Further Hadamard matrices with maximal excess and new $SBIBD(4k^2, 2k^2+k, k^2+k)$

Christos Koukouvinos †
Stratis Kounias †
and
Jennifer Seberry*

Abstract

Let $\sigma(n)$ be the maximal excess of all Hadamard matrices of order n. We find relations between the decompositions of the order $4w$, w the order of four Williamson-type matrices, into squares and t, the order of T-matrices, into squares, in some cases. These are then illustrated and used to find Hadamard matrices with maximal excess for the following new cases:

(ia) $n = (4m+1)^2 + 3$ for $m = 9, 11$ with $\sigma(n) = n(4m+1) = n\sqrt{n-3}$.

(iia) $n = (4m+3)^2 + 3$ for $m = 14, 18, 24$ with $\sigma(n) = n(4m+3) = n\sqrt{n-3}$.

(iiia) $n = 4(2m+1)^2$ for $k = 2m+1 = 31, 49, 55, 57, 61, 85, 87, 91, 93$ with $\sigma(n) = n(4m+2) = n\sqrt{n}$.

This means Hadamard matrices with maximal excess are known for

(ib) $n = (4m+1)^2 + 3$ for $m = 0, 1, 2, 3, 8, 9, 11, 13, 18$

(iib) $n = (4m+3)^2 + 3$ for $m = 0, 1, 2, 3, 4, 5, 12, 14, 18, 24$

(iiib) $n = 4(2m+1)^2 = 4k^2$ for even k when there is an Hadamard matrix of order $2k$ (in particular all $2k \le 210$) and $k \in \{1, 3, 5, \ldots, 45, 49, \ldots, 57, 61, \ldots, 69, 75, 81, \ldots, 95, 99, 115, 117, 625, 3^{2m}, 25 \cdot 3^{2m}, m \ge 0\}$.

and

(iv) $n = 4m(m-1)$, $(\sigma(n) = 4(m-1)^2(2m+1))$ where m is the order of a skew-Hadamard matrix or the order of a conference matrix.

Hence there exist $SBIBD(4k^2, 2k^2+k, k^2+k)$ and regular Hadamard matrices of order $4k^2$ for all k described in (iiib).

UTILITAS MATHEMATICA **36** (1989), pp.135-150.

1 Introduction

An *Hadamard matrix* of order n is an $n \times n$ matrix H with elements $+1, -1$, satisfying $H^T H = HH^T = nI_n$. The sum of the elements of H, denoted by $\sigma(H)$, is called *excess* of H. The maximum excess of H, over all Hadamard matrices of order n, is denoted by $\sigma(n)$, i.e.

$$\sigma(n) = \max \sigma(H) \text{ for all Hadamard matrices of order } n \tag{1}$$

An equivalent notion is the *weight* $w(H)$ which is the number of 1's in H, then $\sigma(H) = 2w(H) - n^2$ and $\sigma(n) = 2w(n) - n^2$, see [9,18,20,27].

Kounias and Farmakis [16] proved that $\sigma(n) = n\sqrt{n}$ when $n = 4(2m+1)^2$ thus satisfying the equality of Best's inequality:

$$\sigma(n) \leq n\sqrt{n}$$

Infinite families of Hadamard matrices satisfying this bound have been found by Seberry [20] and Yamada [29], more have been found by Koukouvinos and Kounias [12].

A *regular Hadamard matrix* has constant row and column sum. These are discussed by Seberry Wallis [22,pp341–346].

A *symmetric balanced incomplete block design* or $SBIBD(v, k, \lambda)$ can be defined as a square matrix of order v with entries 0 or 1, with k 1's in row and column and the inner product of an pair of distinct rows is λ. For more details see Street and Street [24].

An *orthogonal design* $D = x_1A_1 + x_2A_2 + \ldots + x_uA_u$ of order n and type $(s_1, \ldots, s_u)$, written $OD(n; s_1, s_2, \ldots, s_u)$, on the commuting variables $x_1, \ldots, x_u$ is a square matrix with entries $0, \pm x_1, \ldots, \pm x_u$ where x_i or $-x_i$ occurs s_i times in each row and column and distinct rows are formally orthogonal. That is

$$DD^T = \sum_{j=1}^{u} s_j x_j^2.$$

Each A_j is a $(0,1,-1)$-matrix satisfying $A_jA_j^T = s_jI_n$ and is called a *weighing matrix* of weight s_j. A weighing matrix of order n and weight n is called an *Hadamard matrix.*

We define the *excess of the orthogonal design* D as

$$\sigma(D) = \sigma(A_1) + \ldots + \sigma(A_u),$$

where $\sigma(Y_i)$ is the sum of the entries of A_i, this is equivalent to putting all the variables equal to $+1$.

Suitable matrices are matrices with elements +1 and -1 which can be used to replace the variables of ODs to form Hadamard matrices. Of special interest are *Williamson type matrices*, which are 4 matrices, W_1, W_2, W_3, W_4 with elements +1 or -1 of order w which satisfy

$$\sum_{i=1}^{4} W_i W_j^T = 4wI_w$$
$$W_i W_j^T = W_j W_i^T$$

Our construction follows that of Hammer, Levingston and Seberry [9] who formed orthogonal designs $OD(4t; t, t, t, t)$ and then replaced the variables by suitable matrices.

This practice for constructing Hadamard matrices derived from extensions due to Baumert–Hall [1] who found the first $OD(12; 3, 3, 3, 3)$ and Cooper and (Seberry) Wallis [4] who first introduced T-matrices to form $OD(4t; t, t, t, t)$. The variables of these ODs are then replaced by Williamson type matrices of order w to form Hadamard matrices of order $4wt$. These are discussed extensively by Geramita and Seberry [7,pp120–125]. Cohen et al [3] survey the most recent results. This method was also used by Koukouvinos and Kounias [12] to find Hadamard matrices with maximal excess.

The case $n = (2m+1)^2 + 3$ was studied by Kounias and Farmakis [16] (explicitly), Hammer, Levingston and Seberry [9] (implicitly) and Jenkins, Koukouvinos, Kounias, J.Seberry and R.Seberry [10], who showed the two results were equivalent giving

$$\sigma(n) \leq n\sqrt{n-3}. \tag{2}$$

Koukouvinos, Kounias and Seberry [13] have found an infinite family of Hadamard matrices of order $n = 4v$, where $v = q^2 + q + 1$ is a prime and q a prime power, which satisfy this bound (2).

Kounias and Farmakis [16] show that if $n = 4m(m-1)$ then

$$\sigma(n) \leq 4(m-1)^2(2m+1). \tag{3}$$

Kharaghani [11] has given another proof of this bound and further shown that if there is a skew-Hadamard matrix of order m, then there is an Hadamard matrix of order $n = 4m(m-1) \equiv 0 (\bmod 16)$ with maximal excess meeting this bound. Koukouvinos and Seberry [15] have given another proof of Kharaghani's result and also found that if there is a conference matrix of order m, then there is an Hadamard matrix of order $n = 4m(m-1) \equiv 8 (\bmod 16)$ with maximal excess also meeting this bound. Since infinite families of skew-Hadamard matrices and conference matrices are known [19], [22, p452] this means there are infinite families satisfying this bound.

A tabulation of known orders of maximal excess for each order $4t$ of an Hadamard matrix, $t \leq 250$, is given in Jenkins et al [10].

In this paper we find relations between the decompositions of the order $4w$, w the order of four Williamson-type matrices, into squares and t, the order of

T-matrices, into squares, in some cases. These are then illustrated and used to find Hadamard matrices with maximal excess for the following new cases:

(ia) $n = (4m+1)^2 + 3$ for $m = 9, 11$ with $\sigma(n) = n(4m+1) = n\sqrt{n-3}$.

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(iiia) $n = 4(2m+1)^2$ for $k = 2m+1 = 31, 49, 55, 57, 61, 85, 87, 91, 93$ with $\sigma(n) = n(4m+2) = n\sqrt{n}$.

This means Hadamard matrices with maximal excess are known for

(ib) $n = (4m+1)^2 + 3$ for $m = 0, 1, 2, 3, 8, 9, 11, 13, 18$

(iib) $n = (4m+3)^2 + 3$ for $m = 0, 1, 2, 3, 4, 5, 12, 14, 18, 24$

(iiib) $n = 4(2m+1)^2 = 4k^2$ for even k when there is an Hadamard matrix of order $2k$ (in particular all $2k \leq 210$) and $k \in \{1, 3, 5, \ldots, 45, 49, \ldots, 57, 61, \ldots, 69, 75, 81, \ldots, 95, 99, 115, 117, 625, 3^{2m}, 25 \cdot 3^{2m}, m \geq 0\}$.

and

(iv) $n = 4m(m-1)$, $(\sigma(n) = 4(m-1)^2(2m+1))$ where m is the order of a skew-Hadamard matrix or the order of a conference matrix.

2 Method of Construction

We use the results of Table 1 to form circulant matrices (of order t of commuting variables) which are variations of the form

$$\begin{array}{rcrcrcrcr} X & = & AT_1 & + & BT_2 & + & CT_3 & + & DT_4 \\ Y & = & -BT_1 & + & AT_2 & + & DT_3 & - & CT_4 \\ Z & = & -CT_1 & - & DT_2 & + & AT_3 & + & BT_4 \\ W & = & -DT_1 & + & CT_2 & - & BT_3 & + & AT_4 \end{array}$$

To form the circulant matrix, T_j for order t, we first form the first row by replacing the $(1, i)$ element by $+1$ if i is in the set for T_j, by -1 if $-i$ is in the set for T_j and by 0 otherwise. The circulant matrix is then formed by setting the (g, h) element equal to the $(1, (h-g+1) \bmod t)$ element.

These are then used in the Goethals - Seidel or (Seberry)Wallis - Whiteman array to form orthogonal designs $OD(4t; t, t, t, t)$. Such arrays are also called Baumert-Hall arrays of order t. The variables are then replaced by the indicated Williamson-type matrices to form Hadamard matrices of maximal excess. The resultant Hadamard matrices are block-circulant.

Order	Sum of Squares	T_i	Sets
31	$3^2+3^2+3^2+2^2$	T_1	{ 1,5,-8,-9,11,-14,24,25,27 }
		T_2	{ 2,6,10,-12,19,-21,26,-29,30 }
		T_3	{ 4,7,-16,17,-18,20,22,23,-28}
		T_4	{ 3,13,15,-31}
39	$6^2+1^2+1^2+1^2$	T_1	{17,20,-21,23,24,26,35,38 }
		T_2	{14,15,-16,-18,19,22,25,-34,-36,-37,39 }
		T_3	{-4,-7,-8,10,11,13,28,-29,-31,32,33 }
		T_4	{1,2,-3,-5,6,-9,-12,27,30 }
43	$4^2+3^2+3^2+3^2$	T_1	{1,4,-5,6,7,8,9,-13,-14,15,16,-17,-18,21 }
		T_2	{-2,3,10,11,-12,19,20 }
		T_3	{-22,-23,24,26,29,31,34,36,-39,-41,42 }
		T_4	{-25,-27,28,30,32,33,-35,37,-38,40,43 }
49	$4^2+4^2+4^2+1^2$	T_1	{4,6,-18,19,21,-32,34,44,45,-46 }
		T_2	{-8,9,10,12,14,25,26,28,-36,37,38,-40,-42,-48 }
		T_3	{11,13,22,-23,-24,27,39,-41,47,49 }
		T_4	{-1,2,3,5,7,15,-16,-17,-20,29,-30,-31,33,35,-43 }
49	$5^2+4^2+2^2+2^2$	T_1	{1,-2,-3,5,7,-15,16,17,20,-29,30,31,33,35,-43 }
		T_2	{11,-13,-22,23,24,-27,39,41,47,49 }
		T_3	{-4,6,18,-19,-21,32,34,44,45,-46 }
		T_4	{8,-9,-10,12,14,25,26,28,36,-37,-38,-40,-42,48 }
55	$5^2+5^2+2^2+1^2$	T_1	{1,2,-5,7,8,-9,10,11,-23,-24,27,29,30,-31,32,33,-45,-47,48 }
		T_2	{-14,15,17,-36,37,39,51,52,-53,54,55 }
		T_3	{12,13,-16,18,19,-20,21,22,34,35,-38,-40,-41,42,-43,-44 }
		T_4	{-3,4,6,25,-26,-28,-46,49,50 }
57	$4^2+4^2+4^2+3^2$	T_1	{-24,-25,29,30,-31,32,33,-35,36,37,38,53 }
		T_2	{20,21,22,-23,-26,27,-28,-34,49,50,51,-52,54,55,-56,57 }
		T_3	{5,6,-10,11,-12,13,14,-16,17,18,19,40,-41,42,45,-46,-47,-48 }
		T_4	{1,2,3,-4,-7,8,-9,15,-39,43,44 }
61	6^2+5^2	T_1	{2,7,10,17,18,-26,29,-30,31,-32,35,40,-44,-51,55,61 }
		T_2	{3,4,-8,-11,-12,13,14,15,16,19,22,-25,27,-28,36,-37,-38,41, -42,-47,49,52,56,-57,60 }
		T_3	{-1,5,6,-9,-20,21,-23,-24,-33,-34,39,43,45,46,48,-50,-53,54, -58, 59 }
		T_4	{ ϕ }

Table 1: T-matrices used

Order	Sum of Squares	T_i	Sets
67	$8^2+1^2+1^2+1^2$	T_1	{-1,5,9,13,14,15,18,25,27,29,-31,32,-39,43,50,-67 }
		T_2	{2,-8,-12,16,17,23,-40,41,42,-45,-46,-47,-53,54,-56,65,66 }
		T_3	{-6,7,11,19,20,-21,24,-26,-28,-37,38,44,-49,57,-58,-59,61 }
		T_4	{-3,-4,10,22,30,-33,34,-35,36,48,51,-52,-55,-60,-62,63,64 }
85	7^2+6^2	T_1	{1,2,4,-5,-11,-12,14,-15,21,22,24,-25,31,32,-34,35,-41,-42, -44,45,51,52,-54,55,61,62,64,-65,71,72,-74,75,-81 }
		T_2	{3,-13,23,33,-43,53,63,73,82,83 }
		T_3	{-6,-7,-9,10,16,17,-19,20,-26,-27,-29,30,-36,-37,39,-40,-46, -47,-49,50,56,57,-59,60,66,67,69,-70,76,77,-79,80 }
		T_4	{8,18,28,-38,-48,-58,68,-78,-84,85 }
87	$7^2+6^2+1^2+1^2$	T_1	{-2,-3,5,6,10,11,13,-14,15,16,-17,20,21,24,-25,28,29,-62, -65,66,67,70,-73 }
		T_2	{30,-33,-36,-37,38,41,-47,48,51,52,55,-56,74,75,-78,79,82, 83,-86,87 }
		T_3	{1,-4,-7,-8,9,12,18,-19,-22,-23,-26,27,59,-60,61,63,64,68, 69,-71,-72 }
		T_4	{-31,-32,34,35,39,40,42,-43,44,-45,46,-49,-50,-53,54,-57, -58,-76,77,80,81,84,-85 }
91	$5^2+5^2+5^2+4^2$	T_1	{-1,-2,3,5,-6,-8,10,11,13,27,28,-29,-31,32,35,38,53,54,-55, -57,58,-60,62,63,65,-79,-82 }
		T_2	{-4,-7,9,12,30,33,34,-36,-37,-39,56,59,61,64,80,-81,-83, 84,85 }
		T_3	{17,20,,-22,-25,40,41,-42,-44,45,-48,-51,69,72,74,77,86,88, 89,-91 }
		T_4	{-14,-15,16,18,-19,-21,23,24,26,43,46,-47,49,50,52,-66,-67, 68,70,-71,73,-75,-76,-78,87,90 }
93	$6^2+5^2+4^2+4^2$	T_1	{2,3,4,5,-6,7,8,-9,-10,11,12,13,-14,15,-16,17,19,21,23,-25, -27,-29,31,-78 }
		T_2	{1,18,-20,-22,24,-26,-28,30,-63,64,-65,66,67,68,-69,-70,71, 72,-73,74,75,76,77 }
		T_3	{33,34,35,36,-37,38,39,-40,-41,42,43,44,-45,46,-47,-48,-50, -52,-54,56,58,60,-62,-80,82,84,-86,88,90,-92 }
		T_4	{32,-49,51,53,-55,57,59,-61,79,-81,-83,-85,87,89,91,93 }

Table 1 (cont): T-matrices used

Case A: $n = (4m+1)^2 + 3$

Hammer, Levingston and Seberry [9] (implicitly) and Kounias and Farmakis [16] (explicitly) proved that

$$\sigma(n) \leq n(4m+1) = n\sqrt{n-3}$$

and that an Hadamard matrix with maximal excess has row and column sum vectors

$$(4me_{3n/4}, (4m+4)e_{n/4}) \tag{4}$$

or

$$((4m-2)e_{n/4}, (4m+2)e_{3n/4}) \tag{5}$$

where e_s^T is an $s \times 1$ vector of ones and ^{T} denotes the transpose.

For m = 0, 1, 2, 3, 8, 13, 18 Hadamard matrices with this maximal excess have been constructed by Koukouvinos and Kounias [12], and Kounias and Farmakis [16].

We proceed as in [9] and [16] to use the Goethals - Seidel or (Seberry)Wallis - Whiteman array using T-matrices of order t to construct circulant or block circulant (type 1) matrices X, Y, Z, W of commuting variables A, B, C, D which will be replaced by Williamson-type matrices of order w, where $4w = a^2 + b^2 + c^2 + d^2$. We now give more details for the case where the row sums of X, Y, Z and W are $4m + 1, \pm 1, \pm 1$ and ± 1 respectively.

Let T_1, T_2, T_3, T_4 be T-matrices of order t with row sums t_1, t_2, t_3 and t_4. Let X, Y, Z, W be formed from T_1, T_2, T_3, T_4 either by taking

$$\begin{array}{rcrcrcrcr} X & = & AT_1 & + & BT_2 & + & CT_3 & + & DT_4 \\ Y & = & -BT_1 & + & AT_2 & + & DT_3 & - & CT_4 \\ X & = & -CT_1 & - & DT_2 & + & AT_3 & + & BT_4 \\ W & = & -DT_1 & + & CT_2 & - & BT_3 & + & AT_4 \end{array} \tag{6}$$

or

$$\begin{array}{rcrcrcrcr} X & = & -AT_1 & + & BT_2 & + & CT_3 & + & DT_4 \\ Y & = & BT_1 & + & AT_2 & + & DT_3 & - & CT_4 \\ Z & = & CT_1 & - & DT_2 & + & AT_3 & + & BT_4 \\ W & = & DT_1 & + & CT_2 & - & BT_3 & + & AT_4. \end{array} \tag{7}$$

If X, Y, Z, W have row sums $4m + 1$, ± 1, ± 1, ± 1 ($n = 16m^2 + 8m + 4$) and A, B, C, D are replaced by Williamson-type matrices of order w with row sums a, b, c, d ($4w = a^2 + b^2 + c^2 + d^2$) then

$$\begin{array}{rcrcrcrcrcr} at_1 & + & bt_2 & + & ct_3 & + & dt_4 & = & & 4m+1 \\ -bt_1 & + & at_2 & + & dt_3 & - & ct_4 & = & \pm & 1 \\ -ct_1 & - & dt_2 & + & at_3 & + & bt_4 & = & \pm & 1 \\ -dt_1 & + & ct_2 & - & bt_3 & + & at_4 & = & \pm & 1 \end{array} \tag{8}$$

or

$$\begin{array}{rcrcrcrcrcr} -at_1 & + & bt_2 & + & ct_3 & + & dt_4 & = & & 4m+1 \\ bt_1 & + & at_2 & + & dt_3 & - & ct_4 & = & \pm & 1 \\ ct_1 & - & dt_2 & + & at_3 & + & bt_4 & = & \pm & 1 \\ dt_1 & + & ct_2 & - & bt_3 & + & at_4 & = & \pm & 1 \end{array} \tag{9}$$

The rows can have $\pm$ depending on the sizes of the variables in the equations. A systematic, straight forward but tedious calculation shows:

- the equations (8) have two solutions up to equivalence

  (8i) $a = a, b = c = d, p = [(4m + 1)b \pm a]/4w$, $q = [(4m + 1)a \mp 3b]/4w$

  (8ii) $a = b = c = d = 1$, $p = m$, $q = m + 1$, $4w = a^2 + 3b^2, t = q^2 + 3p^2$,

- the equations (9) have one non-trivial solution in the positive integers up to equivalence:

(9i) $q = 0, p = 1, a = 1, b = (4m+1)/3,\ t = 3, 4w = 1 + (4m+1)^2/3$

Lemma 1 *Hadamard matrices with maximal excess $n(4m+1)$ and order $n = (4m+1)^2 + 3$ can be found using the Goethals - Seidel (or (Seberry)Wallis - Whiteman) arrays with T-matrices with row sums $m+1, m, m, m$.*

Comment: These are known (or possible) for the following values where $n = (4m+1)^2 + 3$.

m	n	t	$q^2+p^2+p^2+p^2$	Comment
1	28	7	$2^2+1^2+1^2+1^2$	Known [3]
2	84	21	$3^2+2^2+2^2+2^2$	Known [3]
3	172	43	$4^2+3^2+3^2+3^2$	Not Known
4	292	73	$5^2+4^2+4^2+4^2$	Not Known
5	444	111	$6^2+5^2+5^2+5^2$	Not Known

Lemma 2 *Hadamard matrices with maximal excess $n(4m+1)$ and order $n = 4tw = (4m+1)^2+3$ can be found using Williamson-type matrices with row sums $a, b, b, b, 4w = a^2 + 3b^2$ in Goethals - Seidel (or (Seberry)Wallis - Whiteman) arrays with T-matrices with row sums q, p, p, p where*

$$\begin{aligned} q &= [(4m+1)a \mp 3b]/4w \text{ and} \\ p &= [(4m+1)b \pm a]/4w. \end{aligned}$$

Comment: These are known or possible for

m	$n = 4tw$ $= (4m+1)^2+3$	a	b	$4w$	p	q	t	Comment
2	84	3	1	12	1	2	7	All exist [3]
5	444	3	1	12	2	5	37	All exist [14]
6	628							Impossible as $628 = 4\times$ prime
7	844							Impossible as $844 = 4\times$ prime
8	1092	3	5	84	3	1	13	All exist [3]
8	1092	5	3	52	2	3	21	All exist [3]

Lemma 3 *Hadamard matrices with maximal excess $n(4m+1)$ and order $n = (4m+1)^2 + 3$ can be found using the Goethals - Seidel (or (Seberry)Wallis-Whiteman) array form by using*

$$\begin{aligned} X &= BI + CT + DT^2 \\ Y &= AI + DT - CT^2 \\ Z &= -DI + AT + BT^2 \\ W &= CI - BT + DT^2 \end{aligned}$$

where $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ *is of order 3 and* A, B, C, D *are replaced by Williamson - type matrices with row sums 1,* $(4m+1)/3$, $(4m+1)/3$, $(4m+1)/3$ *respectively.*

Remark: These are known (or possible) for the following values where $n = (4m+1)^2 + 3$.

m	n	t	$4w$	W	$a^2+b^2+b^2+b^2$	Comment
2	84	3	28	7	$1^2+3^2+3^2+3^2$	Exist [22, p.388-9]
5	444	3	148	37	$1^2+7^2+7^2+7^2$	Not known
8	1092	3	364	91	$1^2+11^2+11^2+11^2$	Not known
11	2028	3	676	169	$1^2+15^2+15^2+15^2$	Not known

Results

Using the results in Table 1 we now can construct Hadamard matrices with maximal excess for the two orders $n = 1372$, $m = 9$ and $n = 2028$, $m = 11$.

(i) $m = 9, n = 1372 = 4.7.49$; $t = 49 = 4^2+4^2+4^2+1^2$, $4w = 28 = 3^2+3^2+3^2+1^2$ (see [22, p388-9]).

The following circulant matrices of commuting variables give an $OD(4.49; 49, 49, 49, 49)$:

$$\begin{array}{rcrcrcrcr} X & = & AT_1 & + & BT_2 & + & CT_3 & + & DT_4 \\ Y & = & -BT_1 & + & AT_2 & + & DT_3 & - & CT_4 \\ Z & = & CT_1 & + & DT_2 & - & AT_3 & - & BT_4 \\ W & = & DT_1 & - & CT_2 & + & BT_3 & - & AT_4 \end{array}$$

(ii) $m = 11, n = 2028 = 4.13.39$; $t = 39 = 6^2+1^2+1^2+1^2$, $4w = 52 = 7^2+1^2+1^2+1^2$ (see [22, p388-9]).

The following circulant matrices of commuting variables give an $OD(4.39; 39, 39, 39, 39)$:

$$\begin{array}{rcrcrcrcr} X & = & AT_1 & + & BT_2 & + & CT_3 & + & DT_4 \\ Y & = & -BT_1 & + & AT_2 & + & DT_3 & - & CT_4 \\ Z & = & -CT_1 & - & DT_2 & + & AT_3 & + & BT_4 \\ W & = & -DT_1 & + & CT_2 & - & BT_3 & + & AT_4 \end{array}$$

Case B: $n = (4m+3)^2 + 3$

Hammer, Levingston and Seberry [9] (implicitly) and Kounias and Farmakis [16] (explicitly) proved that

$$\sigma(n) \leq n(4m+3) = n\sqrt{n-3}$$

and the Hadamard matrix with maximal excess has row or column sum vectors

$$((4m+2)e_{3n/4}, (4m+6)e_{n/4}) \tag{10}$$

or

$$(4me_{n/4}, (4m+4)e_{3n/4}). \tag{11}$$

For $m = 0, 1, 2, 3, 4, 5, 12$ Hadamard matrices with this maximal excess have been constructed by Koukouvinos and Kounias [12], and Kounias and Farmakis [16].

We proceed as in case A. In this case X, Y, Z, W have row sums $4m+3$, ± 1, ± 1, ± 1:

If X, Y, Z, W have row sums $4m+3$, ± 1, ± 1, ± 1 ($n = 16m^2 + 24m + 12$) and A, B, C, D are replaced by Williamson-type matrices of order w with row sums a, b, c, d ($4w = a^2 + b^2 + c^2 + d^2$) then

$$\left.\begin{array}{rcrcrcrcl} at_1 & + & bt_2 & + & ct_3 & + & dt_4 & = & 4m+3 \\ -\ bt_1 & + & at_2 & + & dt_3 & - & ct_4 & = & \pm 1 \\ -\ ct_1 & - & dt_2 & + & at_3 & + & bt_4 & = & \pm 1 \\ -\ dt_1 & + & ct_2 & - & bt_3 & + & at_4 & = & \pm 1 \end{array}\right\} \tag{12}$$

or

$$\left.\begin{array}{rcrcrcrcl} -\ at_1 & + & bt_2 & + & ct_3 & + & dt_4 & = & 4m+3 \\ bt_1 & + & at_2 & + & dt_3 & - & ct_4 & = & \pm 1 \\ ct_1 & - & dt_2 & + & at_3 & + & bt_4 & = & \pm 1 \\ dt_1 & + & ct_2 & - & bt_3 & + & at_4 & = & \pm 1 \end{array}\right\} \tag{13}$$

The rows can have $\pm$ depending on the sizes of the variables in the equations. A systematic, straightforward but tedious calculation shows:

- The equations (12) have two solutions up to equivalence

  (12i) $a = a$, $b = c = d$, $p = [(4m+3)b \pm a]/4w$, $q = [(4m+3)a \mp 3b]/4w$;

  (12ii) $a = b = c = d = 1$, $p = m+1$, $q = m$, $4w = a^2 + 3b^2$, $t = q^2 + 3p^2$;

- The equations (13) have one non-trivial solution in the positive integers up to equivalence.

  (13i) $q = 0$, $p = 1$, $a = 1$, $b = (4m+3)/3$, $t = 3$, $4w = 1 + (4m+3)^2/3$.

Lemma 4 *Hadamard matrices with maximal excess $n(4m+3)$ and order $n = (4m+3)^2 + 3$ can be found using the Goethals - Seidel (or (Seberry) Wallis - Whiteman) arrays with T-matrices with row sums $m+1$, $m+1$, $m+1$, m.*

Comment: There are known (or possible) for the following values where $n = (4m+3)^2 + 3$.

m	n	t	$q^2+p^2+p^2+p^2$	Comment
1	52	13	$1^2+2^2+2^2+2^2$	Known [3]
2	124	31	$2^2+3^2+3^2+3^2$	Known [3]
3	228	57	$3^2+4^2+4^2+4^2$	Known [3]
4	364	91	$4^2+5^2+5^2+5^2$	Known [3]
5	532	133	$5^2+6^2+6^2+6^2$	Not Known
6	732	183	$6^2+7^2+7^2+7^2$	Not Known
7	964	241	$7^2+8^2+8^2+8^2$	Not Known
8	1228	307	$8^2+9^2+9^2+9^2$	Not Known

Lemma 5 *Hadamard matrices with maximal excess* $n(4m+3)$ *and order* $n = 4tw = (4m+3)^2+3$ *can be found using Williamson-type matrices with row sums* a, b, b, b, $4w = a^2+3b^2$ *in Geothals - Seidel (or (Seberry)Wallis - Whiteman) arrays with T-matrices with row sums* q, p, p, p, *where*

$$
\begin{aligned}
q &= [(4m+3)a \mp 3b]/4w \\
p &= [(4m+3)b \pm a]/4w.
\end{aligned}
$$

Comment: These are known or possible for

m	$n = 4tw = (4m+3)^2+3$	a	b	$4w$	p	q	t	Comment
3	228	3	1	12	1	4	19	All exist [3]
4	364	1	3	28	2	1	13	All exist [3]
4	364	5	3	52	1	2	7	All exist [3]
5	532	5	1	28	1	4	19	All exist [3]
5	532	7	3	76	1	2	7	All exist [3]
6	732	3	1	12	2	7	61	Not Known
7	964							Impossible as $964 = 4\times$ prime
8	1228							Impossible as $1228 = 4\times$ prime

Lemma 6 *Hadamard matrices with maximal excess* $n(4m+3)$ *and order* $n = (4m+3)^2+3$ *can be found using the Goethals - Seidel (or (Seberry)Wallis - Whiteman) array form by using*

$$
\begin{aligned}
X &= BI + CT + DT^2 \\
Y &= AI + DT - CT^2 \\
Z &= -DI + AT + BT^2 \\
W &= CI - BT + DT^2
\end{aligned}
$$

where $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ *is of order* 3 *and* A, B, C, D *are replaced by Williamson-type matrices with row sums* 1, $(4m+3)/3$, $(4m+3)/3$, $(4m+3)/3$ *respectively.*

Remark: These are known (or possible) for the following values where $n = (4m+3)^2+3$.

m	n	t	$4w$	w	$a^2+b^2+b^2+b^2$	Comment
3	228	3	76	19	$1^2+5^2+5^2+5^2$	Do not exist [22, p388-9]
6	732	3	244	61	$1^2+9^2+9^2+9^2$	Not Known
9	1524	3	508	127	$1^2+13^2+13^2+13^2$	Not Known
12	2604	3	868	217	$1^2+17^2+17^2+17^2$	Not Known

Results:

Using the results in Table 1 we now can construct Hadamard matrices with maximal excess for the three orders $n = 3484$, $m = 14$; $n = 5628$, $m = 18$ and $n = 9804$, $m = 24$.

(i) $m = 14$, $n = 3484 = 4.13.67$; $t = 67 = 8^2+1^2+1^2+1^2$, $4w = 52 = 7^2+1^2+1^2+1^2$ (see [22, p388-9]).

The following circulant matrices of commuting variables give an $OD(4.67; 67, 67, 67, 67)$:

$$\begin{array}{rcrcrcrcr} X &=& AT_1 &+& BT_2 &+& CT_3 &+& DT_4 \\ Y &=& BT_1 &-& AT_2 &-& DT_3 &+& CT_4 \\ Z &=& CT_1 &+& DT_2 &-& AT_3 &-& BT_4 \\ W &=& DT_1 &-& CT_2 &+& BT_3 &-& AT_4 \end{array}$$

(ii) $m = 18$, $n = 5628 = 4.21.67$; $t = 67 = 8^2+1^2+1^2+1^2$, $4w = 84 = 9^2+1^2+1^2+1^2$ (see [22, p388-9]).

The following circulant matrices of commuting variables give an $OD(4.67; 67, 67, 67, 67)$:

$$\begin{array}{rcrcrcrcr} X &=& AT_1 &+& BT_2 &+& CT_3 &+& DT_4 \\ Y &=& -BT_1 &+& AT_2 &+& DT_3 &-& CT_4 \\ Z &=& -CT_1 &-& DT_2 &+& AT_3 &+& BT_4 \\ W &=& -DT_1 &+& CT_2 &-& BT_3 &+& AT_4 \end{array}$$

(iii) $m = 24$, $n = 9804 = 4.43.57$; $t = 57 = 4^2+4^2+4^2+3^2$, $4w = 172 = 7^2+7^2+7^2+5^2$ (see [22, p388-9]).

The following circulant matrices of commuting variables give an $OD(4.57; 57, 57, 57, 57)$:

$$\begin{array}{rcrcrcrcr} X &=& BT_1 &+& CT_2 &+& AT_3 &+& DT_4 \\ Y &=& -AT_1 &-& DT_2 &+& BT_3 &+& CT_4 \\ Z &=& -DT_1 &+& AT_2 &-& CT_3 &+& BT_4 \\ W &=& CT_1 &-& BT_2 &-& DT_3 &+& AT_4 \end{array}$$

Case C: $n = 4(2m+1)^2 = 4k^2$

In this case X, Y, Z, W hav row sums $(2m+1), \pm(2m+1), \pm(2m+1, \pm(2m+1)$. If X, Y, Z, W have row sums $(2m+1)$, $\pm(2m+1)$, $\pm(2m+1)$, $\pm(2m+1)$, $(n = 16m^2+16m+4)$ and A, B, C, D are replaced by Williamson - type matrices of order w with row sums a, b, c, d $(4w = a^2+b^2+c^2+d^2)$ then

$$\left.\begin{array}{rcrcrcrcl} at_1 & + & bt_2 & + & ct_3 & + & dt_4 & = & (2m+1) \\ -bt_1 & + & at_2 & + & dt_3 & - & ct_4 & = & \pm(2m+1) \\ -ct_1 & - & dt_2 & + & at_3 & + & bt_4 & = & \pm(2m+1) \\ -dt_1 & + & ct_2 & - & bt_3 & + & at_4 & = & \pm(2m+1) \end{array}\right\} \quad (14)$$

The rows can have $\pm$ depending on the sizes of the variables in the equations.

It is known (see [16]) that $\sigma(n) \le n(4m+2) = n2k = n\sqrt{n}$ and the corresponding Hadamard matrix has all row and column sums equal to $4m+2 = 2k$, i.e. it is regular (see also [22, p280]).

Seberry [20] proved the following theorem:

Theorem 7 (Seberry) *Hadamard matrices of order* $4k^2$ *with maximal excess* $8k^3$ *exist for*

(i) k *even* $k \le 210$, *or an Hadamard matrix of order* $2k$ *exists,*

(ii) $k \in \{$ 1, 3, 5, ..., 29, 33, ..., 41,*45,51,53,61,...,69,75,81,83,89,95,* 99, 625, 3^{2m}, $5^2.3^{2m}$, $m \ge 0\}$.

This means that regular Hadamard matrices of order $4k^2$ *and* $SBIBD(4k^2, 2k^2 \pm k, k^2 \pm k)$ *also exist for these* k *values.*

In this section we extend these values to include $k \in \{$ 31, 43, 49, 55, 57, 85, 87, 91, 93$\}$ and find a new construction for $k = 61$.

Lemma 8 *Hadamard matrices with maximal excess* $n(4m+2)$ *and order* $n = 4(2m+1)^2$ *can be found using Williamson-type matrices with row sums* a, b, c, d $(4(2m+1) = a^2+b^2+c^2+d^2)$ *in Goethals - Seidel (or (Seberry)Wallis - Whiteman) arrays with T-matrices with row sums* t_1, t_2, t_3, t_4 $(2m+1 = t_1^2 + t_2^2 + t_3^2 + t_4^2)$.

Results:

Using the results in Table 1 we now can construct Hadamard matrices with maximal excess for the twelve orders in Table 2.

m	n	$\sigma(n)$	T-matrix squares				Williamson squares				Method	Ref
			x_1	x_2	x_3	x_4	a	b	c	d		
15	$4 \cdot 31^2 =$ 3844	$8 \cdot 31^3$	3	3	3	2	5	5	7	5	1G	[25]
21	$4 \cdot 43^2 =$ 7396	$8 \cdot 43^3$	4	3	3	3	5	7	7	7	1G	[25,30]
24	$4 \cdot 49^2 =$ 9604	$8 \cdot 49^3$	5	4	2	2	3	9	9	5	1G	[25]
27	$4 \cdot 55^2 =$ 12100	$8 \cdot 55^3$	5	5	2	1	3	9	11	3	1G	[25]
28	$4 \cdot 57^2 =$ 12996	$8 \cdot 57^3$	4	4	4	3	7	7	9	7	1G	[25]
30	$4 \cdot 61^2 =$ 14884	$8 \cdot 61^3$	6	5			1	11	1	11	2G	[25,28]
42	$4 \cdot 85^2 =$ 28900	$8 \cdot 85^3$	7	6			1	13	1	13	2G	[25]
43	$4 \cdot 87^2 =$ 30276	$8 \cdot 87^3$	7	6	1	1	1	13	13	3	1G	[25]
45	$4 \cdot 91^2 =$ 33124	$8 \cdot 91^3$	5	5	5	4	9	9	11	9	1G	[25]
46	$4 \cdot 93^2 =$ 34596	$8 \cdot 93^3$	6	-5	-4	-4	19	1	3	1	2G	[25]
57	$4 \cdot 115^2 =$ 52900	$8 \cdot 115^3$	1	1	7	8	15	3	1	15	1G	[10,25]
58	$4 \cdot 117^2 =$ 54756	$8 \cdot 117^3$	7	6	4	4	7	13	13	9	1G	[10,25]

Table 2: Constructions of Hadamard matrices with maximal excess $4k^2$

Remark: For the construction of Williamson-type matrices of order 93 we have used (31, 25, 20) and (31, 16, 8) cyclic difference sets with row sums of the corresponding incidence matrices 19 and 1 respectively in Seberry Wallis construction [23]. So the Williamson-type matrices of order 93 have row sums 19, 3, 1, 1.

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†Department of Mathematics
University of Thessaloniki
Thessaloniki, 54006
Greece

* Department of Computer Science
University College
University of New South Wales
Australian Defence Force Academy
Canberra, A.C.T. 2600
AUSTRALIA