Or four (1,-1) sequences $A = \{X, U, Y, V\}$ where

$$
\begin{aligned}
X &= \{x_1 = 1, x_2, x_3, \ldots, x_m, x_{m+1}, x_m, \ldots, x_3, x_2, x_1 = 1\}, \\
U &= \{u_1 = 1, u_2, u_3, \ldots, u_m, u_{m+1}, u_m, \ldots, u_3, u_2, -1\}, \\
Y &= \{y_1, y_2, \ldots, y_m, -y_m, \ldots, -y_2, -y_1\}, \\
V &= \{v_1, v_2, \ldots, v_m, -v_m, \ldots, -v_2, -v_1\},
\end{aligned}
$$

which have $N_A = 0$ and $8m + 2$ is the sum of two squares will be called *Turyn sequences* of length $n+1, n+1, n, n$ (they have weights $n+1, n+1, n, n$ also).

Length	Sequences
m = 1	{{1-1}, {1 1}, {1}, {1}}
m = 2	{{1 1 1}, {1 1-1}, {1-1}, {1-1}}
m = 3	{{1 1-1-1}, {1 1-1 1}, {1 1 1}, {1-1 1}}
m = 4	{{1 1-1 1 1}, {1 1 1 1-1}, {1 1-1-1}, {1-1 1-1}}
m = 5	{{1 1 1-1-1-1}, {1 1-1 1-1 1}, {1 1-1 1 1}, {1 1-1 1 1}}
m = 6	{{1 1 1-1 1 1 1}, {1 1-1-1-1 1-1}, {1 1-1 1-1-1}, {1 1-1 1-1-1 }}
m = 7	{{1 1-1 1-1 1-1-1}, {1 1 1 1-1-1-1 1}, {1 1 1-1 1 1 1}, {1-1-1 1-1-1 1}}
m = 12	{{1 1 1 1-1 1-1 1-1 1 1 1 1}, {1 1 1-1-1 1-1 1-1-1 1 1-1}, {1 1 1-1 1 1-1-1 1-1-1-1}, {1 1 1-1-1 1-1 1 1-1-1-1}}
m = 14	{{1 1-1 1 1 1-1 1-1 1 1 1-1 1 1}, {1 1 1-1 1 1-1-1-1 1 1-1 1 1-1}, {1 1 1 1-1-1 1-1 1 1-1-1-1-1}, {1-1-1-1-1 1-1 1-1 1 1 1 1-1}}

Table 2: Turyn Sequences of Lengths m+1, m+1, m, m

Known Turyn sequences are given in Table 2.

Geramita and Seberry [4, p142-143] quote Robinson and Seberry(Wallis) results giving such sequences where the longer sequence is of length 2, 3, 4, 5, 6, 7, 8, 13, 15 (though the result for 5 has a typographical error and the last sequence should be 1-1-), that they cannot exist for 11, 12, 17 or 18 and a complete machine search showed they do not exist for lengths 9, 10, 14 or 16. So in fact the first resolved cases (and unresolved since 1976) are lengths 19, 20, 21.

A sequence $X = \{x_1, \ldots, x_n\}$ will be called *skew* if n is even and $x_i = -x_{n-i+1}$, and *symmetric* if n is odd and $x_i = x_{n-i+1}$.

Theorem 9 (Turyn) *Suppose $A = \{X, U, Y, V\}$ are Turyn sequences of lengths $m+1, m+1, m, m$, where X is skew and Y is symmetric for m even and X is symmetric and Y is skew for m odd. Then there are T-sequences of lengths $2m+1$ and $4m+3$.*

Proof: We use the notation A/B as before to denote the interleaving of two sequences $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_{m-1}\}$.

$$A/B = \{a_1, b_1, a_2, b_2, \ldots, b_{m-1}, a_m\}.$$

Let 0_t be a sequence of zeros of length t. Then

$$T_1 = \{\{\tfrac{1}{2}(X+Y), 0_m\}, \{\tfrac{1}{2}(X-Y), 0_m\}, \{0_{m+1}, \tfrac{1}{2}(Y+V)\}, \{0_{m+1}, \tfrac{1}{2}(Y-V)\}\}$$

and

$$T_2 = \{\{1, 0_{4m+2}\}, \{0, X/Y, 0_{2m+1}\}, \{0, 0_{2m+1}, U/0_m\}, \{0, 0_{2m+1}, 0_{m+1}/V\}\}$$

are T-sequences of lengths $2m+1$ and $4m+3$ respectively.

Corollary 10 *There are T-sequences constructed from Turyn sequences of lengths 3, 5, 7, 9, 11, 13, 15, 19, 23, 25, 27, 29, 31, 51, 59.*

Theorem 11 *If X and Y are Golay sequences of length r, then writing 0_r for the vector of r zeros, $T = \{\{1, 0_r\}, \{0, \frac{1}{2}(X+Y)\}, \{0, \frac{1}{2}(X-Y)\}, \{0_{r+1}\}\}$ are T-sequences of length $r+1$.*

Corollary 12 (Turyn) *There exist T-sequences of lengths $1 + 2^a 10^b 26^c$, where a, b, c are non-negative integers.*

Corollary 13 *There exist T-sequences of lengths 3, 5, 7, ..., 33, 41, 51, 53, 59, 65, 81, 101.*

A desire to fill the gaps in the list in Corollary 13 leads to the following idea.

Lemma 14 *Suppose $X = \{A, B, C, D\}$ are 4-complementary sequences of length $m+1, m+1, m, m$ respectively, and weight k. Then*

$$Y = \{\{A, C\}, \{A, -C\}, \{B, D\}, \{B, -D\}\}$$

are 4-complementary sequences of length $2m+1$ and weight $2k$. Further, if $\frac{1}{2}(A+B)$ and $\frac{1}{2}(C+D)$ are also (0,1,-1) sequences, then, with 0_t the sequence of t zeros,

$$Z = \{\{\tfrac{1}{2}(A+B), 0_m\}, \{\tfrac{1}{2}(A-B), 0_m\}, \{0_{m+1}, \tfrac{1}{2}(C+D)\}, \{0_{m+1}, \tfrac{1}{2}(C-D)\}\}$$

are 4-complementary sequences of length $2m+1$ and weight k. If A, B, C, D are $(1,-1)$ sequences, then Z consists of T-sequences of length $2m+1$.

Lemma 15 *If there are Turyn sequences of length $m+1, m+1, m, m$ there are base sequences of lengths $2m+2, 2m+2, 2m+1, 2m+1$.*

Proof: Let X, U, Y, V be the Turyn sequences as in Table 2. Then

$$E = \{1, X/Y\}, \quad F = \{-1, X/Y\}, \quad G = \{U/V\}, \quad H = \{U/-V\}$$

are 4-complementary base sequences of lengths $2m+2, 2m+2, 2m+1, 2m+1$ respectively.

Corollary 16 *There are base sequences of lengths $m+1, m+1, m, m$ for m*

(a) t, $2t+1$ where there are Turyn sequences of length $t+1$, $t+1$, t, t.

(b) 9,11,13,25,29

(c) for lengths g where there are Golay sequences of length g.

(d) 17 (Seberry–Sproul), 23 (Turyn) given in Tables 1 and 3.

Now Cooper–(Seberry)Wallis–Turyn have shown how 4 disjoint complementary sequences of length t and zero non-periodic (or periodic) autocorrelation function can be used to form $OD(4t; t, t, t, t)$ (formerly called Baumert–Hall arrays) [3],[4].

First the sequences (variously called T-sequences or Turyn sequences – but this latter has two different usages) are turned into T-matrices and then Cooper–(Seberry)Wallis construction, described below, can be applied.

The appropriate theorem for the construction of Hadamard matrices is (it is implied by Williamson, Baumert–Hall, Welch, Cooper–Wallis, Turyn):

Theorem 17 *Suppose there exists an $OD(4t; t, t, t, t)$ and 4 suitable matrices A, B, C, D of order w and entries 1 or -1 which satisfy*

$$AA^T + BB^T + CC^T + DD^T = 4wI_w,$$

$$XY^T = YX^T \text{ for } X, Y \in \{A, B, C, D\}.$$

Then there is an Hadamard matrix of order $4wt$ (Williamson matrices are suitable matrices with the extra condition that $X^T = X$ for $X \in \{A, B, C, D\}$.

n matrices $X_1, X_2, \ldots, X_n$ of order m are said to be *suitable* if they have elements +1 or -1 and they satisfy

$$u_1X_1X_1^T + u_2X_2X_2^T + \ldots + u_nX_nX_n^T = (\sum u_i)mI_m$$

$$X_iX_j^T = X_jX_i^T, \quad i, j = 1, \ldots, n$$

where there is an $OD(\sum u_i; u_1, u_2, \ldots, u_n)$. They are used in the following theorem.

Theorem 18 (Geramita–Seberry) *If there is an $OD(\sum u_i; u_1, u_2, \ldots, u_n)$ and suitable matrices of order m then there is an Hadamard matrix of order $(\sum u_i)m$.*

2 *OD*s and Hadamard matrices

Welch's $OD(20; 5, 5, 5, 5)$ composed of block *circulant* matrices is:

$$
\left[\begin{array}{ccccc|ccccc|ccccc|ccccc}
-D & B & -C & -C & -B & C & A & -D & -D & -A & -B & -A & C & -C & -A & A & -B & -D & D & -B \\
-B & -D & B & -C & -C & -A & C & A & -D & -D & -A & -B & -A & C & -C & -B & A & -B & -D & D \\
-C & -B & -D & B & -C & -D & -A & C & A & -D & -C & -A & -B & -A & C & D & -B & A & -B & -D \\
-C & -C & -B & -D & B & -D & -D & -A & C & A & C & -C & -A & -B & -A & -D & D & -B & A & -B \\
B & -C & -C & -B & -D & A & -D & -D & -A & C & -A & C & -C & -A & -B & -B & -D & D & -B & A \\
\hline
-C & A & D & D & -A & -D & -B & -C & -C & B & -A & B & -D & D & B & -B & -A & -C & C & -A \\
-A & -C & A & D & D & B & -D & -B & -C & -C & B & -A & B & -D & D & -A & -B & -A & -C & C \\
D & -A & -C & A & D & -C & B & -D & -B & -C & D & B & -A & B & -D & C & -A & -B & -A & -C \\
D & D & -A & -C & A & -C & -C & B & -D & -B & -D & D & B & -A & B & -C & C & -A & -B & -A \\
A & D & D & -A & -C & -B & -C & -C & B & -D & B & -D & D & B & -D & -A & -C & C & -A & -B \\
\hline
B & -A & -C & C & -A & A & B & -D & D & B & -D & -B & C & C & B & -C & A & -D & -D & -A \\
-A & B & -A & -C & C & B & A & B & -D & D & B & -D & -B & C & C & -A & -C & A & -D & -D \\
C & -A & B & -A & -C & D & B & A & B & -D & C & B & -D & -B & C & -D & -A & -C & A & -D \\
-C & C & -A & B & -A & -D & D & B & A & B & C & C & B & -D & -B & -D & -D & -A & -C & A \\
-A & -C & C & -A & B & B & -D & D & B & A & -B & C & C & B & -D & A & -D & -D & -A & -C \\
\hline
-A & -B & -D & D & -B & B & -A & C & -C & -A & C & A & D & D & -A & -D & B & C & C & -B \\
-B & -A & B & -D & D & -A & B & -A & C & -C & -A & C & A & D & D & -B & -D & B & C & C \\
D & -B & -A & -B & -D & -C & -A & B & -A & C & D & -A & C & A & D & C & -B & -D & B & C \\
-D & D & -B & -A & -B & C & -C & -A & B & -A & D & D & -A & C & A & C & C & -B & -D & B \\
-B & -D & D & -B & -A & -A & C & -C & -A & B & A & D & D & -A & C & B & C & C & -B & -D
\end{array}\right]
$$

Lemma 19 *Let X_i, $i = 1, \ldots, 4$, be T-matrices with row sum (and column sum) x_i, respectively. Then*

$$\sum_{i=1}^{4} x_i^2 = n.$$

Proof: $X_i J = x_i J$: so considering $\sum_{i=1}^{4} X_i X_i^T J = nJ$ gives the result.

The following result, in a slightly different form, was also discovered by R.J.Turyn.

Theorem 20 (Cooper–Wallis) *Suppose there exist T-matrices (or T-sequences) X_i, $i = 1, \ldots, 4$ of order n. Let a, b, c, d be commuting variables.*

Then

$$
\begin{aligned}
A &= aX_1 + bX_2 + cX_3 + dX_4 \\
B &= -bX_1 + aX_2 + dX_3 - cX_4 \\
C &= -cX_1 - dX_2 + aX_3 + bX_4 \\
D &= -dX_1 + cX_2 - bX_3 + aX_4
\end{aligned}
$$

can be used in the Goethals-Seidel (or Wallis–Whiteman) array to obtain an $OD(4n; n, n, n, n)$.

Proof: By straightforward verification.

Example. Let

$$X_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ X_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \ X_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ X_4 = 0.$$

Then X_1, X_2, X_3, X_4 are T-matrices of order 3, and the $OD(12; 3, 3, 3, 3)$ is

$$\left[\begin{array}{ccc|ccc|ccc|ccc}
a & b & c & -b & a & d & -c & -d & a & -d & c & -b \\
c & a & b & a & d & -b & -d & a & -c & c & -b & -d \\
b & c & a & d & -b & a & a & -c & -d & -b & -d & c \\
\hline
b & -a & -d & a & b & c & -d & -b & c & c & -a & d \\
-a & -d & b & c & a & b & -b & c & -d & -a & d & c \\
-d & b & -a & b & c & a & c & -d & -b & d & c & -a \\
\hline
c & d & -a & d & b & -c & a & b & c & -b & d & a \\
d & -a & c & b & -c & d & c & a & b & d & a & -b \\
-a & c & d & -c & d & b & b & c & a & a & -b & d \\
\hline
d & -c & b & -c & a & -d & b & -d & -a & a & b & c \\
-c & b & d & a & -d & -c & -d & -a & b & c & a & b \\
b & d & -c & -d & -c & a & -a & b & -d & b & c & a
\end{array}\right]$$

We will not give the proof here which can be found in Wallis [25, p.360] but will just quote the results given there. Cyclotomy may be used in constructing these arrays including the orders t =13,19,25,31,37,41 and 61.

A most important theorem which shows how Welch $OD(20; 5, 5, 5, 5)$ can be used is now given. One of us (Yamada) has reported that Ono and Sawade [15] have found a Welch type $OD(36; 9, 9, 9, 9)$. This is being translated at present.

Theorem 21 (Turyn) *Suppose there is a Welch type OD of order $4s$ constructed of sixteen circulant (or type 1) $s \times s$ blocks. Further suppose there are T-matrices of order t. Then there is an $OD(4st; st, st, st, st)$.*

Proof: Since the Welch type OD, W, is constructed of sixteen circulant (or type 1) blocks, we may write the OD as (N_{ij}), $i, j = 1, 2, 3, 4$, where each N_{ij} is circulant (or type 1).

Since $WW^T = s(a^2 + b^2 + c^2 + d^2)I_4$, where a, b, c, d are the commuting variables, we have

$$N_{i1}N_{j1}^T + N_{i2}N_{j2}^T + N_{i3}N_{j3}^T + N_{i4}N_{j4}^T =$$

$$\begin{cases} s(a^2 + b^2 + c^2 + d^2)I_s, & i = j, \quad i = 1, 2, 3, 4 \\ 0 & , \quad i \neq j. \end{cases}$$

Suppose the T-matrices are T_1, T_2, T_3, T_4. Then form the matrices

$$\begin{aligned}
A &= T_1 \times N_{11} + T_2 \times N_{21} + T_3 \times N_{31} + T_4 \times N_{41} \\
B &= T_1 \times N_{12} + T_2 \times N_{22} + T_3 \times N_{32} + T_4 \times N_{42} \\
C &= T_1 \times N_{13} + T_2 \times N_{23} + T_3 \times N_{33} + T_4 \times N_{43} \\
D &= T_1 \times N_{14} + T_2 \times N_{24} + T_3 \times N_{34} + T_4 \times N_{44}
\end{aligned}$$

Now
$$AA^T + BB^T + CC^T + DD^T = st(a^2 + b^2 + c^2 + d^2)I_{st},$$

and since A, B, C, D are type 1, they can be used in the Wallis–Whiteman generalization of the Goethals–Seidel array to obtain the desired result.

Since the OD of order 20 given by Welch is constructed of sixteen circulant blocks, we have:

Corollary 22 *Suppose there are T-matrices of order t. Then there is $OD(20t; 5t, 5t, 5t, 5t)$ and an $OD(36t; 9t, 9t, 9t, 9t)$.*

As we have seen, Baumert and Hall's array of order 3, discovered to obtain the Hadamard matrix of order 156, has led to one of the most powerful constructions for Hadamard matrices.

In fact, to prove the Hadamard conjecture it would be sufficient to prove:

Conjecture 23 *There exists $OD(4t; t, t, t, t)$ for every positive integer t.*

3 On Yang's theorems on T-sequences.

Let $\mathbf{Z}$ be the rational integer ring and x, y independent variables. $R = \mathbf{Z}[x, y]$ denotes the polynomial ring in x, y. Furthermore we consider the residue class ring $\mathcal{R} = R/(xy - 1)R$. We have $\mathcal{R} = \mathbf{Z}[x, x^{-1}]$.

Let $U = (u_0, \ldots, u_{n-1})$ be a sequence of length n such that $u_i \in \mathbf{Z}$. Define the polynomial $U(x)$ in $\mathcal{R}$ associated with a sequence $U = (u_1, \ldots, u_n)$
$$U(x) = \sum_{i=0}^{n-1} u_i x^i.$$

For the reverse $U^* = (u_{n-1}, u_{n-2}, \ldots, u_0)$ of a sequence $U = (u_0, \ldots, u_{n-1})$, the corresponding polynomial is
$$U^*(x) = \sum_{i=0}^{n-1} u_i x^{n-1-i} = x^{n-1} \sum_{i=0}^{n-1} u_i x^{-i} = x^{n-1} U(x^{-1}).$$

For a sequence U, we note that the non-periodic auto-correlation function of the reverse U^* is the same as that of a sequence U: $N_U(j) = N_{U^*}(j)$.

Now Yang [29], [30], [31] found that base sequences can be multiplied by 3,7,13 and $2^a 10^b 26^c$, $a, b, c \geq 0$ so we call these integers Yang numbers. If y is a Yang number and there are base sequences of lengths $m + p, m + p, m, m$ then there are 4 complementary T-sequences of length $y(2m + p)$. This is of most interest when $2m + p$ is odd.

We reprove and restate Yang's theorems from [30] to illustrate why they work.

Multiplying by $2g+1$ where g is the length of a Golay sequence

Theorem 24 (Yang [30]) *Let A, B, C, D be base sequences of length $m+p$, $m+p$, m, m, p: odd and $F=(f_k)$ and $G=(g_k)$ be Golay sequences of length s. Then the following Q, R, S, T become 4-complementary sequences (namely, the sum of non-periodic auto-correlation functions is 0).*

$$\begin{aligned}
Q &= (Af_s, Cg_1; 0,0; Af_{s-1}, Cg_2; 0,0; \ldots; Af_1, Cg_s; 0,0; -B^*, 0)\\
R &= (Bf_s, Dg_s; 0,0; Bf_{s-1}, Dg_{s-1}; 0,0; \ldots; Bf_1, Dg_1; 0,0; A^*, 0)\\
S &= (0,0; Ag_s, -Cf_1; 0,0; Ag_{s-1}, -Cf_2; \ldots; 0,0; Ag_1, -Cf_s; 0, -D^*)\\
T &= (0,0; Bg_1, -Df_1; 0,0; Bg_2, -Df_2; \ldots; 0,0; Bg_s, -Df_s; 0, C^*)
\end{aligned}$$

Furthermore if we define sequences
$X=(Q+R)/2$, $Y=(Q-R)/2$, $V=(S+T)/2$, $W=(S-T)/2$,
then these sequences become T-sequences of length $t(2s+1)$, $t=2m+p$.

Proof: Let $A(x)=\sum_{i=0}^{m+p-1} a_i x^i$, $B(x)=\sum_{i=0}^{m+p-1} b_i x^i$, $C(x)=\sum_{i=0}^{m+p-1} c_i x^i$, $D(x)=\sum_{i=0}^{m+p-1} d_i x^i$ be the corresponding polynomials associated with base sequences $A=(a_i)$, $B=(b_i)$, $C=(c_i)$, $D=(d_i)$.

Let $F(x)$ and $G(x)$ be the corresponding polynomials associated with Golay sequences $F=(f_i)$ and $G=(g_i)$:

$$F(x)=\sum_{i=0}^{s-1} f_{i+1}x^i, \quad G(x)=\sum_{i=0}^{s-1} g_{i+1}x^i.$$

We define the polynomials $f(x)$ and $g(x)$ by using $F(x)$ and $G(x)$,

$$\begin{aligned}
f(x) &= F(x^{2t}) = \sum_{i=0}^{s-1} f_{i+1}x^{2ti},\\
g(x) &= G(x^{2t}) = \sum_{i=0}^{s-1} g_{i+1}x^{2ti}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
f^*(x) &= \sum_{i=0}^{s-1} f_{i+1}x^{2t(s-1-i)} = x^{2t(s-1)}f(x^{-1}),\\
g^*(x) &= \sum_{i=0}^{s-1} g_{i+1}x^{2t(s-1-i)} = x^{2t(s-1)}g(x^{-1}).
\end{aligned}$$

Now we define the polynomials

$$Q(x) = A(x)f^*(x) + C(x)g(x)x^M - B^*(x)x^{2st},$$

$$
\begin{aligned}
R(x) &= B(x)f^*(x) + D(x)g^*(x)x^M + A^*(x)x^{2st}, \\
S(x) &= A(x)g^*(x)x^t - C(x)f(x)x^{M+t} - D^*(x)x^{2st+M}, \\
T(x) &= B(x)g(x)x^t - D(x)f(x)x^{m+t} + C^*(x)x^{2st+M},
\end{aligned}
$$

where $M = m + p$.

For convenience sake, we shall use the same A to represent $A(x)$. A^{-1} stands for $A(x^{-1})$.

$$
\begin{aligned}
Q(x)Q(x^{-1}) &= QQ^{-1} \\
&= (Af^* + Cgx^M - B^*x^{2st})(A^{-1}f^{*^{-1}} + C^{-1}g^{-1}x^{-M} \\
&\quad -B^{*^{-1}}x^{-2st}), \\
&= AA^{-1}f^*f^{*^{-1}} + AC^{-1}f^*g^{-1}x^{-M} - AB^{*^{-1}}f^*x^{-2st} \\
&\quad +CA^{-1}gf^{*^{-1}}x^M + CC^{-1}gg^{-1} \\
&\quad -CB^*gx^{M-2st} - B^*A^{-1}f^{*^{-1}}x^{2st} \\
&\quad -B^*C^{-1}g^{-1}x^{2st-M} + B^*B^{*^{-1}},
\end{aligned}
$$

$$
\begin{aligned}
R(x)R(x^{-1}) &= RR^{-1} \\
&= (Bf^* + Dg^*x^M + A^*x^{2st})(B^{-1}f^{*^{-1}} + D^{-1}g^{*-1}x^{-M} \\
&\quad +A^{*^{-1}}x^{-2st}) \\
&= BB^{-1}f^*f^{*^{-1}} + BD^{-1}f^*g^{*^{-1}}x^{-M} + BA^{*^{-1}}f^*x^{-2st} \\
&\quad +DB^{-1}g^*f^{*^{-1}}x^M + DD^{-1}g^*g^{*^{-1}} \\
&\quad +DA^{*^{-1}}g^*x^{M-2st} + A^*B^{-1}f^{*^{-1}}x^{2st} \\
&\quad +A^*D^{-1}g^{*^{-1}}x^{2st-M} + A^*A^{*^{-1}},
\end{aligned}
$$

$$
\begin{aligned}
S(x)S(x^{-1}) &= SS^{-1} \\
&= (Ag^* - Cfx^M + D^*x^{(2s-1)t+M})x^t(A^{-1}g^{*^{-1}} - C^{-1}f^{-1}x^{-M} \\
&\quad -D^{*^{-1}}x^{-(2s-1)t-M})x^{-t} \\
&= AA^{-1}g^*g^{*^{-1}} - AC^{-1}g^*f^{-1}x^{-M} - AD^{*^{-1}}g^*x^{-(2s-1)t-M} \\
&\quad -CA^{-1}g^{*^{-1}}fx^M + CC^{-1}ff^{-1} \\
&\quad -CD^{*^{-1}}fx^{-(2s-1)t} \\
&\quad +D^*A^{-1}g^{*^{-1}}x^{(2s-1)t+M} - D^*C^{-1}f^{-1}x^{(2s-1)t} + D^*D^{*^{-1}}, \\
T(x)T(x^{-1}) &= TT^{-1} \\
&= (Bg - Dfx^M + C^*x^{(2s-1)t+M}(B^{-1}g^{-1} - D^{-1}f^{-1}x^{-M} \\
&\quad +C^{*^{-1}}x^{-(2s-1)t-M}) \\
&= BB^{-1}gg^{-1} - BD^{-1}f^{-1}gx^{-M} + BC^{*^{-1}}gx^{-(2s-1)t-M} \\
&\quad -DB^{-1}fg^{-1}x^M + DD^{-1}ff^{-1} - DC^{*^{-1}}fx^{-(2s-1)t} \\
&\quad +C^*B^{-1}g^{-1}x^{(2s-1)t+M} - C^*D^{-1}f^{-1}x^{(2s-1)t} + C^*C^{*^{-1}}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&QQ^{-1}+RR^{-1}+SS^{-1}+TT^{-1}\\
=\;& AA^{-1}f^*f^{*^{-1}}+CC^{-1}gg^{-1}+B^*B^{*^{-1}}+BB^{-1}f^*f^{*^{-1}}+DD^{-1}g^*g^{*^{-1}}\\
&+A^*A^{*^{-1}}+AA^{-1}g^*g^{*^{-1}}\\
&+CC^{-1}ff^{-1}+D^*D^{*^{-1}}+BB^{-1}gg^{-1}+DD^{-1}ff^{-1}\\
&+C^*C^{*^{-1}}\\
=\;& AA^{-1}(f^*f^{*^{-1}}+g^*g^{*^{-1}})+BB^{-1}(f^*f^{*^{-1}}+gg^{-1})+CC^{-1}(ff^{-1}+gg^{-1})\\
&+DD^{-1}(ff^{-1}+g^*g^{*^{-1}})\\
&+A^*A^{*^{-1}}+B^*B^{*^{-1}}+C^*C^{*^{-1}}+D^*D^{*^{-1}}\\
=\;& (AA^{-1}+BB^{-1}+CC^{-1}+DD^{-1})(ff^{-1}+gg^{-1})+AA^{-1}+BB^{-1}\\
&+CC^{-1}+DD^{-1}\\
=\;& (AA^{-1}+BB^{-1}+CC^{-1}+DD^{-1})(ff^{-1}+gg^{-1}+1)\\
=\;& 2t(2s+1).
\end{aligned}
$$

If we define

$$
\begin{aligned}
X(x) &= (Q(x)+R(x))/2,\\
Y(x) &= (Q(x)-R(x))/2,\\
V(x) &= (S(x)+T(x))/2,\\
W(x) &= (S(x)-T(x))/2.
\end{aligned}
$$

then we have

$$
\begin{aligned}
&X(x)X(x^{-1})+Y(x)Y(x^{-1})+V(x)V(x^{-1})+W(x)W(x^{-1})\\
&\quad= \frac{1}{4}\{(Q+R)(Q^{-1}+R^{-1})+(Q-R)(Q^{-1}-R^{-1})\\
&\quad+ (S+T)(S^{-1}+T^{-1})+(S-T)(S^{-1}-T^{-1})\}\\
&\quad= \frac{1}{2}(QQ^{-1}+RR^{-1}+SS^{-1}+TT^{-1})\\
&\quad= t(2s+1),
\end{aligned}
$$

and the corresponding sequences for $X(x), Y(x), Z(x), W(x)$ are disjoint. □

Note: The interesting case for Yang's theorem is for base sequences of lengths $m+p, m+p, m, m$ where p is odd for then Yang's theorem produces T-sequences of odd length $3(2m+p)$.

Restatement 25 (Yang) *Suppose E, F, G, H are base sequences of lengths $m+p, m+p, m, m$. Define $A=\frac{1}{2}(E+F)$, $B=\frac{1}{2}(E-F)$, $C=\frac{1}{2}(G+H)$ and $D=\frac{1}{2}(G-F)$ to be suitable (pairwise disjoint) sequences. Then the following sequences are disjoint T-sequences of length $3(2m+p)$:*

$$
\begin{array}{rcllllll}
X &=& A & C; & 0 & 0'; & B^* & 0'\\
Y &=& B & D; & 0 & 0'; & -A^* & 0'\\
Z &=& 0 & 0'; & A & -C; & 0 & D^*\\
W &=& 0 & 0'; & B & -D; & 0 & -C^*
\end{array}
$$

and

$$\begin{array}{rclcccccc} X &=& B^* & 0'; & A & C; & 0 & 0' \\ Y &=& -A^* & 0'; & B & D; & 0 & 0' \\ Z &=& 0 & D^*; & 0 & 0'; & A & -C \\ W &=& 0 & -C^*; & 0 & 0'; & B & -D \end{array}$$

where 0 and $0'$ *are the zero sequences of lengths* $m+p$ *and* m *respectively.*

Multiplying by 7 and 13

The next two theorems can be used recursively but as the sequences produced are of equal lengths the next recursive use of the theorems gives sequences of (equal) even length.

Theorem 26 (Yang) *Let* (E, F, G, H) *be the base sequences of length* $2m+p = t$ $(TBS(t))$. *Define the suitable sequences* $A = \frac{1}{2}(E+F)$, $B = \frac{1}{2}(E-F)$, $C = \frac{1}{2}(G+H)$, *and* $D = \frac{1}{2}(G-H)$ *of lengths* $m+p, m+p, m$ *and* m. *Then the following* X, Y, Z, W *are 4-disjoint T-sequences of length* $7t$ $(RD(7t))$:

$$\begin{array}{rcl} X &=& (\bar{A}, C;\ \ 0,0;\ \ A, D;\ \ 0,0;\ \ A, C;\ \ 0,0;\ \ \bar{B}', 0), \\ Y &=& (\bar{B}, D;\ \ 0,0;\ \ B, \bar{C};\ \ 0,0;\ \ B, D;\ \ 0,0;\ \ A', 0), \\ Z &=& (0,0;\ \ A, \bar{C};\ \ 0,0;\ \ \bar{B}, \bar{C};\ \ 0,0;\ \ A, C;\ \ 0, \bar{D}'), \\ W &=& (0,0;\ \ B, \bar{D};\ \ 0,0;\ \ A, \bar{D};\ \ 0,0;\ \ B, D;\ \ 0, C'). \end{array}$$

(in Yang's paper on the Lagrange identity for polynomials and δ-codes of length $7t$ *and* $13t$ *[31]).*

Theorem 27 (Yang) *Let* (A, B, C, D) *be the base sequences of length* $2m+p = t$ $(TBS(t))$. *Define the suitable sequences* $A = \frac{1}{2}(E+F)$, $B = \frac{1}{2}(E-F)$, $C = \frac{1}{2}(G+H)$, *and* $D = \frac{1}{2}(G-H)$ *of lengths* $m+p, m+p, m$ *and* m. *Then the following* X, Y, Z, W *are 4-disjoint T-sequences of length* $13t$ $(RD(13t))$.

$$\begin{array}{rl} Q = & (A, D';\ \bar{A}, \bar{C};\ \bar{A}, D';\ \bar{A}, C;\ \bar{A}, D';\ A, \bar{C};\ 0, C; \\ & \quad 0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0), \\ R = & (\bar{B}, C';\ B, D;\ B, C';\ B, \bar{D};\ B, C';\ \bar{B}, D;\ 0, \bar{D}; \\ & \quad 0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0), \\ S = & (0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0;\ \bar{A}, 0; \\ & \quad A, C;\ B', \bar{C};\ \bar{A}, \bar{C};\ B', \bar{C};\ A, \bar{C};\ B', C), \\ T = & (0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0;\ 0,0;\ B, 0; \\ & \quad \bar{B}, \bar{D};\ A', D;\ B, D;\ A', D;\ \bar{B}, D;\ A', \bar{D}). \end{array}$$

(in Yang's paper on the Lagrange identity for polynomials and δ-codes of length $7t$ *and* $13t$ *[31]).*

Yang shows how to multiply by 11. The sequences obtained are not disjoint and so cannot be used in another iteration.

Theorem 28 (Yang) *Let (E, F, G, H) be the base sequences of length $2m+p = t$ $(TBS(t))$. Define the suitable sequences $A = \frac{1}{2}(E+F)$, $B = \frac{1}{2}(E-F)$, $C = \frac{1}{2}(G+H)$, and $D = \frac{1}{2}(G-H)$ of lengths $m+p, m+p, m$ and m. Then the following (Q, R, S, T) are 4 complementary sequences of length $11t$.*

$$\begin{array}{rcllllllllll}
Q & = & (A,C; & \bar{A},\bar{C}; & \bar{B}',\bar{C}; & \bar{A},C; & 0,\bar{D}; & & & & \\
 & & & & & 0,D; & 0,D; & 0,0; & 0,0; & 0,0; & 0,0), \\
R & = & (\bar{A},\bar{C}; & A,D'; & A,\hat{C}; & \bar{A},C; & B,0; & & & & \\
 & & & & & B,0 & B,0; & 0,0; & 0,0; & 0,0; & 0,0), \\
S & = & (0,0; & 0,0; & 0,0; & 0,0; & A,0; & & & & \\
 & & & & & A,0; & A,0; & B,D; & \bar{B},C'; & \bar{B},D; & B,\bar{D}), \\
T & = & (0,0; & 0,0; & 0,0; & 0,0; & 0,C; & & & & \\
 & & & & & 0,C; & 0,C; & \bar{B},\bar{D}; & B,D; & \bar{A}',D; & B,\bar{D}).
\end{array}$$

However Theorem 28 becomes less important when the next result is known.

4 T-sequences and T-matrices of length up to 200

From Table 2 and Golay sequences we can construct 4-disjoint T-sequences for all lengths up to 33, and also for 51 and 59. These are given in Table 3 with the corresponding decomposition into squares.

We now give an example showing how to use Yang's theorem to find sequences $3(2m+p)$ and how the results can be clearly inequivalent as they correspond to different decompositions of $3(2m+p)$ into four squares.

Example Suppose we have $m = 1$. Then four suitable sequences of lengths 2, 2, 1, 1 are

$$\begin{array}{ll}
1 & 0 \\
0 & 1 \\
1 & \\
0 &
\end{array}$$

Note that A and B are disjoint and C and D are disjoint.

Suppose we want to use these in Yang's construction to multiply by 3 finding sequences of length 9. The construction gives the T-sequences

$$\begin{array}{rcl}
X & = & A\,C;\ 0\,0;\ B^*\,0 \\
Y & = & B\,D;\ 0\,0;\ -A^*\,0 \\
Z & = & 0\,0;\ A-C;\ 0\,D^* \\
W & = & 0\,0;\ B-D;\ 0-C^*
\end{array}$$

Use the suitable sequences $A = 1\,0, B = 0\,1, C = 1, D = 0$ to get

$$\begin{array}{rcccccccccc}
X & = & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
Y & = & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
Z & = & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
W & = & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1
\end{array}$$

Length	Sum of Squares	Sequences
3	$1^2 + 1^2 + 1^2$	1 0 0, 0 1 0, 0 0 1, 0 0 0
5	$2^2 + 1^2$	1 1 0 0 0, 0 0-1 1 0, 0 0 0 0 1, 0 0 0 0 0
7	$2^2 + 1^2 + 1^2 + 1^2$	-1 1 1 0 1 0 0, 0 0 0 1 0 0 0, 0 0 0 0 0 1 0, 0 0 0 0 0 0 1
9	3^2	1 1 0 1 0 0 0 0 0, 0 0 1 0 0-1 0 0 0, 0 0 0 0 1 0-1 0 0, 0 0 0 0 0 0 0 1-1
9	$2^2 + 2^2 + 1^2$	1 1 1-1 0 0 0 0 0, 0 0 0 0 1 1-1 1 0, 0 0 0 0 0 0 0 0 1, 0 0 0 0 0 0 0 0 0
11	$3^2 + 1^2 + 1^2$	1-1-1 0 1 0 1 0 0 0 0, 0 0 0 1 0 1 0 1 1-1 0, 0 0 0 0 0 0 0 0 0 0 1, 0 0 0 0 0 0 0 0 0 0 0
13	$3^2 + 2^2$	0 0 1 0 1 0 1 0 0 0 0 0 0, 1 1 0-1 0 1 0 0 0 0 0 0 0, 0 0 0 0 0 0 0 1 1-1 1-1-1, 0 0 0 0 0 0 0 0 0 0 0 0 0
13	$2^2 + 2^2 + 2^2 + 1^2$	-1 0 1 1 0 1 0 0 0 0 0 0 0, 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1, 0 0 0 0 0 0 0 1 1-1 0-1 1 0
15	$3^2 + 2^2 + 1^2 + 1^2$	0 1 1-1 1 1 0 0 0 0 0 0 0 0 0, 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1-1 0-1 0, 0 0 0 0 0 0 0 0 0 0 1 0 0-1 0 1
17	$4^2 + 1^2$	0 1 1-1 1 1 1 1-1 0 0 0 0 0 0 0 0, 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0, 0 0 0 0 0 0 0 0 0 0 1 1-1 1-1-1-1 1
17	$3^2 + 2^2 + 2^2$	-1 1 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0, 0 0 1-1-1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 1, 0 0 0 0 0 0 0 0-1 1 0 1 1 1 0-1 0
19	$4^2 + 1^2 + 1^2 + 1^2$	1-1 1 1 0-1 1 1 1 0 0 0 0 0 0 0 0 0 0, 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0-1, 0 0 0 0 0 0 0 0 0 0 1 1 1-1 0 0-1 1-1 0
19	$3^2 + 3^2 + 1^2$	0 1 1 1 1-1-1 1-1 1 0 0 0 0 0 0 0 0 0, 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 1 0-1, 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0, 0 0 0 0 0 0 0 0 0 0 0 0-1 0 1 0-1 0 1 0
21	$4^2 + 2^2 + 1^2$	0 1-1-1 1 1 1 1 1 1-1 0 0 0 0 0 0 0 0 0 0, 0_{11}-1 1 1 1-1 1-1 1 1 1-1, 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0, 0_{21}
21	$4^2 + 2^2 + 1^2$	1 1-1 0 1 0 1 0 0 0 0 0 0 0-1 0 0 0 0 0 0, 0 0 0-1 0 1 0_9 1-1-1 0 0 0, 0 0 0 0 0 0 0 1 1-1 0-1 0-1 0 0 0 0 0 1 0, 0_{10}-1 0-1 0 0 0 0 0-1 0-1
21	$3^2 + 2^2 + 2^2 + 2^2$	-1 0 1 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0, 0 1 0-1 1-1 1 0 1 0_{12} 0_{11} 1 0 0 0 0 0 0 0 1 1, 0 0 0 0 0 0 0 0 0 0-1 0 1-1-1 1 1 1 1 0 0
23	$3^2 + 3^2 + 2^2 + 1^2$	0 1 1 1 1 1-1-1 1-1 1-1 0 0 0 0 0 0 0 0 0 0 0, 0_{13} 1 0 1 0-1 0 1 0 1 0, 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0-1 0 1 0-1 0 1, 1 0_{22}
25	5^2	see text
25	5^2	0 1-1 1 1 0 0 0 0 0 0-1 1 1 1 0 0 0 0 0 0 0 1 0 0, 1 0 0 0 0 0 0 0 0 0-1 0 0 0 0 0 0 0 0 0 1-1 0 0 0, 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0-1-1, 0 0 0 0 0 0 1-1 1 1 0 0 0 0 0 0 1-1-1-1 0 0 0 0 0
25	$4^2 + 3^2$	1 1 1 0-1 1-1 1-1 0 1 1 0 0_{12}, 0 0 0 1 0 0 0 0 0 1 0 0 1 0_{12}, 0_{13} 0 0 0 0 1 0 0-1 0 0 0 0, 0_{13} 1 1 1-1 0 1-1 0 1-1-1-1
25	$4^2 + 2^2 + 2^2 + 1^2$	1 1 1 0 0 0 0 0 0-1 1 1 0 0_{13}, 0 0 0 1 1-1 1-1 1 0 0 0 0_{13}, 0_{12} 1 1 1 1-1 1-1-1 1 1-1-1 0, 0_{13} 0 0 0 0 0 0 0 0 0 0 0 1
27	$5^2 + 1^2 + 1^2$	0_{13} 1 0 1-1 1-1-1 1 1-1 1 1 1 1, 1 0_{26}, 0 1 1 1 -1-1 1 1 1-1 1-1-1 0-1 0_{12}, 0_{27}
27	$4^2 + 3^2 + 1^2 + 1^2$	0 0 0 0 0 0 0 0 0 1 1 1-1 0 0 0 0 0 0 0 0 0 0 1-1 1 1, 1 1 1-1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0, 0 0 0 0 1 1 1-1 1 0 0 0 0 0 0 0 0 0 0 0 1-1-1-1 0 0 0 0, 0 0 0 0 0 0 0 0 0 0 0 0 0-1 1 1-1 1 0 0 0 0 0 0 0 0 0
27	$3^2 + 3^2 + 3^2$	1 0-1 0 0 0 0 0 0 1 1 0 1 0 1 0 0-1 0 0 0 0 0 0 0 0 0, 0 1 0 1 1 0 0 0 0 0 0 1 0-1 0-1 1 0 0 0 0 0 0 0 0 0 0, 0 0 0 0 0 0-1 1 0 0 0 0 0 0 0 0 0 0 1 1 0 1 0-1 0 0 1, 0 0 0 0 0 1 0 0-1 0 0 0 0 0 0 0 0 0 0 0-1 0 1 0-1 1 0
29	$5^2 + 2^2$	1 1 0 0 1 1-1 0-1 1 1 0 0 1 0 0_{14}, 0 0-1 1 0 0 0 1 0 0 0 1-1 0 1 0_{14}, 0_{15} 1 0 0 0-1 0 0 0 0 1 0 0 0-1, 0_{15} 0 1 1 1 0-1 1-1 1 0-1-1-1 0

Table 3: 4-disjoint T-sequences

Length	Sum of Squares	Sequences
29	$4^2 + 3^2 + 2^2$	1-1 0 0 1 1 0 0 1-1 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0, 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0-1-1 0 1 1 0 0 0 1 1 0 1-1 1, 0 0-1 1 0 0 1 1 0 0 1-1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0, 0 0 0 0 0 0 0 0 0 0 0 0 0 0-1 0 0 1 0 0 1 1-1 0 0-1 0 0 0
31	$5^2 + 2^2 + 1^2 + 1^2$	0 1 1 1 1-1 1 1-1-1 1 1 1-1 1-1 0_{15}, 0_{16} 1 0 1 0 1 0 1 0-1 0-1 0-1 0 1, 0_{16} 0-1 0 1 0 1 0-1 0 1 0 1 0-1 0, 1 0_{30}
31	$3^2 + 3^2 + 3^2 + 2^2$	T-matrices are known, see text.
33	$5^2 + 2^2 + 2^2$	1 1 0 0-1 0 1 1-1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1-1 0 0 0 0 0 0 0, 0 0 0 0 0 0 0 0 0 0-1-1 0 0 1 0 1 1-1 1 1 0 0 0 0 0 0 0 0 0 0 0, 0 0 0 0 0 0 0 0 0 0 0 0 0 1-1 0-1 0 0 0 0 0 0 0 0 0 0 0 1 1-1 1 1, 0 0 1-1 0-1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0-1 0 0 1 1 0 0 0 0 0
33	$4^2 + 4^2 + 1^2$	Let X, Y be Golay sequences each of length 16 and row sum 4. The sequences are X 0_{17}, 0_{16} Y 0, 0_{32} 1, 0_{33}
33	$4^2 + 3^2 + 2^2 + 2^2$	0_{11} 0 0-1 1 0 1 0_{11} 1 1-1 1 1, 1 1 0 0-1 0 1 1-1 1 1 0_{11}-1 0-1 1 0 0 0_5 0_{11}-1-1 0 0 1 0 1 1-1 1 1 0_{11}, 0 0-1 1 0 1 0 0 0 0 0 0_{11} 0-1 0 0 1 1 0_5
35	$5^2 + 3^2 + 1^2$	1 1 0 1 1 0 0 0 1-1 0_3 1-1 0 1 0_{18}, 0 0-1 0 0 1 1-1 0 0-1 1 1 0 0 1 0 1 0_{17}, 0_{18} 0-1 0-1 0 1 0 0-1 0 0 1 0 1 0 1 0, 0_{18} 1 0-1 0 1 0-1-1 0-1 1 0 1 0 1 0-1
35	$4^2 + 3^2 + 3^2 + 1^2$	0_7 1-1 1 0_5 1 1 0-1 1 0_7 1 1-1 0_5, 0_5 1 1 0_{10}-1 0_7 1 1 0_6 1-1, 0 0-1 1-1 0_7 1 0_9 1 1-1 0_6 1 1 0 0, -1-1 0_8 1 1 0 1-1 0_5 1 1 0_8-1 0 0 0 0
37	$6^2 + 1^2$	T-matrices are known, see text
37	$5^2 + 2^2 + 2^2 + 2^2$	
37	$4^2 + 4^2 + 2^2 + 1^2$	
39	$6^2 + 1^2 + 1^2 + 1^2$	0_{16} 1 0 0 1-1 0 1 1 0 1 0 0 0 0 0 0 0 0 0 1 0 0 1 0, 0_{13} 1 1-1 0-1 1 0 0 1 0 0 1 0_8-1 0-1-1 0 1, 0 0 0-1 0 0-1-1 0 1 1 0 1 0_{14} 1-1 0-1 1 1 0 0 0 0 0 0, 1 1-1 0-1 1 0 0-1 0 0-1 0_{14} 1 0 0 1 0 0 0 0 0 0 0 0 0
39	$5^2 + 3^2 + 2^2 + 1^2$	0 0 1 0 1 0 1 0_{20} 1 0-1 0 1 1 0_6, 0_{15} 1 0 1 0 1 0_{13}-1-1 1-1 1 1, 0_{13} 1 1 0-1 0 1 0-1-1 1-1 1 1 0_{13}, -1-1 0 1 0-1 0-1-1 1-1 1 1 0_{13} 1 0 1 0 1 0_8
41	$6^2 + 2^2 + 1^2$	Let X, Y be Golay sequences each of length 10 and row sums 4 and 2. The sequences are 0 X 0_{10} Y 0_{10}, 0_{11} X 0_{10} -Y, 1 0_{40}, 0_{41}
41	$6^2 + 2^2 + 1^2$	1-1-1 1 1 1 1 1 1-1-1 1 1-1 1-1 1 1 1 1-1 0_{21}, 0_{21} 1-1-1 1 1 1 1 1 1-1 1-1-1 1-1 1-1-1-1 1, 0_{20} 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0, 0_{41}
41	$5^2 + 4^2$	T-matrices are known, see text.
41	$4^2 + 4^2 + 3^2$	
43	$5^2 + 4^2 + 1^2 + 1^2$	
43	$5^2 + 3^2 + 3^2$	
43	$4^2 + 3^2 + 3^2 + 3^2$	
45	$6^2 + 3^2$	0_{17} 1 0 0-1 0 1 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 1-1 1 1 0, 1 1 0 1-1 0-1 0 0 1 1-1 1 1 0_{16}-1 0 1 0 0-1 0 0 0 0 0 0 0 0 0, 0_{15} 1-1 0 1-1 0-1 0 0-1-1 1-1-1 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1, 0 0-1 0 0 1 0-1 1 0 0 0 0 0 1 0_{16} 1 0 1-1 0-1-1 0 0 0 0 0 0 0
45	$6^2 + 2^2 + 2^2 + 1^2$	-1 0 1 0 0-1 0 0 0 0 0 0 0 0 0 1 1 0 1-1 0-1 0 1 0 0 0 0 0 1 0_{15}, 0 1 0 1-1 0-1-1 0 0 0 0 0 0 0 0 0-1 0 0 1 0-1 0 1 1-1 1 1 0_{16}, 0 0 0 0 0 0 0 0 0 1 1-1 1 1 0_{16} 1 1 0 1-1 0-1 0-1 0 0 0 0 0-1, 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0_{27} 1 0 0-1 0 1 0 1 1-1 1 1 0

Table 3(cont): 4-disjoint T-sequences

Length	Sum of Squares	Sequences
45	$5^2 + 4^2 + 2^2$	1 1 0 1-1 0-1 0 0 1 1-1 1 1 0_{16} 1 0-1 0 0 1 0 0 0 0 0 0 0 0 0, 0_{15}-1-1 0-1 1 0 1 0 0 1 1-1 1 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1, 0 0-1 0 0 1 0-1 1 0 0 0 0 0 1 0_{15}-1 0-1 1 0 1 1 0 0 0 0 0 0 0, 0_{17}-1 0 0 1 0-1-1 0 0 0 0 0-1 0 0 0 0 0 0 0 0 0 1 1-1 1 1 0
45	$4^2 + 4^2 + 3^2 + 2^2$	0_{15}-1-1 0-1 1 0 1 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 1-1 1 1 0, 1 1 0 1-1 0-1 0 1 0 0 0 0 0 1 0_{15} 1 0-1 0 0 1 0 0 0 0 0 0 0 0 0, 0 0-1 0 0 1 0-1 0 1 1-1 1 1 0_{17}-1 0-1 1 0 1 1 0 0 0 0 0 0 0, 0_{17} 1 0 0-1 0 1 0 1 1-1 1 1 0 0 0 0 0 0 0 0 0-1 0 0 0 0 0-1
47	$6^2 + 3^2 + 1^2 + 1^2$	
47	$5^2 + 3^2 + 3^2 + 2^2$	1-1-1 0 0-1 1-1 0_8 1-1-1 0 0-1-1 0_{24}, 0 0 0-1 1 0 0 0-1-1-1 1 1 1 1 1 1 0 0 0 1-1 0 0 1 0_{23}, 0_{26}-1 0 1 0 0 0 0 1-1 1 1 1 0 0 0 0 1 0-1 0 0, 0_{24} 1 1 0-1 0-1 1 1-1 0 0 0 0 0-1 1-1-1 0-1-1 1
49	7^2	-1-1 1 0 1 0 1 0_7 1 1-1 0 0 1 0_8 1 1-1 0 1 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0, 0 0 0 1 0 1 0_{11}-1-1 0-1 0_{10}-1 0 1 0 0 0 0 0 0 0 0 0-1 1 1 0 0 0, 0_7 1 1-1 0-1 0-1 0_{10} 1-1 0-1 0 0 0 0 0 0 0 1 1-1 0 1 0 1 0 0 0 0 0-1 0, 0_{10}-1 0-1 0 0 0 0 0 0 0 0 1 1-1 0 0-1 0_{11}-1 0 1 0 0 0 0 0 1 0 1
49	$6^2 + 3^2 + 2^2$	0_{10} 1 0 1 0_8 1 1-1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0-1 0 0 0 0 0 1 0 1, -1-1 1 0 1 0 1 0_7 1 1-1 0 0-1 0_8 1 1-1 0 1 0 1 0_7-1 0 0 0 0 0 0, 0 0 0-1 0-1 0_{11} 1-1 0-1 0_{10} 1 0-1 0 0 0 0 0 0 0 0 0-1 1 1 0 0 0, 0 0 0 0 0 0 0 1 1-1 0-1 0-1 0_{10}-1-1 0-1 0_7 1 1-1 0 1 0 1 0 0 0 0 0 1 0,
49	$5^2 + 4^2 + 2^2 + 2^2$	-1-1 1 0 1 0 1 0_7 1 1-1 0 0 1 0_8 1 1-1 0 1 0 1 0_7-1 0 0 0 0 0 0, 0_{10} 1 0-1 0_8 1 1-1 0 0-1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1, 0 0 0-1 0 1 0_{11} 1-1 0-1 0_{10} 1 0 1 0 0 0 0 0 0 0 0 0-1 1 1 0 0 0, 0_7-1-1 1 0 1 0 1 0_{10} 1 1 0 1 0 0 0 0 0 0 0-1-1 1 0-1 0-1 0 0 0 0 0 1 0
49	$4^2 + 4^2 + 4^2 + 1^2$	0 0 0 1 0 1 0_{11}-1 1 0 1 0_{10}-1 0 1 0 0 0 0 0 0 0 0 0 1 1-1 0 0 0, 0_7-1 1 1 0 1 0 1 0_{10} 1 1 0 1 0 0 0 0 0 0 0-1 1 1 0-1 0-1 0 0 0 0 0-1 0, 0_{10} 1 0 1 0_8 1-1-1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0-1 0 0 0 0 0 1 0 1, -1 1 1 0 1 0 1 0_7 1-1-1 0 0-1 0_8 1-1-1 0 1 0 1 0_7-1 0 0 0 0 0 0
51	$7^2 + 1^2 + 1^2$	0 1 1 1 1 1 1 1 1-1-1 1 1 1 1-1-1 1-1-1 1 1-1 1-1 1-1 1 1 0_{25}, 1 0_{50} 0_{26} 1 0 1 0 1 0-1 0-1 0 1 0-1 0 1 0-1 0-1 0 1 0 1 0-1, 0_{26} 0 1 0 1 0 1 0-1 0-1 0 1 0-1 0 1 0 1 0-1 0-1 0-1 0,
51	$5^2 + 5^2 + 1^2$	-1 1 0 1 1 1 0-1 0 0 0 1-1-1 1 0_{19} 1 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0, 0_{19} 1 0 0 0 1 0 1-1 1 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0-1 1 1-1 0 0, 0 0-1 0 0 0-1 0-1-1 1 0 0 0 0 1 1 0_{18}-1 0 1 1 1 0 1-1 0 0 0 0 0 0 0 0, 0_{17}-1 1 0 1 1 1 0-1 0 0 0-1 1 1-1 0 0 0 0 0 0 0 0 0 0 0-1-1 0 0 0 0-1 1
51	$5^2 + 4^2 + 3^2 + 1^2$	1 0 0 0 0 0 0 0 0 1 1 1-1-1-1 1-1 0_{17} 1-1 1 1-1 1 1 1 0 0 0 0 0 0 0 0 0, 0_{18} 1 1 1-1 1 1-1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1-1 1 1 1-1-1-1, 0 1 1 1-1 1 1-1 1 0_{33}-1 0 0 0 0 0 0 0 0, 0_{17} 1 0 0 0 0 0 0 0 0-1-1-1 1 1 1-1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
53	$7^2 + 2^2$	
53	$6^2 + 4^2 + 1^2$	Let X, Y be Golay sequences each of length 26 and row sums 6 and 4. The sequences are 0 X 0_{26}, 0_{27} Y, 1 0_{52}, 0_{53}
53	$6^2 + 3^2 + 2^2 + 2^2$	
55	$7^2 + 2^2 + 1^2 + 1^2$	0_{11} 1-1 1 1 1 1 0 1 0 0 0-1 0_{11} 1-1 1 1 1 0-1 0 0 0 1 0 0 0 0 0 0 0 0-1 1 1 0, 0_{16} 1 0-1-1 1 0_{17} 1 0 1 1-1 0 0 0 0 0 0 0 0-1 0 0 0 1, -1 1-1-1-1 0-1 0 0 0 1 0_{11} 1-1 1 1 1 0-1 0 0 0 1 0_{11} 1 0 0 0 0 0 0 0 0 0 0, 0 0 0 0 0 1 0-1-1 1 0_{17}-1 0-1-1 1 0_{13} 1 1 1-1 1 0 0 0 0 0

Table 3(cont): 4-disjoint T-sequences

Length	Sum of Squares	Sequences
55	$6^2 + 3^2 + 3^2 + 1^2$	0 1 0 0 1 1-1 1-1 1 1 0_{12}-1 0 0-1-1-1 1-1 1 1 0_{13}-1-1 0 1 0 0 0 0 0, -1 0 1 1 0_{18} 1 0-1-1 0_{18} 1 1 0 0 1 0 0 0 0 0 0, 0_{12} 1 0 0 1 1-1 1-1 1 1 0_{12} 1 0 0 1 1 1-1 1-1-1 0 0 0 0 0 0 0 0 0 0 0, 0 0 0 0 0 0 0 0 0 0 0-1 0 1 1 0_{18}-1 0 1 1 0_{13} 1 1-1 1-1
55	$5^2 + 5^2 + 2^2 + 1^2$	1 1 0 0-1 0 1 1-1 1 1 0_{11}-1-1 0 0 1 0 1 1-1 1 1 0_{11}-1 0-1 1 0 0 0 0 0 0 0, 0 0 1-1 0-1 0_{18}-1 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0-1-1 0 0 0 0 0, 0_{11} 1 1 0 0-1 0 1 1-1 1 1 0_{11} 1 1 0 0-1 0-1-1 1-1-1 0 0 0 0 0 0 0 0 0 0 0, 0 0 0 0 0 0 0 0 0 0 0 0 0 1-1 0-1 0_{18} 1-1 0-1 0 0 0 0 0 0 0 0 0 0 0 0-1-1 1-1-1
57	$7^2 + 2^2 + 2^2$	0 1 1 1 1-1-1 1-1 1 1 0 1 0 1 0 1 0-1 0_{28} 1 0_9, 1 0_{10}-1 0 1 0-1 0 1 0_{20}-1 1-1 1 1-1-1-1-1 0_{10}, 0_{19} 1 0_{10} 1 0-1 0 1 0-1 0_{10} 0 1 0-1 0-1 0-1 0-1 0_{20} 1 1 1 1-1-1 1-1 1-1 0-1 0-1 0-1 0 1 0_{10} 0 1 0-1 0 1 0-1 0,
57	$6^2 + 4^2 + 2^2 + 1^2$	see text
57	$5^2 + 4^2 + 4^2$	see text
57	$4^2 + 4^2 + 4^2 + 3^2$	0 0 0 0 1 1 0 0 0-1 1-1 1 1 0-1 1 1 1 0_{20} 1-1 1 0 0 1-1-1-1 0 0 0 0 0 0 0 0 0, 1 1 1-1 0 0-1 1-1 0 0 0 0 0 1 0_{23}-1 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0, 0_{23}-1-1 0 0 0 1 1-1 1 1 0-1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0, 0_{19} 1 1 1-1 0 0-1 1-1 0 0 0 0 0-1 0_{14} 1 1 1-1 0 1 1-1 1
59	$7^2 + 3^2 + 1^2$	0 1 1 1 1-1 1 1 1 1-1 1-1-1 1 1-1-1 1 1 1 1 1-1 1-1-1-1 1-1 1 0_{29}, 0_{28} 1 0 1 0 1 0-1 0 1 0 1 0-1 0-1 0-1 0 1 0 1 0-1 0 1 0 1 0-1, 1 0_{58}, 0_{28} 0 1 0-1 0-1 0-1 0-1 0 1 0-1 0 1 0-1 0 1 0 1 0 1 0 1 0-1 0
59	$5^2 + 5^2 + 3^2$	
59	$5^2 + 4^2 + 3^2 + 3^2$	
61	$7^2 + 2^2 + 2^2 + 2^2$	
61	$6^2 + 5^2$	T-matrices are known, see text
61	$6^2 + 4^2 + 3^2$	
61	$5^2 + 4^2 + 4^2 + 2^2$	
63	$7^2 + 3^2 + 2^2 + 1^2$	see text
63	$7^2 + 3^2 + 2^2 + 1^2$	0 1 0 0 0 0 0 0 0 1 1-1 0 1 0 0 0 0 1 0 1 0_{23}-1-1-1-1 1 1-1 0 1 0_{10}, -1 0 1-1-1 1 1 1 1 1 0 0 0 1 0-1 1-1 1 0 1 0_{22} 1 1 0 0 0 0 0 0 0 0 1 0_{11}, 0_{22}-1 0 0 0 0 0 0 0-1-1-1 0 1 0 0 0 0 1 0 1 0_{12} 1 0 1-1 1-1 0 1 0, 0_{21}-1 0 1-1-1 1 1 1 1 1 0 0 0-1 0 1-1 1-1 0-1 0_{12} 1 0 1 0 0 0 0 1 0-1
63	$6^2 + 5^2 + 1^2 + 1^2$	see text
63	$6^2 + 5^2 + 1^2 + 1^2$	0 0-1 0-1 1 0 0 1 0_{11} 1 0 1 0 1-1 0_{12} 1 0 1 1 0 0 1 0_{10} 1 0-1-1 0 0 0 0, -1-1 0 1 0 0 1-1 0_{10} 1 1 0-1 0-1 0 0-1 0_9 1 1 0-1 0 0 1-1 0_{10} 1 0 1 0_6, 0_{11} 1 0 1-1 0 0-1 0_9-1-1 0 1 0-1 0 0-1 0_{11} 1 0 1 1 0 0 1 0_6 1-1 0, 0_9 1 1 0-1 0 0-1 1 0_{12} 1 0 1 0-1 1 0_{10} 1 1 0-1 0 0 1-1 0_6 1 0 0 1
63	$6^2 + 3^2 + 3^2 + 3^2$	0_9 1 1 0 1 0-1 0 0 1 0_{10} 0-1 0 1-1 0 0 1 0_9 1 1 0 1 0 1 0 0-1 0_6-1 1 0, -1-1 0-1 0 1 0 0-1 0_9 1 1 0 1 0 0-1 1 0_{10} 1 1 0 1 0 1 0 0-1 0_9 1 0-1 0_6, 0 0-1 0 1 0-1 1 0_{12} 1 0-1-1 0 0 1 0_{11} 1 0-1 0-1 1 0_{11} 1 0 1 1 0 0 0 0, 0_{11} 1 0-1 0 1-1 0_{10} 1 1 0 1 0 0 1-1 0_{12} 1 0-1 0-1 1 0_5 0-1 0 0 1
63	$5^2 + 5^2 + 3^2 + 2^2$	1 0-1 0 0 0-1 0 1 0 0 0-1 0 1 0 0 0 1 0_5 1 0_5 1 0_5 1 0_5 1 0 1... 0 0 0 1 0 1 0 0 0-1 0-1 0 0 0-1 0 0 0 1 0_5-1 0_5-1 0_5 1 1 0_4 1-1 0_4 1 1 0_4 1-1 0 0 0 0 1 0_5 1 0_5-1 0_5 1 0 0 0 0 1 0-1 0_3-1 0 1 0_3-1 0 1 0_4 1 1 0_4-1 1 0_4 1 1 0_4-1 1... 0 0 0-1 0-1 0 0 0-1 0-1 0 0 0 1 0 1 0_3, 0 0 0 0-1 0_5 1 0_5 1 0 0 0 0-1 0_5 1 0_5-1 0_5 1 0_6 1 0_5 1 0_5-1 0 0 0 1

Table 3(cont): 4-disjoint T-sequences

Length	Sum of Squares	Sequences
65	$8^2 + 1^2$	0_{33} 1 1 1 1 1-1-1 1 1-1 1-1 1 1-1-1 1 1 1 1 1 1-1-1 1-1 1-1 1-1-1 1 1 1, 0_{32} 1 0_{32}, 1 1 1 1 1-1-1 1 1-1 1-1 1 1 1-1-1-1-1-1-1-1 1 1-1 1-1 1-1 1 1-1-1 0_{33}, 0_{65}
65	$7^2 + 4^2$	0_{31} 1-1 0 0 0-1 1-1-1 0 0 1 1 1 0 1-1 1 1 0 0 1 1 1 0-1 1 1 1 0 0 1-1-1, 1 0 0 1 1-1 0 0 0 0-1 0 0 1 1-1 0 0 0 0-1 0 0 1 1 1 0_{39}, 0-1 1 0 0 0 1-1 1 1 0 1-1 0 0 0 1-1-1-1 0 1-1 0 0 0-1 1 1 1 0 0 0-1-1 0_{30}, 0_{30}-1 0 0 0 0 1 0 0 0 0-1 1 0 0 0-1 0 0 0 0-1 1 0 0 0 1 0 0 0 0-1 1 0 0 0
65	$6^2 + 5^2 + 2^2$	0_{16} 1 0 0 1 0 1 0 0 1 0_{17} 1 0 0 1 0-1 0 0-1 0 0 0 0 0 0 0 0 0 1 0 1 1 0-1, 0 0 0 1 0 0 1 0 1 0 0 1 0_{17}-1 0 0-1 0 1 0 0 1 0_{15} 1-1 0-1 1 1 0_{6}, 1 1-1 0-1 1 0-1 0 1 1 0 1 0_{13}-1-1 1 0 1-1 0-1 0 1 1 0 1 0_{13}-1 0 0-1 0_{9}, 0_{13} 1 1-1 0-1 1 0-1 0 1 1 0 1 0_{13} 1 1-1 0-1 1 0 1 0-1-1 0-1 0_{8}-1 0 0-1 0
65	$6^2 + 4^2 + 3^2 + 2^2$	-1-1 0 0 0 1 1 0 1-1 1 1 0 0 0 1 1 0-1 1 1 1 0 0 0-1-1 0 1-1 0 0 0-1 1 0_{30}, 0 0-1-1 1 0 0 1 0 0 0 0 1-1 1 0 0 1 0 0 0 0 1-1 1 0 0-1 0_{37}, 0_{30}-1-1 0 0 0 1 1 0 1-1 1 0 0-1 1-1-1 0-1 1 1 0 0-1 1 1 1 0-1 1 1 0 0 1-1, 0_{32} 1 0 0 0 0-1 0 0 0 1 1 0 0 0 0 1 0 0 0 1 1 0 0 0 0-1 0 0 0 1 1 0 0
67	$8^2 + 1^2 + 1^2 + 1^2$	T-matrices are known, see text
67	$7^2 + 4^2 + 1^2 + 1^2$	
67	$7^2 + 3^2 + 3^2$	
67	$5^2 + 5^2 + 4^2 + 1^2$	
69	$8^2 + 2^2 + 1^2$	0 0 0 1-1 0 0 1 1 0 0 1-1 0 1 1 1 0 1-1 1 0-1 0_{24} 1 1 0 0-1 1 0 0 1 1-1 0_{31}, 0_{23}-1 1 1 0 0 1-1 0 0 1 1 0 0-1 0 0 0 1 0 0 0 1 0_{13} 1 0-1 1-1 0-1-1-1 0 1, 1-1-1 0 0-1 1 0 0-1-1 0 0-1 0 0 0 1 0 0 0 1 0_{24} 1 0 0 1 1 0 0-1 1 0_{14}, 0_{26} 1-1 0 0 1 1 0 0 1 1 0-1-1-1 0-1 1-1 0 1 0_{13}-1 0 0 0-1 0 0 0 1 0
69	$7^2 + 4^2 + 2^2$	0 0 0 1-1 0 0 1 1 0 0 1-1 0 1 1 1 0 1-1 1 0-1 0_{24}-1-1 0 0 1-1 0 0-1-1 1 0_{11}, -1 1 1 0 0 1-1 0 0 1 1 0 0-1 0 0 0 1 0 0 0 1 0_{23} 1 0 0 1 1 0 0-1 1 0_{14}, 0_{26} 1-1 0 0 1 1 0 0 1 1 0-1-1-1 0-1 1-1 0 1 0_{13}-1 0 0 0-1 0 0 0 1 0, 0_{23}-1 1 1 0 0 1-1 0 0 1 1 0 0 1 0 0 0-1 0 0 0-1 0_{13}-1 0 1-1 1 0 1 1 1 0-1
69	$6^2 + 4^2 + 4^2 + 1^2$	0 1 1 1 1 1-1-1 1-1 1-1 1 0 1 0-1 0 1 0-1 0 1 0_{34} 1 0_{11}, 0_{24} 1 1 1 1 1-1-1 1-1 1-1-1 0-1 0 1 0-1 0 1 0-1 0_{13} 1 0 1 0-1 0 1 0 1 0, 0_{23}-1 0_{12} 1 0 1 0-1 0 1 0 1 0_{13} 1 0-1 0 1 0-1 0 1 0 1, 1 0_{12} 1 0 1 0-1 0 1 0 1 0_{24} 1-1 1-1 1 1-1-1-1-1-1 0_{12}
69	$6^2 + 5^2 + 2^2 + 2^2$	1 0_{11} 1 0 1 0-1 0 1 0-1 0 1 0_{23}-1 1-1 1-1-1 1 1 1 1 1 0_{12}, 0 1 1 1 1 1-1-1 1-1 1-1 0 1 0 1 0-1 0 1 0 1 0_{35}-1 0_{11}, 0_{23} 1 0_{11}-1 0-1 0 1 0-1 0 1 0-1 0_{13} 1 0 1 0-1 0 1 0 1 0, 0_{24}-1-1-1-1-1 1 1-1 1-1 1 0 1 0 1 0-1 0 1 0 1 0_{13} 1 0-1 0 1 0-1 0 1 0 1
71		Not known for any decomposition into squares.
73		Not known for any decomposition into squares.
75	$8^2 + 3^2 + 1^2 + 1^2$	0_{16} 1 1 0 0 1-1 1-1 1 0-1 0 1 1 0_{16} 1 1 0 0 1-1 1 1-1 0 1 0-1-1 0_{10} 1 0 1 0 0, 0 1 1 0 0 1-1 1-1 1 0-1 0 1 1 0_{16}-1-1 0 0-1 1-1-1 1 0-1 0 1 1 0_{18} 1 1 0 0-1 0_{7}, -1 0 0 1 1 0 0 0 0 0 1 0 1 0_{17} 1 0 0-1-1 0 0 0 0 0 1 0 1 0_{17}-1 1-1 0 0-1-1 0_{8}, 0_{15}-1 0 0 1 1 0 0 0 0 0 1 0 1 0_{17}-1 0 0 1 1 0 0 0 0 0-1 0-1 0_{10}-1-1 0 1 0-1 1
75	$7^2 + 5^2 + 1^2$	0_{16} 1 1 0 0 1-1 1 0 0 1 0 1 0 0_{18} 1 1 0 0 1-1 1 0 0-1 0-1 0_{10} 1 1 0-1 0 1-1, 0 1 1 0 0 1-1 1 0 0 1 0 1 0_{18}-1-1 0 0-1 1-1 0 0 1 0 1 0_{20} 1 1 0 0-1 0_{7}, 1 0 0-1-1 0 0 0 1-1 0 1 0-1-1 0_{15}-1 0 0 1 1 0 0 0 1-1 0 1 0-1-1 0_{15} 1-1 1 0 0 1 1 0_{8}, 0_{15} 1 0 0-1-1 0 0 0 1-1 0 1 0-1-1 0_{15} 1 0 0-1-1 0 0 0-1 1 0-1 0 1 1 0_{10} 1 0 1 0 0
75	$7^2 + 4^2 + 3^2 + 1^2$	0 1 0 1-1 1 0-1-1 1 0 1 1 0 1 0 0 0 1 1 0 0 0 1 0_{28}-1 0 0 0 1 0 0 0-1 0 1 0_{12}, -1 0 1 0_{3}-1 0_{3} 1 0 0 1 0 1 1-1 0 0 1-1-1 0-1 0_{25} 1 1 0 1-1-1 0 1-1 1 0 1 0_{13}, 0_{26}-1 0-1 1-1 0 1 1-1 0-1-1 0 1 0 0 0 1 1 0 0 0 1 0_{14}-1 0-1-1 1 0 0-1 1 1 0 1, 0_{25}-1 0 1 0_{3}-1 0 0 0 1 0 0-1 0-1-1 1 0 0-1 1 1 0 1 0_{14} 1 0 0 0 1 1 0 0 0 1 0

Table 3(cont): 4-disjoint T-sequences

Length	Sum of Squares	Sequences
75	$5^2 + 5^2 + 5^2$	0 1 1 0 0 1-1 1 0 0 1 0 1 0_{18}-1-1 0 0-1 1-1 0 0 1 0 1 0_{20} 1 1 0 0-1 0_7, 1 0 0-1-1 0 0 0-1 1 0-1 0 1 1 0_{15}-1 0 0 1 1 0 0 0-1 1 0-1 0 1 1 0_{15} 1-1 1 0 0 1 1 0_8, 0_{16} 1 1 0 0 1-1 1 0 0 1 0 1 0_{18} 1 1 0 0 1-1 1 0 0-1 0-1 0_{10}-1-1 0 1 0-1 1,
75	$5^2 + 5^2 + 4^2 + 3^2$	0_{15} 1 0 0-1-1 0 0 0-1 1 0-1 0 1 1 0_{15} 1 0 0-1-1 0 0 0 1-1 0 1 0-1-1 0_{10} 1 0 1 0 0 1 1 1 1-1 1-1-1 1 1 1-1-1 0 1 1 1 0 0 0 0 0 0-1 1 1 0_{25}-1 0_{24}, 0_{12} 1 0 0 0 1 1-1 1-1 1 0_{28}-1-1 1 1-1-1 1-1 1 1 1 1 1 0_{12}, 0_{25}-1-1-1-1 1-1 1 1-1-1 1 1 0 1 1 1 0 0 0 0 0 0-1 1 1 0_{16} 1-1 1-1 1 1 1 0 0 0, 0_{37} 1 0 0 0-1-1 1-1 1-1 0_{16} 1 1-1 0 0 0 0 0 0 0 1 1 1
77	$8^2 + 3^2 + 2^2$	0_{11} 1-1 1 1 1 1 0-1 0 0 0 1 0_{16} 1-1 0 0 0 1 0_{11} 1-1 1 1 1 0 1 0 0 0-1 0_7-1 1 1 0, -1 1-1-1-1 0 1 0 0 0-1 0_{11} 1-1 1 1 1 0 0-1-1 1 0_{12} 1-1 1 1 1 0 1 0_3-1 0_{11} 1 0_{10}, 0_{16}-1 0 1 1-1 0_{12} 1-1 1 1 1 0 0 1 1-1 0_{17}-1 0-1-1 1 0 0 0 0 0 0 0-1 0 0 0 1, 0 0 0 0 0 1 0-1-1 1 0_{17}-1-1 0 0 0 1 0_{16}-1 0-1-1 1 0_{13} 1 1 1-1 1 0 0 0 0 0
77	$6^2 + 5^2 + 4^2$	0_{11} 1-1 1 1 1 1 0 0 1 1-1 0_{17}-1 0 1 1-1 0_{12} 1-1 1 1 1 1 0 0-1-1 1 0_7-1 0 0 0 1, 0 0 0 0 0-1-1 0 0 0 1 0_{16} 1 0 1 1-1 0_{17} 1-1 0 0 0 1 0_{12} 1 1 1-1 1 0 0 0 0 0, 0_{16} 1 1 0 0 0-1 0_{11} 1-1 1 1 1 0 1 0 0 0-1 0_{16} 1-1 0 0 0 1 0_7 1-1-1 0,
77	$6^2 + 6^2 + 2^2 + 1^2$	1-1 1 1 1 0 0 1 1-1 0_{12}-1 1-1-1-1 0 1 0_3-1 0_{11}-1 1-1-1-1 0 0 1 1-1 0_{12} 1 0_{10} -1-1 0 0 1 0 1 1-1 1 1 1 0_{11} 1 1 0 0-1 0_{17} 1 1 0 0-1 0 1 1-1 1 1 0_{11}-1 0-1 1 0_7, 0 0-1 1 0 1 0_{18} 1-1 0-1 1 1-1 1 1 0_{13} 1-1 0-1 0_{17} 1 0 0-1-1 0 0 0 0 0, 0_{11}-1-1 0 0 1 0 1 1-1 1 1 0_{13}-1 1 0 1 1 1-1 1 1 0_{11}-1-1 0 0 1 0-1-1 1-1-1 0_{11}, 0_{13}-1 1 0 1 0_{16} 1 1 0 0-1 0_{19}-1 1 0 1 0_{11} 1 1-1 1 1
77	$6^2 + 4^2 + 4^2 + 3^2$	0-1 0 0-1-1-1 1-1 1 1 0_{12} 1 0 0 1 1 0_{17} 1 0 0 1 1-1 1-1 1 1 0_{13}-1-1 0 1 0_5, 1 0-1-1 0_{18}-1 0 1 1 0 0 1-1 1-1-1 0_{11}-1 0 1 1 0_{18} 1 1 0 0 1 0 0 0 0 0 0, 0_{12} 1 0 0 1 1 1-1 1-1-1 0_{11} 1 0-1-1 0 0 1-1 1-1-1 0_{12} 1 0 0 1 1-1 1-1 1 1 0_{11}, 0_{11}-1 0 1 1 0_{19} 1 0 0 1 1 0_{16}-1 0 1 1 0_{13} 1 1-1 1-1
79		Not known for any decomposition into squares.
81	9^2	0-1-1-1 1 1 1-1 1 0_{10} 1 1 1-1 0_{14} 1 1 1-1 0_{14} 1 1 1-1 1 1-1 1 0_{13} 1 0 0 0 0, -1 0_{17} 1 0 0 0 0 1 1-1 1 0_9 1 0 0 0 0-1-1 1-1 0_9 1 0_{17} 1-1-1-1 0 0 0 0 0, 0_{10} 1 1 1-1-1-1 1-1 0_9-1 0_4-1-1 1-1 0_9 1 0_4-1-1 1-1 0_{10} 1 1 1-1 1 1-1 1 0_9, 0_9 1 0_{18}-1-1-1 1 0_{14} 1 1 1-1 0_{13} 1 0_{13}-1 1-1-1
81	$8^2 + 4^2 + 1^2$	0_{41} 1 1 1 1-1-1-1 1 1 1-1 1 1 1-1 1 1-1 1-1-1 1-1... 1 1-1 1 1-1 1 1 1 1-1 1 1 1-1-1-1-1, 1 1 1 1-1-1-1 1 1 1 1-1 1 1 1-1 1 1-1 1-1 1-1 1-1-1... 1-1-1 1-1-1-1 1-1-1-1 1 1 1 1 1 0_{41}, 0_{40} 1 0_{40}, 0_{81}
81	$8^2 + 3^2 + 2^2 + 2^2$	-1 0_4 1 1-1 1 0_9 1 0_4-1-1 1-1 0_9 1 0_4 1 1-1 1 0_9 1 0_4 1 1-1 1 0_9-1 1 1 1 0_5, 0-1-1-1 1 0_{14} 1 1 1-1 0_{14} 1 1 1-1 0_{14} 1 1 1-1 0_{17}-1 0 0 0 0, 0_{10} 1 1 1-1 0_{14}-1-1-1 1 0_{14} 1 1 1-1 0_{14} 1 1 1-1 0 0 0 0 0 0 0 0 0-1 1-1-1, 0_9-1 0_4 1 1-1 1 0_9-1 0_4 1 1-1 1 0_9 1 0_4 1 1-1 1 0_9-1 0 0 0 0-1-1 1-1 0_9
81	$7^2 + 4^2 + 4^2$	0 0-1 0-1 1 0 0 1 0_{11} 1 0 1 0-1 1 0_{12} 1 0 1 0 1-1... 0_{12} 1 0 1 1 0 0 1 0_{10} 1 0-1-1 0 0 0 0, -1-1 0 1 0 0 1-1 0_{10} 1 1 0-1 0-1 0 0-1 0_9 1 1 0-1... 0 1 0 0 1 0_9 1 1 0-1 0 0 1-1 0_{10} 1 0 1 0_6, 0_{11} 1 0 1-1 0 0-1 0_9 1 1 0-1 0-1 0 0-1 0_9-1-1 0 1... 0-1 0 0-1 0_{11} 1 0 1 1 0 0 1 0 0 0 0 0 0 1-1 0, 0_9 1 1 0-1 0 0-1 1 0_{12} 1 0 1 0-1 1 0_{12}-1 0-1 0-1 1... 0_{10} 1 1 0-1 0 0 1-1 0 0 0 0 0 0 1 0 0 1

Table 3(cont): 4-disjoint T-sequences

Length	Sum of Squares	Sequences
81	$6^2 + 6^2 + 3^2$	0_9 1 1 1-1 0 0 1-1 0_{10}-1-1-1 1 0 0 1-1 0_{10} 1 1 1-1 0 0 1-1 0_{10} 1 1 1-1...
		0 0-1 1 0 0 0 0 0 0 1 0 0 1.
		0_{13} 1 1 0 0 1 0_{13} 1 1 0 0 1 0_{13}-1 1 0 0 1 0_{13} 1-1 0 0-1 0 0 0 0 0 0 1-1 0.
		-1-1-1 1 0 0-1 1 0_{10} 1 1 1-1 0 0-1 1 0_{10} 1 1 1-1 0 0 1-1 0_{10} 1 1 1-1...
		0 0-1 1 0_{10}-1 0 0 0 0 0 0 0 0.
		0 0 0 0-1-1 0 0-1 0_{13} 1 1 0 0 1 0_{13} 1-1 0 0-1 0_{13} 1-1 0 0-1 0_{10}-1 1 1 1 0_4
81	$6^2 + 5^2 + 4^2 + 2^2$	1 0_{13} 1 0 1 0-1 0-1 0-1 0 1 0-1 0_{27} 1-1 1-1 1 1-1-1 1 1 1 1 1 1 0_{14}.
		0 1 1 1 1 1-1-1 1 1-1 1-1 1 0 1 0 1 0-1 0 1 0-1 0-1 0_{41}-1 0_{13}.
		0_{27} 1 0_{13}-1 0-1 0 1 0 1 0 1 0-1 0 1 0_{15}-1 0-1 0 1 0-1 0 1 0 1 0.
		0_{28} 1 1 1 1 1-1-1 1 1-1 1-1 1 0-1 0-1 0 1 0-1 0 1...
		0 1 0_{15} 1 0-1 0 1 0 1 0 1 0-1 0-1
83		Not known for any decomposition into squares.
85	$9^2 + 2^2$	1 1 0_8-1-1 0 1-1 0_5 1 1 0_8 1 1 0-1 1 0_5-1-1 0-1 1...
		0_5 1 1 0_8 1 1 0 1-1 0_5 1 1 0_8 1 0 0 0 0.
		0 0 1 1-1 0_7-1 0_9 1 1-1 0_7 1 0_9-1 0_9 1-1 1 0_7 1 0_9 1-1 1 0 0 0 0 0 0-1-1 0 0.
		0_7-1-1 1 0_5 1 1 0-1 1 0_7-1-1 1 0_5-1-1 0 1-1 0_5-1-1...
		0-1 1 0_7 1-1 1 0_5 1 1 0 1-1 0_7 1-1 1 0_5.
		0 0 0 0 0-1-1 0_{10} 1 0_7-1-1 0_{10}-1 0_9-1 0_7 1 1 0_{10} 1 0_7 1 1 0 0 0 0 0 0 0 1-1
85	$8^2 + 4^2 + 2^2 + 1^2$	0 0 1 0_9-1 1 1 0_7 1 0_9 1-1-1 0_7-1-1-1 0_7 1 0_9 1 1 1 0_7 1 0_8 1-1 0 0.
		1-1 0-1-1 0_5-1 1 0_8 1-1 0-1-1 0_5 1-1 0_8-1 1 0_8 1-1...
		0 1 1 0_5 1-1 0_8 1-1 0 1 1 0_5 1 0 0 0 0.
		0_5-1 1 0 1 1 0_7-1 1 1 0_5-1 1 0 1 1 0_7 1-1-1 0_7 1 1 1...
		0_5 1-1 0 1 1 0_7-1-1-1 0_5 1-1 0 1 1 0_5.
		0_7 1 0_7 1-1 0_{10} 1 0_7-1 1 0_8-1 1 0_{10}-1 0_7 1-1 0_{10}-1 0 0 0 0 0 1 1
85	$7^2 + 6^2$	1 1 0 1-1 0_5-1-1 0 1-1 0_5 1 1 0 1-1 0_5 1 1 0-1 1 0_5-1-1 0-1 1...
		0_5 1 1 0-1 1 0_5 1 1 0 1-1 0_5 1 1 0-1 1 0_5-1 0_4.
		0 0 1 0_9-1 0_9 1 0_9 1 0_9-1 0_9 1 0_9 1 0_9 1 0_8 1 1 0 0.
		0_5-1-1 0-1 1 0_5 1 1 0-1 1 0_5-1-1 0-1 1 0_5-1-1 0 1-1 0_5-1-1 0-1 1...
		0_5 1 1 0-1 1 0_5 1 1 0 1-1 0_5 1 1 0-1 1 0_5.
		0_7 1 0_9 1 0_9 1 0_9-1 0_9-1 0_9-1 0_9 1 0_9-1 0 0 0 0 0-1 1
85	$7^2 + 4^2 + 4^2 + 2^2$	-1 1 0 1 1 1 0-1 0-1 1 0 0 0 0 1 1 0_{17} 1-1 0-1-1-1 0 1 0-1 1...
		0 0 0 0 1 1 0_{17} 1 0 1 0 0 0 1 0_{10}.
		0_{17}-1 1 0 1 1 1 0-1 0-1 1 0 0 0 0 1 1 0_{17}-1 1 0 1 1 1 0-1 0 1-1...
		0 0 0 0-1-1 0_{11}-1 1 1-1 0 0.
		0_{19} 1 0 0 0 1 0 1 0 0-1 1 1-1 0_{21} 1 0 0 0 1 0 1 0 0 1-1-1 1...
		0 0 0 0 0 0 0 0 0 0 0 0-1-1 0 0 0 0-1 1.
		0 0-1 0 0 0-1 0-1 0 0 1-1-1 1 0_{21} 1 0 0 0 1 0 1 0 0 1-1-1 1...
		0_{20}-1 0 1 1 1 0 1-1 0 0 0 0 0 0 0 0
85	$6^2 + 6^2 + 3^2 + 2^2$	0 0 1 0 0 0 1 0 1-1 1 0 0 0 0 1 1 0_{19}-1 0 0 0-1 0-1-1 1 0 0 0 0 1 1...
		0_{18} 1 0-1-1-1 0-1 1 0_8.
		-1 1 0 1 1 1 0-1 0 0 0-1 1 1-1 0_{19} 1-1 0-1-1-1 0 1 0 0 0-1 1 1-1...
		0_{19} 1 0 1 0 0 0 1 0_{10}.
		0_{19} 1 0 0 0 1 0 1-1 1 0 0 0 0 1 1 0_{19} 1 0 0 0 1 0 1 1-1 0 0 0 0-1-1...
		0_{11} 1-1-1 1 0 0.
		0_{17}-1 1 0 1 1 1 0-1 0 0 0-1 1 1-1 0_{19}-1 1 0 1 1 1 0-1 0 0 0 1-1-1 1...
		0_{11} 1 1 0 0 0 0 1-1

Table 3(cont): 4-disjoint T-sequences

Length	Sum of Squares	Sequences
87	$9^2 + 2^2 + 1^2 + 1^2$	0_{30}-1-1 0 1 1 0 0 0 1 1 0 1-1 1 1-1 0 0 1 1 0 0 1-1 0 0 1 1... 0_{17}-1 1 0 0 1 1 0 0 1-1 0 0, 0_{29} 1 0 0-1 0 0-1-1 1 0 0 1 0 0 0 0 0 1-1 0 0-1-1 0 0-1 1... 0_{17} 1 1 0 0-1 1 0 0 1 1 0 0-1 1, 0 1 1 0-1-1 0 0 0-1-1 0-1 1-1 1-1 0 0 1 1 0 0 1-1 0 0 1 1... 0_{32}-1 0 0-1 1 1 0 0 1 0 0-1 0_{14}, -1 0 0 1 0 0 1 1-1 0 0-1 0 0 0 0 0 1-1 0 0-1-1 0 0-1 1... 0_{31} 1-1 1 0 1 1 0 0 0 1 1 0-1-1 0_{15}
87	$7^2 + 6^2 + 1^2 + 1^2$	0-1-1 0 1 1 0 0 0 1 1 0 1-1 1 1-1 0 0 1 1 0 0 1-1 0 0 1 1... 0_{32}-1 0 0-1 1 1 0 0 1 0 0-1 0_{14}. 0_{29} 1 0 0-1 0 0-1-1 1 0 0 1 0 0 0 0 0-1 1 0 0 1 1 0 0 1-1... 0_{17} 1 1 0 0-1 1 0 0 1 1 0 0-1 1, 1 0 0-1 0 0-1-1 1 0 0 1 0 0 0 0 0 1-1 0 0-1-1 0 0-1 1... 0_{31} 1-1 1 0 1 1 0 0 0 1 1 0-1-1 0_{15}. 0_{30}-1-1 0 1 1 0 0 0 1 1 0 1-1 1-1 1 0 0-1-1 0 0-1 1 0 0-1-1... 0_{17}-1 1 0 0 1 1 0 0 1-1 0 0
87	$7^2 + 5^2 + 3^2 + 2^2$	1 1 0 0 1 1-1 0-1 1 1 0 0 1 0 1 0 0 0-1 0 0 0 0 1 0 0 0-1... 0_{29} 1 0-1 1 0 0 0 1 0 0 0 1-1 0_{16}, 0 0 1-1 0 0 0-1 0 0 0-1 1 0-1 0-1-1-1 0 1-1 1-1 0 1 1 1... 0_{31} 1 0 0 1 1-1 0-1 1 1 0 0 1 1 0_{14}, 0_{29} 1 1 0 0 1 1-1 0-1 1 1 0 0 1 0-1 0 0 0 1 0 0 0 0-1 0 0 0 1... 0_{16}-1-1-1 0 1-1 1-1 0 1 1 1 0, 0_{31}-1 1 0 0 0 1 0 0 0 1-1 0 1 0-1-1-1 0 1-1 1-1 0 1 1 1... 0_{16} 1 0 0 0-1 0 0 0 0 1 0 0 0-1
87	$6^2 + 5^2 + 5^2 + 1^2$	0_{29} 1 0 0-1 0 0-1-1 1 0 0 1 0 0 0 1-1 0 0 1 1 0 0 1-1 0 0 1 1... 0_{17}-1 1 0 0 1 1 0 0 1-1 0 0, 0_{30}-1-1 0 1 1 0 0 0 1 1 0 1-1 1 0 0 1-1 0 0-1-1 0 0-1 1... 0_{17} 1 1 0 0-1 1 0 0 1 1 0 0-1 1, 0-1-1 0 1 1 0 0 0 1 1 0 1-1 1 0 0-1 1 0 0 1 1 0 0 1-1... 0_{34}-1 0 0-1 1 1 0 0 1 0 0-1 0_{14}. -1 0 0 1 0 0 1 1-1 0 0-1 0 0 0 1-1 0 0 1 1 0 0 1-1 0 0 1 1... 0_{29}-1 1-1 0-1-1 0 0 0-1-1 0 1 1 0_{15}
89		Not known for any decomposition into squares.
91	$9^2 + 3^2 + 1^2$	0 0 0-1 0 0-1-1 0 1 1 0 1 0_{16} 1 0 0 1 0 1 0 0 1 0_{17} 1 0 0 1-1 0 1 1 0 1... 0_{14} 1-1 0-1 1 1 0 0 0 0 0 0, 1 1-1 0-1 1 0 0 1 0 0 1 0_{14}-1-1 1 0 1-1 0 1 0-1-1 0-1... 0_{13}-1-1 1 0 1-1 0 0 1 0 0 1 0_{14} 1 0 0 1 0_9, 0_{16} 1 0 0 1 1 0-1-1 0-1 0_{13} 1 1-1 0-1 1 0 1 0-1-1 0-1... 0_{16} 1 0 0 1-1 0 1 1 0 1 0_8-1 0 0-1 0, 0_{13}-1-1 1 0 1-1 0 0-1 0 0-1 0_{17} 1 0 0 1 0-1 0 0-1... 0_{14}-1-1 1 0 1-1 0 0 1 0 0 1 0_8 1 0 1 1 0-1
91	$8^2 + 5^2 + 1^2 + 1^2$	0 0 0-1 1 0 1 0 0 0 1 0 1 0 0 0 0 1 1 0 1 0 0 0 1 0-1... 0 0 0 0 1 1 0 1 0 0 0-1 0 1 0 0 0 0 0 0-1 0_{43}, -1 1 1 0 0 1 0 1-1-1 0-1 0-1 1-1-1 0 0 1 0 1-1-1 0 1... 0 1 1-1-1 0 0 1 0-1 1 1 0-1 0-1 0_4 1 0 1 0_{42}, 0_{45} 1 0_6-1 0-1 0 0 1 1-1... 0 1 0 0 0 0 1 0 1 0 0 1 1-1 0 1 0 0 0 0-1 0 1 0 0 1 1-1 0-1 0, 0_{42}-1 1 1 0 0 0 0 1-1-1 0-1 0-1-1 0 0 0 1 0 1-1 1 1 0 1... 0 1-1 0 0 0 1 0 1 1-1-1 0 1 0 1-1 0 0 0-1 0-1

Table 3(cont): 4-disjoint T-sequences

Length	Sum of Squares	Sequences
91	$8^2 + 3^2 + 3^2 + 3^2$	0_{13} 1 1-1 0-1 1 0-1 0 1 1 0 1 0_{16} 1 0 0 1-1 0 1 1 0 1... 0_{13} 1 1-1 0-1 1 0 1 0-1-1 0-1 0_{8} 1 0 0 1 0, 0_{16} 1 0 0 1 0-1 0 0-1 0_{14}-1-1 1 0 1-1 0 0-1 0 0-1 0_{17}... 1 0 0 1 0 1 0 0 1 0 0 0 0 0 0 0 0 1 0 1 1 0-1, 1 1-1 0-1 1 0-1 0 1 1 0 1 0_{13}-1-1 1 0 1-1 0 0 1 0 0 1... 0_{14}-1-1 1 0 1-1 0-1 0 1 1 0 1 0_{13}-1 0 0-1 0_{9}, 0 0 0-1 0 0-1 0 1 0 0 1 0_{17} 1 0 0 1 1 0-1-1 0-1 0_{16}... 1 0 0 1 0 1 0 0 1 0_{15}-1 1 0 1-1-1 0 0 0 0 0 0
91	$7^2 + 5^2 + 4^2 + 1^2$	0 1-1 0-1-1-1 1 0 0 0 0 1 0_{14}-1 1 0 1 1 1 0-1 1 1-1 0_{15}... -1 1 0 1 1 1 1 0 0 0 0 1 0_{16} 1 0 0-1 0_{6}, -1 0 0 1 0_{4}-1 1 1-1 0_{14} 1 0 0-1 0 0 0-1 0 0 0 0-1 0_{13}... 1 0 0-1 0 0 0 0-1 1 1-1 0_{14} 1 1 1 0 1-1 0_{7}, 0_{14}-1 1 0 1 1 1-1 0 0 0 0-1 0_{13}-1 0 0 1 0 0 0-1 0 0 0 0-1... 0_{14}-1 1 0 1 1 1 1 0 0 0 0 1 0_{8} 1-1-1 1 0, 0_{13} 1 0 0-1 0 0 0 0 1-1-1 1 0_{15}-1 1 0 1 1 1 0 1-1-1 1... 0_{14} 1 0 0-1 0 0 0 0-1 1 1-1 0_{8} 1 0 0 0 0 1
91	$5^2 + 5^2 + 5^2 + 4^2$	-1-1 1 0 1-1 0-1 0 1 1 0 1 0_{13} 1 1-1 0-1 1 0 0 1 0 0 1... 0_{14} 1 1-1 0-1 1 0-1 0 1 1 0 1 0_{13}-1 0 0-1 0_{9}, 0 0 0-1 0 0-1 0 1 0 0 1 0_{17} 1 0 0 1 1 0-1-1 0-1 0_{16}... 1 0 0 1 0 1 0 0 1 0_{15} 1-1 0-1 1 1 0 0 0 0 0 0, 0_{16} 1 0 0 1 0-1 0 0-1 0_{14} 1 1-1 0-1 1 0 0-1 0 0-1... 0_{17} 1 0 0 1 0 1 0 0 1 0 0 0 0 0 0 0 0 1 0 1 1 0-1, 0_{13}-1-1 1 0 1-1 0-1 0 1 1 0 1 0_{16} 1 0 0 1-1 0 1 1 0 1... 0_{13}-1-1 1 0 1-1 0 1 0-1-1 0-1 0_{8} 1 0 0 1 0
93	$9^2 + 2^2 + 2^2 + 2^2$	0-1 0 0-1 1 0-1 1 0 1 1 0 0 1 1 1 0 0-1 0-1 1 1 1-1 0 1 0 0 1 0_{23}... 1 1 0 0 1 0 0-1 0 0-1 1 0 1 0_{15}. -1 0-1 1 0 0 1 0 0-1 0 0-1-1 0 0 0-1 1 0 1 0 0 0 0 0-1 0 1 1 0_{32}... 1 1 0 0 1 1 0 1-1 0 1-1 0 0-1 0_{16}. 0_{32} 1 0 0 1-1 0 1-1 0-1-1 0 0-1-1 1 0 0-1 0-1 1 1 1-1 0 1 0 0 1... 0_{17} 1 1 0-1 0 0 0 0 0 1 0 1-1 0, 0_{31} 1 0 1-1 0 0-1 0 0 1 0 0 1 1 0 0 0-1 1 0 1 0 0 0 0 0-1 0 1 1... 0_{17}-1 0 0-1 0 1-1-1-1 1 0 1 0 0-1
93	$8^2 + 5^2 + 2^2$	0 1 1 1 1-1 1 1-1-1 1 1 1-1 1-1 1 0 1 0 1 0 1 0-1 0-1 0-1 0 1 0_{46} 1 0_{15}. -1 0_{16}-1 0 1 0 1 0-1 0 1 0 1 0-1 0_{32}-1 1-1 1 1 1-1-1 1 1-1 1 1 1 1 0_{16}. 0_{32} 1 1 1 1-1 1 1-1-1 1 1 1-1 1-1-1 0-1 0-1 0-1 0 1... 0 1 0 1 0-1 0_{17} 1 0-1 0-1 0 1 0-1 0-1 0 1 0, 0_{31}-1 0_{16} 1 0-1 0-1 0 1 0-1 0-1 0 1 0_{17} 1 0-1 0-1 0-1 0 1 0 1 0 1 0 1
93	$8^2 + 4^2 + 3^2 + 2^2$	1 0_{15} 1 0 1 0 1 0 1 0-1 0-1 0-1 0 1 0_{31}-1 1-1 1 1 1-1-1 1 1-1 1 1 1 1 1 0_{16}. 0 1 1 1 1-1 1 1-1-1 1 1 1-1 1-1 0 1 0-1 0-1 0 1 0-1 0-1 0 1 0_{47}-1 0_{15}, 0_{31}-1 0_{15} 1 0 1 0 1 0 1 0-1 0-1 0-1 0 1 0_{17}-1 0 1 0 1 0-1 0 1 0 1 0-1 0, 0_{32} 1 1 1 1-1 1 1-1-1 1 1 1-1 1-1 0-1 0 1 0 1 0-1 0 1 0 1 0-1... 0_{16}-1 0 1 0 1 0 1 0-1 0-1 0-1 0-1
93	$7^2 + 6^2 + 2^2 + 2^2$	0 1 1 1 1-1 1 1-1-1 1 1 1-1 1-1 0-1 0 1 0 1 0-1 0 1 0 1 0-1 0_{47} 1 0_{15}, -1 0_{15} 1 0 1 0 1 0 1 0-1 0-1 0-1 0 1 0_{31}-1 1-1 1 1 1... -1-1 1 1-1 1 1 1 1 1 0_{16}, 0_{32} 1 1 1 1-1 1 1-1-1 1 1 1-1 1-1 0 1 0-1 0-1 0 1 0-1 0-1 0 1... 0_{17}-1 0 1 0 1 0 1 0-1 0-1 0-1 0-1, 0_{31} 1 0_{15} 1 0 1 0 1 0 1 0-1 0-1 0-1 0 1 0_{17} 1 0... -1 0-1 0 1 0-1 0-1 0 1 0

Table 3(cont): 4-disjoint T-sequences

Length	Sum of Squares	Sequences
93	$6^2 + 5^2 + 4^2 + 4^2$	0 1 1 1 1-1 1 1-1-1 1 1 1-1 1-1 1 0 1 0 1 0 1 0-1 0-1 0-1 0 1 0_{26}-1 0_{15}, 1 0_{16} 1 0-1 0-1 0 1 0-1 0-1 0 1 0_{32}-1 1-1 1 1 1-1-1 1 1-1 1 1 1 1 1 0_{16}, 0_{32} 1 1 1 1-1 1 1-1-1 1 1 1-1 1-1-1 0-1 0-1 0-1 0 1 0 1 0 1 0-1... 0_{17}-1 0 1 0 1 0-1 0 1 0 1 0-1 0,
95	$9^2 + 3^2 + 2^2 + 1^2$	0_{31} 1 0_{16}-1 0 1 0 1 0-1 0 1 0 1 0-1 0_{17} 1 0-1 0-1 0-1 0 1 0 1 0 1 0 1 1 0_{9} 1 0 1 0 1 0 1 0-1... 0_{19}-1 0_{9} 1 0 1 0 1 0 1 0-1 0_{19} 1-1 1-1-1 1 1 1 1 0 0 0 0 0 0 0 0 0 0, 0_{20} 1 1 1 1-1-1 1-1 1 0 1 0-1 0 1 0-1 0_{21} 1 1 1 1-1-1 1-1 1 0-1 0 1 0-1... 0 1 0_{11} 1 0-1 0-1 0-1 0-1, 0_{19} 1 0_{9} 1 0 1 0 1 0 1 0-1 0_{19} 1 0 0 0 0 0 0 0 0 0-1 0-1 0-1 0-1 0 1... 0_{11}-1 0 1 0-1 0 1 0, 0-1-1-1-1 1 1-1 1-1 0-1 0 1 0-1 0 1 0_{21} 1 1 1 1-1-1 1-1 1 0-1 0 1 0-1... 0 1 0_{29} 1 0 0 0 0 0 0 0 0 0
95	$7^2 + 6^2 + 3^2 + 1^2$	0 0 0 0 1 1 0 0 0-1 1-1 1 1 0-1 1 1 1 0_{23}-1-1 0 0 0 1 1-1 1 1... 0-1 1 1 1 0_{20} 1-1 1 0 0 1-1-1-1 0_{9}, 1 1 1-1 0 0-1 1-1 0 0 0 0 0 1 0_{23}-1-1-1 1 0 0 1-1 1 0 0 0 0 0 1... 0_{23}-1 0 0 0 1 1 0_{13}, 0_{23} 1 1 0 0 0-1 1-1 1 1 0-1 1 1 1 0_{23} 1 1 0 0 0-1-1 1-1-1 0... 1-1-1-1 0_{14}-1 0 0 0 0,
95	$6^2 + 5^2 + 5^2 + 3^2$	1 1 1 0_{4} 1-1 0 1 1 1 0-1 0 1-1 1 0_{19}-1-1-1 0 0 0 0-1 1 0 1 1 1... 0-1 0 1-1 1 0_{19} 1 0 0-1 1-1-1 0_{12}, 0 0 0 1 1-1 1 0 0-1 0 0 0 1 0-1 0_{25}-1-1 1-1 0 0 1 0 0 0 1 0-1... 0_{23}-1 1 0 0 0 0 1 1 1 0_{9}, 0_{19} 1 1 1 0 0 0 0 1-1 0 1 1 1 0-1 0 1-1 1 0_{19} 1 1 1 0 0 0 0 1-1... 0-1-1-1 0 1 0-1 1-1 0_{13} 1 0-1 0 0 0, 0_{22} 1 1-1 1 0 0-1 0 0 0 1 0-1 0_{25} 1 1-1 1 0 0-1 0 0 0-1 0 1... 0_{13} 1-1 1 0-1 0 1 1 1
99	$9^2 + 4^2 + 1^2 + 1^2$	-1 1-1-1-1 0 0 1 1-1 0_{12} 1-1 1 1 1 0 0 1 1-1 0_{12} 1-1 1 1 1... 0_{2}-1-1 1 0_{12} 1-1 1 1 1 0_{2} 1 1-1 0_{12} 1 0_{10}, 0_{11} 1-1 1 1 1 0_{2}-1-1 1 0_{12}-1 1-1-1-1 0_{2}-1-1 1 0_{12} 1-1 1 1 1... 0_{2}-1-1 1 0_{12} 1-1 1 1 1 0 0 1 1-1 0_{7} 1 0_{3}-1, 0 0 0 0 0 1 1 0 0 0-1 0_{16}-1-1 0 0 0 1 0_{16}-1 1 0 0 0-1... 0_{16}-1 1 0 0 0-1 0_{12} 1 1 1-1 1 0 0 0 0 0, 0_{16} 1 1 0 0 0-1 0_{16} 1 1 0 0 0-1 0_{16}-1 1... 0 0 0-1 0_{16} 1-1 0 0 0 1 0 0 0 0 0 0 0 0 1-1-1 0
99	$9^2 + 3^2 + 3^2$	1 0 0 1 1-1 0_{5} 1-1-1 0 0 0 1 1 1 1 1-1-1 1 1-1 1-1 1 1-1-1... 0_{33} 1 1 1 0 0 0-1 1-1 1-1 0 0 0 1 1 0_{17}, 0 1 1 0 0 0-1 1-1 1-1 0 0 0 1 1 1 0_{52} 1 1-1 0 0 0 0 0 1-1-1 0 0-1 0_{16}, 0_{33}-1 0 0-1-1 1 0 0 0 0 0-1 1 1 0 0 0 1 1 1 1 1-1-1 1 1-1 1-1 1 1-1-1 0_{32}, 0_{34} 1 1 0 0 0-1 1-1 1-1 0 0 0 1 1 1 0_{33} 1 1-1-1 1-1 1-1-1 1 1-1-1-1-1-1
99	$8^2 + 5^2 + 3^2 + 1^2$	0 0 0 1 0 1 0 0 0-1 0 1 0 0 0 1 0 1 0 0 0 1 0-1 0 0 0 1 0 1... 0 0 0-1 0 1 0 0 0-1 0-1 0 0 0-1 0 1 0 0 0 1 0 1 0 0 0-1 0 1 0 0 0 1... 0 1 0 0 0 1 0-1 0 0 0-1 0-1 0 0 0 1 0-1 0 0 0 1 0 1 0 0 0 1 0-1 0 0 0, 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0-1 0 0 0 0 0 1 0 0 0 0 0 1... 0 0 0 0-1 0 0 0 0 0 1 0 0 0 0 0-1 0 0 0 0 0-1 0 0 0 0 0-1 0 0 0 0 0... 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0-1 0 0 0 0 0 1 0 0 0 1, 0-1 0 0 0 0 0 1 0 0 0 0 0-1 0 0 0 0 0-1 0 0 0 0 0-1 0 0 0 0 0 1 0 0 0... 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0-1 0 0 0 0 0 1 0 0 0 0 0 1 0 0... 0 0 0-1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0-1 0, -1 0-1 0 0 0 1 0-1 0 0 0-1 0-1 0 0 0-1 0 1 0 0 0-1 0-1 0 0 0 1 0-1 0... 0 0 1 0 1 0 0 0 1 0-1 0 0 0 1 0 1 0 0 0-1 0 1 0 0 0 1 0 1 0 0 0 1 0-1... 0 0 0-1 0-1 0 0 0 1 0-1 0 0 0 1 0 1 0 0 0 1 0-1 0 0 0 1 0 0

Table 3(cont): 4-disjoint T-sequences

Length	Sum of Squares	Sequences
99	$7^2 + 7^2 + 1^2$	-1 1-1-1-1 0-1 0_3 1 0_{11} 1-1 1 1 1 1 0-1 0_3 1 0_{11} 1-1 1 1 1 1 0 1... 0_3-1 0_{11} 1-1 1 1 1 1 0-1 0_3 1 0_{11} 1 0_{10}, 0_{11} 1-1 1 1 1 0 1 0_3-1 0_{11}-1 1-1-1-1 0 1 0_3-1 0_{11} 1-1 1 1 1 1 0 1... 0_3-1 0_{11} 1-1 1 1 1 1 0-1 0_3 1 0_7-1 1 1 1 0, 0 0 0 0 0-1 0 1 1-1 0_{17} 1 0-1-1 1 0_{17} 1 0 1 1-1 0_{17} 1 0 1 1-1... 0_{13}-1-1-1 1-1 0 0 0 0 0, 0_{16} 1 0-1-1 1 0_{17} 1 0-1-1 1 0_{17}-1 0-1-1 1 0_{17} 1 0 1 1-1 0_7-1 0 0 0 1
99	$7^2 + 5^2 + 5^2$	-1 0 1 0 0 0 1 0_5-1 0 1 0 0 0-1 0_5-1 0 1 0 0 0 1 0_5 1 0-1 0 0 0 1 0_5 1... 0_5-1 0-1 0 0 0 1 0_5 1 0 1 0 0 0-1 0_5 1 0 1 0 0 0 1 0_5 1 0 1 0 0 0-1 0 0, 0-1 0_5 1-1 0_4-1 0_5-1 1 0_4-1 0_5 1-1 0_4 1 0_5 1-1 0_4 1 1 0_4-1... 0_5 1 1 0_4 1 0_5-1-1 0_4 1 0_5 1 1 0_4 1 0_5 1 0, 0_3 1 0-1 0_4 1-1 0_3 1 0-1 0_4-1 1 0 0 0 1 0-1 0_4 1-1 0_3-1 0 1 0_4 1-1 0_4-1-1... 0 0 0-1 0-1 0_4-1-1 0 0 0 1 0 1 0_4 1 1 0 0 0 1 0 1 0_4-1-1 0 0 0 1 0 1 0 0 0, 0_4 1 0_4 1 0_6 1 0_4-1 0_6 1 0_4 1 0_6-1 0_4 1 0_5-1 0_6-1 0 0 0 0-1... 0_6 1 0_4 1 0_6 1 0_4-1 0_6 1 0 0 0 1
99	$7^2 + 5^2 + 4^2 + 3^2$	-1-1 0 0 1 0 1 1-1 1 1 0_{11} 1 1 0 0-1 0_{17} 1 1 0 0-1... 0_{17} 1 1 0 0-1 0 1 1-1 1 1 0_{11}-1 0-1 1 0_7, 0 0 1-1 0-1 0_{18}-1 1 0 1-1-1 1-1-1 0_{13}-1 1 0 1 1 1-1 1 1... 0_{13}-1 1 0 1 0_{17}-1 0 0 1 1 0 0 0 0 0, 0_{11}-1-1 0 0 1 0 1 1-1 1 1 0_{13} 1-1 0-1 1 1-1 1 1 0_{13}-1 1 0 1 1 1-1 1 1... 0_{11}-1-1 0 0 1 0-1-1 1-1-1 0_{11}, 0_{13}-1 1 0 1 0_{16} 1 1 0 0-1 0_{17}-1-1 0 0 1 0_{19}-1 1 0 1 0_{11} 1 1-1 1 1

Table 3(cont): 4-disjoint T-sequences

Remark: Multiple solutions have been found for many of these lengths. Full details are available in a Technical Report CSADFA88/6 from the authors.

Corresponding to the decomposition $3^2 + 0^2 + 0^2 + 0^2$.

Using $A = 1\,0, B = 0\,1, C = 0, D = 1$ gives

$$\begin{array}{rcl} X & = & 1\ 0\ 0 \quad 0\ 0\ \ 0 \quad 1\ \ \ 0\ 0 \\ Y & = & 0\ 1\ 1 \quad 0\ 0\ \ 0 \quad 0\ -1\ 0 \\ Z & = & 0\ 0\ 0 \quad 1\ 0\ \ 0 \quad 0\ \ \ 0\ 1 \\ W & = & 0\ 0\ 0 \quad 0\ 1\ -1 \quad 0\ \ \ 0\ 0 \end{array}$$

The sequences correspond to the decomposition $2^2 + 1^2 + 2^2 + 0^2$.

Some detailed results

$t = 13$ and $t = 19$

Koukouvinos and Kounias [12] had found T-matrices corresponding to the decompositions

$$\begin{array}{rcl} 13 & = & 2^2 + 2^2 + 2^2 + 1^2 \\ 19 & = & 4^2 + 1^2 + 1^2 + 1^2. \end{array}$$

but we now give T-sequences corresponding to these decompositions

$t = 25$

Hunt and (Seberry) Wallis (see Geramita and Seberry p125) give T-matrices (not T-sequences) which are type 1 and defined over the abelian group $EA(25)$) which correspond to the decomposition

$$25 = 5^2 + 0^2 + 0^2 + 0^2$$

$t = 31$

Hunt and (Seberry) Wallis (See Geramita and Seberry p125) give T-matrices which correspond to the decomposition

$$31 = 3^2 + 3^2 + 3^2 + 2^2.$$

The rows of these T-matrices are:

1 0 0 0 1 0 0 −1 −1 0 1 0 0 −1 0 0 0 0 0 0 0 0 0 1 1 0 1 0 0 0 0
0 1 0 0 0 1 0 0 0 1 0 −1 0 0 0 0 0 0 1 0 −1 0 0 0 0 1 0 0 −1 1 0
0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 −1 1 −1 0 1 0 1 1 0 0 0 0 −1 0 0 0
0 0 1 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 −1

$t = 37$

Hunt and (Seberry) Wallis (see Geramita and Seberry p125) give T-matrices (not T-sequences) which are type 1 and defined over the abelian group $EA(37)$) which correspond to the decomposition

$$37 = 6^2 + 1^2 + 0^2 + 0^2$$

$t = 41$

Hunt and (Seberry) Wallis (see Geramita and Seberry p125) give T-matrices (not T-sequences) which are type 1 and defined over the abelian group $EA(41)$ which corresponds to the decomposition $41 = 5^2 + 4^2 + 0^2 + 0^2$.

$t = 43$

T-sequences are not yet known of length 43 for any decomposition of 43 into squares.

$t = 57$

Use the sequences

E	=	1	0	0	0	0	0	0	0	0	0
F	=	0	1	1	1	1	-1	-1	1	-1	1
G	=	0	1	0	-1	0	1	0	-1	0	
H	=	1	0	1	0	1	0	1	0	-1	

with $A = F, B = E, C = G, D = H$ in Yang's Theorem (see Restatement 2.2) to obtain decomposition

$$57 = 4^2 + 1^2 + 6^2 + 2^2$$

and with $A = E, B = F, C = G, D = H$ in Yang's Theorem (see Restatement 2.2) to obtain the decomposition

$$57 = 4^2 + 5^2 + 4^2 + 0^2$$

$t = 59$

Use Theorem 9 (Turyn) on the Turyn sequences of lengths 15,15,14,14 given in Table 2.

$t = 61$

Hunt and (Seberry) Wallis (see Geramita and Seberry p125) give T-matrices (not T-sequences) which are type 1 and defined over the abelian group $EA(61)$) which correspond to the decomposition

$$61 = 6^2 + 5^2 + 0^2 + 0^2$$

$t = 63$

Let X and Y be Golay sequences of length 10 with row sums 4 and 2. Use the sequences

$$\begin{aligned} A &= 1\,0_{20} \\ B &= 0\,X\,Y \\ C &= -X\,Y \\ D &= 0_{20} \end{aligned}$$

in Yang's Theorem (see Restatement 2.2) to obtain the decomposition

$$63 = 7^2 + 3^2 + 1^2 + 2^2$$

Use the sequences

A	=	0	1	1	1	-1
B	=	1	0	0	0	0
C	=	1	1	-1	1	
D	=	0	0	0	0	

in Yang's Theorem (see Theorem 3.1) to obtain the decomposition

$$63 = 5^2 + 1^2 + 1^2 + 6^2$$

$t = 67$

T-sequences are not yet known of length 67 for any decomposition of 67 but two inequivalent T-matrices of order 67 have been found by Kazue Sawade for

$$67 = 8^2 + 1^2 + 1^2 + 1^2$$

$t = 71, 73, 79, 83, 89, 97$

Neither T-sequences nor T-matrices are known for these lengths for any decomposition of t into squares.

5 Summary

The sequences constructed in section 4 and table 3 can be used with the Cooper-(Seberry)Wallis construction [3] to obtain orthogonal designs $OD(4t; t, t, t, t)$ for

$t = 1, 3, \ldots, 41, 45, \ldots, 65, 67, 69, 75, 77, 81, 85, 87, 91, 93, 95, 99, 101, 105, 111, 115, 117, 119, 123, 125, 129, 133, 141, \ldots, 147, 153, 155, 159, 161, 165, 169, 171, 175, 177, 183, 185, 189, 195, 201, 203, 205, 209$.

Williamson matrices which can be used with these orthogonal designs to form Hadamard matrices are the subject of a separate report. The paper (Seberry [18]) gives a recent listing of the orders for which Williamson matrices are known.

References

[1] Terence Andres, (1977), *Some Combinatorial Properties of Complementary Sequences*, MSc. Thesis, University of Manitoba.

[2] L.Baumert and Marshall Hall Jr, (1965), A new construction for Hadamard matrices, *Bull. Amer. Math. Soc.*, **71**, 169-170.

[3] Joan Cooper and Jennifer Seberry Wallis, (1972), A construction for Hadamard arrays, *Bull. Austral. Math. Soc.*, **7**, 269-278.

[4] A.V.Geramita and Jennifer Seberry, (1979), *Orthogonal Designs: Quadratic Forms and Hadamard Matrices,* Marcel Dekker, New York - Basel.

[5] M.J.E. Golay, (1961), Complementary sequences, *IRE Transactions on Information Theory* **IT-7**, 82-87.

[6] M.J.E. Golay, (1962), Note on complementary series, *Proceedings of the IRE*, 84.

[7] Malcom Griffin, (1977), There are no Golay sequences of length 2.9^t *Aequationes Math.* **15**, 73-77.

[8] I.M.I. Habbab and L.F. Turner, (1986), New class of M-ary communication systems using complementary sequences, *IEE Proceedings*, **133**, **Pt.F**(3).

[9] Mark James, (1987), *Golay Sequences*, Honours Thesis, University of Sydney.

[10] S. Jauregui, Jr. (1962), Complementary sequences of length 26, *IRE Transactions on Information Theory*, **IT-7**, 323.

[11] Z.Kiyasu (1985), Private communication to K.Sawade

[12] C. Koukouvinos and S. Kounias, (1987), Optimal designs in linear models and Hadamard matrices, *Proc. Second International Tampere Conference in Statistics*, ed. T. Pukkila and S. Puntanen, Dept. Math., Univ. Tampere, 531-540.

[13] Joseph Kruskal (1961), Golay's complementary series, *IRE Transactions on Information Theory*, **IT-7**, 273-276.

[14] R.B. Lakein and J. Seberry Wallis, (1975), On the matrices used to con-

struct Baumert-Hall arrays, in *Combinatorial Mathematics III*, Vol 452, Lecture Notes in Mathematics, Springer-Verlag, Berlin - Heidelberg - New York, 156-170.

[15] T.Ono and K.Sawade, (1984), A construction of Baumert–Hall–Welch array of order 36, *Sugaku* (in Japanese), **36**, 172-174.

[16] Peter J. Robinson and Jennifer Seberry Wallis, (1978), A note on using sequences to construct orthogonal designs, *Colloq. Math. Janos Bolyai*, **18**, 911-932.

[17] Kazue Sawade, (1985), A Hadamard matrix of order 268, *Graphs and Combinatorics*, **1**, 185-187.

[18] Jennifer Seberry, (1988), Constructing Hadamard matrices from orthogonal designs, University College UNSW, ADFA, TR.

[19] Bernard Schweitzer (1971), *Generalized Complementary Code Sets*, Ph.D. Thesis, U.C.L.A.

[20] Y. Taki, H. Miyakawa, M. Hatori, and S. Namba (1969), Even shift orthogonal sequences, *IEEE Transactions on Information Theory* **IT-7** 295-300.

[21] C.C. Tseng, (1971), Signal multiplexing in surface-wave delay lines using orthogonal pairs of Golay's complementary sequences, *IEEE Transactions on Sonics and Ultrasonics* **SU-18**(2), 103-107.

[22] C.C. Tseng and C.L. Liu, (1972), Complementary sets of sequences, *IEEE Transactions on Information Theory* **IT-18**(5).

[23] Richard Turyn, (1963), Ambiguity functions of complementary sequences, *IEEE Transactions on Information Theory*, **IT-9**, 47.

[24] R.J. Turyn, (1974), Hadamard matrices, Baumert-Hall units, four-symbol sequences, pulse compressions and surface wave encodings, *J. Combinatorial Theory, Ser. A* **16**, 313-333.

[25] Jennifer Seberry Wallis, (1972), Hadamard matrices, Part IV of W.D. Wallis, Anne Penfold Street, and Jennifer Seberry Wallis, *Combinatorics: Room squares, sum free sets and Hadamard matrices,* Lecture Notes in Mathematics, Vol 292, Springer-Verlag, Berlin-Heidelberg-New York.

[26] Jennifer (Seberry) Wallis, (1973), Hadamard matrices of order $28m, 36m$ and $44m$, *J. Combinatorial Theory*, **15**, 323-328.

[27] George Welti, (1960), Quaternary codes for pulsed radar, *IRE Transactions on Information Theory*, **IT-6**, 400-408.

[28] Earl Glen Whitehead Jr, (1978), Autocorrelation of (+1,-1) sequences, in *Combinatorial Mathematics,* (eds D.A. Holton and Jennifer Seberry), Vol 686, Lecture Notes in Mathematics, Springer-Verlag, Berlin - Heidelberg - New York, 329-326.

[29] C.H.Yang, (1982), Hadamard matrices and δ-codes of length $3n$, *Proc. Amer. Math. Soc*, **85**, 375-378.

[30] C.H.Yang, (1983), A composition theorem for δ-codes, *Proc. Amer. Math. Soc.*, **89**, 375-378.

[31] C.H.Yang, (1983), Lagrange identity for polynomials and δ-codes of lengths $7t$ and $13t$, *Proc. Amer. Math. Soc.*, **88**, 746-750.

1. Department of Computer Science *
University College
The University of New South Wales
Australian Defence Force Academy
Canberra, A.C.T. 2600
Australia

2. Department of Mathematics †
University of Thessaloniki
Thessaloniki 54006
Greece

3. Department of Mathematics ‡
Tokyo Woman's Christian University
Zempukuji, Suginami-Ku, Tokyo 167
Japan