

## ***SBIBD*( $4k^2, 2k^2 + k, k^2 + k$ ) and Hadamard Matrices of Order $4k^2$ with Maximal Excess Are Equivalent**

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**Abstract.** We show that an *SBIBD*( $4k^2, 2k^2 + k, k^2 + k$ ) is equivalent to a regular Hadamard matrix of order  $4k^2$  which is equivalent to an Hadamard matrix of order  $4k^2$  with maximal excess.

We find many new *SBIBD*( $4k^2, 2k^2 + k, k^2 + k$ ) including those for even  $k$  when there is an Hadamard matrix of order  $2k$  (in particular all  $2k \leq 210$ ) and  $k \in \{1, 3, 5, \dots, 29, 33, \dots, 41, 45, 51, 53, 61, \dots, 69, 75, 81, 83, 89, 95, 99, 625, 3^{2m}, 25 \cdot 3^{2m}, m \geq 0\}$ .

### **1. Introduction**

An Hadamard matrix of order  $n$  is an  $n \times n$  matrix  $H$  with elements  $+1, -1$ , satisfying  $H^T H = H H^T = nI_n$ . The sum of the elements of  $H$ , denoted by  $\sigma(H)$ , is called excess of  $H$ . The maximum excess of  $H$ , over all Hadamard matrices of order  $n$ , is denoted by  $\sigma(n)$ , i.e.

$$\sigma(n) = \max \sigma(H) \quad \text{for all Hadamard matrices of order } n \quad (1)$$

An equivalent notion is the weight  $w(H)$  which is the number of 1's in  $H$ , then  $\sigma(H) = 2w(H) - n^2$  and  $\sigma(n) = 2w(n) - n^2$ , see [5, 10, 16, 25].

Kounias and Farmakis [13] proved that  $\sigma(n) = n\sqrt{n}$  when  $n = 4(2m + 1)^2$  thus satisfying the equality of Best's inequality:

$$\sigma(n) \leq n\sqrt{n}$$

A regular Hadamard matrix has constant row and column sum. These are discussed by Seberry Wallis [24, pp. 341–346].

A symmetric balanced incomplete block design or *SBIBD*( $v, k, \lambda$ ) can be defined as a square matrix of order  $v$  with entries 0 or 1, with  $k$  1's in row and column and the inner product of an pair of distinct rows is  $\lambda$ . For more details see Street and Street [17].

An orthogonal design  $D = x_1 A_1 + x_2 A_2 + \dots + x_u A_u$  of order  $n$  and type  $(s_1, \dots, s_u)$ , written  $OD(n; s_1, s_2, \dots, s_u)$ , on the commuting variables  $x_1, \dots, x_u$  is a square matrix with entries  $0, \pm x_1, \dots, \pm x_u$  where  $x_i$  or  $-x_i$  occurs  $s_i$  times in each row and column and distinct rows are formally orthogonal. That is

$$DD^T = \sum_{j=1}^u s_j x_j^2.$$

Each  $A_j$  is a  $(0, 1, -1)$ -matrix satisfying  $A_j A_j^T = s_j I_n$  and is called a *weighing matrix* of weight  $s_j$ . A weighing matrix of order  $n$  and weight  $n$  is called an *Hadamard matrix*.

We define the *excess of the orthogonal design*  $D$  as

$$\sigma(D) = \sigma(A_1) + \cdots + \sigma(A_u),$$

where  $\sigma(A_i)$  is the sum of the entries of  $A_i$ , this is equivalent to putting all the variables equal to  $+1$ .

*Suitable matrices* are matrices with elements  $+1$  and  $-1$  which can be used to replace the variables of *ODs* to form Hadamard matrices. Of special interest are *Williamson type matrices*, which are 4 matrices,  $W_1, W_2, W_3, W_4$  with elements  $+1$  or  $-1$  of order  $w$  which satisfy

$$\sum_{i=1}^4 W_i W_i^T = 4wI_w$$

$$W_i W_j^T = W_j W_i^T$$

Our construction follows that of Hammer, Levingston and Seberry [8] who formed orthogonal designs  $OD(4t; t, t, t, t)$  and then replaced the variables by suitable matrices.

This practice for constructing Hadamard matrices derived from extensions due to Baumert-Hall [1] who found the first  $OD(12; 3, 3, 3, 3)$  and Cooper and (Seberry) Wallis [4] who first introduced *T*-matrices to form  $OD(4t; t, t, t, t)$ . The variables of these *ODs* are then replaced by Williamson type matrices of order  $w$  to form Hadamard matrices of order  $4wt$ . These are discussed extensively by Geramita and Seberry [7, pp. 120–125]. Cohen, Rubie, Koukouvinos, Seberry and Yamada [3] survey the most recent results. This method was also used by Koukouvinos and Kounias [12] to find Hadamard matrices with maximal excess.

## 2. The Equivalence Theorem

**Theorem 1.** *There is an Hadamard matrix of order  $n = 4s^2$  and maximal excess  $n\sqrt{n} = 8s^3$  if and only if there is an  $SBIBD(4s^2, 2s^2 + s, s^2 + s)$ .*

*Proof.* If there is an *SBIBD*,  $B$ , with parameters  $(4s^2, 2s^2 + s, s^2 + s)$  then  $A = 2B - J$  has elements  $+1$  and  $-1$ .  $A$  has  $2s^2 + s$  elements  $+1$  in each row (and column) and  $2s^2 - s$  elements  $-1$  in each row (and column). Thus the row (column) sum of each row (column) of  $A$  is  $2s^2 + s - (2s^2 - s) = 2s$ . Thus the excess of  $A = 4s^2 \times 2s = 8s^3 =$  number of rows (columns) of  $A$  times the row (column) sum of  $A$ .

Further

$$\begin{aligned} AA^T &= (2B - J)(2B - J)^T \\ &= 4BB^T - 2JB^T - 2BJ + J^2 \\ &= (s^2I + (s^2 + s)J) - 4(2s^2 + s)J + 4s^2J \\ &= 4s^2I \end{aligned}$$

Thus  $A$  is an Hadamard matrix.

Conversely, let  $A$  be an Hadamard matrix of order  $n = 4s^2$  and maximal excess  $8s^3$ .

Let the column sum of the  $i$ th column of  $A$  be  $x_i$ . Then all  $x_i \geq 0$ , otherwise that entire column could be negated giving an Hadamard matrix with greater excess

$$\sum_{i=1}^n x_i = 8s^3, \quad x_i \geq 0 \text{ all } i \tag{2}$$

since the sum of the column sums is the excess. Now let  $e$  be the  $1 \times n$  matrix of ones. Since  $A$  is an Hadamard matrix we have

$$AA^T = 4s^2I$$

$$eAA^Te^T = 4s^2ee^T = 16s^4 = (x_1 x_2 \dots x_n)(x_1 x_2 \dots x_n)^T.$$

So

$$\sum_{i=1}^n x_i^2 = 16s^4. \tag{3}$$

The only solution to (2) and (3) is

$$x_1 = x_2 = \dots = x_n = 2s.$$

[Suppose  $x_i = 2s + t_i$ , then

$$\sum_{i=1}^n x_i = 8s^3 + \sum_{i=1}^n t_i = 8s^3. \quad \text{So } \sum_{i=1}^n t_i = 0.$$

Also

$$\sum_{i=1}^n x_i^2 = 16s^4 = 16s^4 + 4s \sum_{i=1}^n t_i + \sum_{i=1}^n t_i^2.$$

Thus

$$\sum_{i=1}^n t_i^2 = 0,$$

so  $t_i = 0$  for all  $i$ .]

But this means each column of  $A$  has  $2s^2 + s$  elements  $+1$  and  $2s^2 - s$  elements  $-1$ . Now, since  $A$  is an Hadamard matrix, the columns of  $A$  are orthogonal, so if two columns are written

$$\begin{array}{cccc} \underbrace{1 \dots 1}_x & \underbrace{1 \dots 1}_{2s^2 + s - x} & \underbrace{-1 \dots -1}_{2s^2 + s - x} & \underbrace{-1 \dots -1}_{4s^2 - 2(2s^2 + s - x) - x} \\ \underbrace{1 \dots 1}_x & \underbrace{-1 \dots -1}_{2s^2 + s - x} & \underbrace{1 \dots 1}_{2s^2 + s - x} & \underbrace{-1 \dots -1}_{4s^2 - 2(2s^2 + s - x) - x} \end{array}$$

where  $x, 2s^2 + s - x, 2s^2 + s - x, -2s + x$  are the number of columns of each type. Now since the rows are orthogonal

$$x - (2s^2 + s - x) - (2s^2 + s - x) - 2s + x = 0$$

$$4s^2 + 4s = 4x$$

$$x = s^2 + s$$

Thus  $A$  has  $2s^2 + s$  elements  $+1$  in each column and  $s^2 + s$  elements  $+1$  in any column overlapping with elements  $+1$  in every other column. A similar argument can be used for the rows. Thus  $B = \frac{1}{2}(A + J)$  is an  $SBIBD(4s^2, 2s^2 + s, s^2 + s)$ .  $\square$

In Seberry Wallis [24, p. 343] it is pointed out that Goethals and Seidel [9] and Shrikhande and Singh [19] have established:

**Theorem 2.** *If there exists a  $BIBD(2k^2 - k, 4k^2 - 1, 2k + 1, k, 1)$  then there exists a symmetric Hadamard matrix with constant diagonal of order  $4k^2$ .*

Moreover Shrikhande [18], [21] has studied these designs and showed they exist for  $k = 2^t, t \geq 1$ . They are also known for  $k = 3, 5, 6, 7$  [7].

In Seberry Wallis [24, pp. 344–346] it is established that symmetric Hadamard matrices with constant diagonal thus exist for  $2^{2t}, t \geq 1, 36, 100, 144, 196$  (after Theorem 5.14 of [24]) and using results of (Seberry) Wallis-Whiteman [23] and Szekeres [20] they are shown to exist with the extra property of regularity for  $4 \cdot 5^2, 4 \cdot 13^2, 4 \cdot 29^2, 4 \cdot 51^2$ , and  $4 \left( 2 \left( \frac{p-3}{4} \right) + 1 \right)^2$ , for  $p \equiv 3 \pmod{4}$  a prime power (after Theorem 5.15 of [24]).

*Remark 1.* Now a Theorem of Goethals and Seidel [9] (see Geramita and Seberry [8]) tells us that if there is an Hadamard matrix with constant diagonal of order  $4k$  there is a regular symmetric Hadamard matrix with constant diagonal of order  $4(2k)^2$ . So an Hadamard matrix of order  $4t$  gives a regular symmetric Hadamard matrix of order  $4k^2, k = 2t$ . In particular known results give these matrices for  $2t \leq 210$ .

*Remark 2.* Now combining these results, and noting that regular symmetric Hadamard matrices with constant diagonal of orders  $4s^2$  and  $4t^2$  give a regular symmetric Hadamard matrix with constant diagonal with order  $4(2st)^2$ , we have them for orders  $4k^2$  for

- (i) all even  $k \leq 210$ , all even  $2t$  when there is an Hadamard matrix of order  $4t$ ;
- (ii)  $k \in \{1, 3, 5, 9, 11, 13, 15, 21, 23, 25, 29, 33, 35, 39, 41, 45, 51, 53, 63, 65, 69, 75, 81, 83, 89, 95, 99, 105, 111, 113, 119, 125, 131, 135, 141, 153, 155, 165, 173, 179, 183, 189, 191, 209\}$ .

We now wish to establish the existence of some of the remaining undecided cases.

We first note a theorem given by Seberry Wallis: [24, p. 280]

**Theorem 3.** *A regular Hadamard matrix  $H$  of order  $4k^2$  with row sum  $\pm 2k$  exists if and only if there exists an  $SBIBD(4k^2, 2k^2 \pm k, k^2 \pm k)$ .*

We observe that the stipulation that the row sum is  $\pm 2k$  is unnecessary for if the matrix is regular it must have constant row sum,  $x$ , say.

Thus  $eH^T = (x, \dots, x)$  where  $e$  is the  $1 \times 4k^2$  matrix of ones. Now  $H^T H = 4k^2 I$ , so

$$16k^4 = 4k^2 ee^T = eH^T H e^T = (x, \dots, x)(x, \dots, x)^T = 4k^2 x^2.$$

Thus  $x = \pm 2k$ . The matrix with constant row sum  $-2k$  is the negative of the matrix with constant row sum  $2k$ . □

We can now combine the results obtained so far as

**Theorem 4** (Equivalence Theorem). *The following are equivalent:*

- (i) *there exists an Hadamard matrix of order  $4k^2$  with maximal excess ( $8k^3$ );*
- (ii) *there exists a regular Hadamard matrix of order  $4k^2$ ;*
- (iii) *there is an SBIBD( $4k^2, 2k^2 + k, k^2 + k$ ) (and its complement the SBIBD( $4k^2, 2k^2 - k, k^2 + k$ )).*

This result was also observed by Best [1].

We now wish to consider the undecided cases. First we look at a known family of Williamson matrices.

### 3. Matrices of Order $4q^2, 2q^2 - 1$ a Prime Power

We show that if  $p \equiv 1 \pmod{4}$  is a prime power,  $p = 2q^2 - 1$ , then the Hadamard matrix found as in Hammer, Levingston and Seberry [10, p. 244] with excess  $2(p + 1)(x + y)$ ,  $p = x^2 + y^2$  has

$$\sigma(2(p + 1)) > 2(p + 1)(x + y).$$

Since  $p = x^2 + y^2 = 2q^2 - 1$  the excess is  $4q^2(x + y)$  and the order is  $4q^2$ . But an Hadamard matrix of order  $4q^2$  has maximal excess  $8q^3$ . So we consider  $x + y$ .

Now  $x = (2q^2 - 1 - y^2)^{1/2}$  so  $E = x + y$  is maximal for  $\frac{dE}{dx} = 0$  or  $x = y$ . But that means

$$x = y = (q^2 - 0.5)^{1/2}$$

As  $x$  is an integer this means  $x = y < q$  so  $x + y < 2q$  and the excess  $2(p + 1)(x + y) < 8q^3$ . So this method cannot give maximal excess for matrix orders  $4q^2$ .

In some cases the construction gives quite high excess. The results are tabulated in Table 1.

Table 1

$q$	$2q^2 - 1 = x^2 + y^2$ (prime)	Hadamard order $= 4q^2$	Maximal Excess $= 4q^2 \cdot 2q$	Found Excess $= 4q^2(x + y)$
43	$3697 = 36^2 + 49^2$	$4.43^2$	$4.43^2 \cdot 86$	$4.43^2 \cdot 85$
49	$4801 = 24^2 + 65^2$	$4.49^2$	$4.49^2 \cdot 98$	$4.49^2 \cdot 89$
59	$6961 = 20^2 + 81^2$	$4.59^2$	$4.59^2 \cdot 118$	$4.59^2 \cdot 101$
69	$9521 = 40^2 + 89^2$	$4.69^2$	$4.69^2 \cdot 138$	$4.69^2 \cdot 129$
73	$10657 = 64^2 + 81^2$	$4.73^2$	$4.73^2 \cdot 146$	$4.73^2 \cdot 145$
85	$14449 = 7^2 + 120^2$	$4.85^2$	$4.85^2 \cdot 170$	$4.85^2 \cdot 127$
87	$15137 = 41^2 + 116^2$	$4.87^2$	$4.87^2 \cdot 174$	$4.87^2 \cdot 157$
91	$16561 = 81^2 + 100^2$	$4.91^2$	$4.91^2 \cdot 182$	$4.91^2 \cdot 181$

#### 4. The Hammer – Levingston – Seberry Construction Revisited

Hammer, Levingston and Seberry [10] suggested (following Cooper and (Seberry) Wallis [4]) using 4 circulant (or type 1) matrices of order  $t$ ,  $X_1, X_2, X_3, X_4$ , with entries 0, +1, -1 row sums  $x_1, x_2, x_3, x_4$  respectively satisfying

$$\begin{cases} \text{(i)} & \sum_{i=1}^4 X_i X_i^T = tI_t, \\ \text{(ii)} & X_i J = x_i J, \\ \text{(iii)} & X_i * X_j = 0, i \neq j, \\ \text{(iv)} & \sum_{i=1}^4 X_i \text{ is a } (1, -1)\text{-matrix,} \\ \text{(v)} & x_1^2 + x_2^2 + x_3^2 + x_4^2 = t. \end{cases}$$

These matrices are called  $T$ -matrices.

This means  $\sigma(X_i)$ , the excess of  $X_i$  is  $tx_i$ ,  $i = 1, 2, 3, 4$ , because each  $X_i$  is circulant (or type 1 = block circulant).

Let  $y_1, y_2, y_3, y_4$  be commuting variables and

$$U = \begin{bmatrix} -y_1 & y_2 & y_3 & y_4 \\ y_2 & y_1 & y_4 & -y_3 \\ y_3 & -y_4 & y_1 & y_2 \\ y_4 & y_3 & -y_2 & y_1 \end{bmatrix} = (u_{ij}), \quad V = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & -y_1 & -y_4 & y_3 \\ y_3 & y_4 & -y_1 & -y_2 \\ y_4 & -y_3 & y_2 & -y_1 \end{bmatrix} = (v_{ij}).$$

Now we can form  $A_i$  by either choosing

$$A_i = \sum_{k=1}^4 u_{ik} X_k, \quad i = 1, 2, 3, 4,$$

or

$$A_i = \sum_{k=1}^4 v_{ik} X_k, \quad i = 1, 2, 3, 4.$$

$A_i$  will be circulant (or type 1) according as  $X_i$  is circulant (or type 1).

Now the elements of  $A_i$  are variables, so the excess is a linear expression in  $x_i$  (constants) and  $y_i$  (variables). Depending on which coefficients are used (the  $u_{ik}$  or  $v_{ik}$ ) the excesses of the  $A_i$  will be:

*Case 1.*

$$\sigma(A_1) = (-y_1 x_1 + y_2 x_2 + y_3 x_3 + y_4 x_4)t$$

$$\sigma(A_2) = (y_2 x_1 + y_1 x_2 + y_4 x_3 - y_3 x_4)t$$

$$\sigma(A_3) = (y_3 x_1 - y_4 x_2 + y_1 x_3 + y_2 x_4)t$$

$$\sigma(A_4) = (y_4 x_1 + y_3 x_2 - y_2 x_3 + y_1 x_4)t$$

*Case 2.*

$$\sigma(A_1) = (y_1 x_1 + y_2 x_2 + y_3 x_3 + y_4 x_4)t$$

$$\sigma(A_2) = (y_2 x_1 - y_1 x_2 - y_4 x_3 + y_3 x_4)t$$

$$\sigma(A_3) = (y_3 x_1 + y_4 x_2 - y_1 x_3 - y_2 x_4)t$$

$$\sigma(A_4) = (y_4 x_1 - y_3 x_2 + y_2 x_3 - y_1 x_4)t$$

Write

$$G = \begin{bmatrix} -A_1 & A_2R & A_3R & A_4R \\ A_2R & A_1 & A_4^T R & -A_3^T R \\ A_3R & -A_4^T R & A_1 & A_2^T R \\ A_4R & A_3^T R & -A_2^T R & A_1 \end{bmatrix},$$

$$H = \begin{bmatrix} A_1 & A_2R & A_3R & A_4R \\ -A_2R & A_1 & A_4^T R & -A_3^T R \\ -A_3R & -A_4^T R & A_1 & A_2^T R \\ -A_4R & A_3^T R & -A_2^T R & A_1 \end{bmatrix}$$

where  $R$  is the back diagonal matrix and  $A_1, A_2, A_3, A_4$  are circulant matrices (or type 1).

Now if the matrices from Case 1 are used in  $G$  we get

$$\begin{aligned} \sigma(OD) &= 2\sigma(A_1) + 2\sigma(A_2) + 2\sigma(A_3) + 2\sigma(A_4) \\ &= 2t(y_1y_2y_3y_4) \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \end{aligned}$$

Call this case 1G. If the matrices from Case 1 are used in  $H$  we get

$$\sigma(OD) = 4\sigma(A_1) = 4t(y_1y_2y_3y_4) \begin{bmatrix} -x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Call this case 1H.

If the matrices of Case 2 are used in  $G$  we get

$$\sigma(OD) = 2t(y_1y_2y_3y_4) \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Call this Case 2G. While if the matrices from Case 2 are used in  $H$  we get

$$\sigma(OD) = 4t(y_1y_2y_3y_4) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Call this Case 2H.

Case 1H is never used as for positive  $x_i$  and  $y_i$  (which can always be assumed as a row or matrix with negative row sum or excess can be just negated to get a row or matrix with positive row sum or excess).

If each of the variables  $y_i$  is replaced by 1 we get the excesses

$4t(x_1 + x_2 + x_3 + x_4)$ ,  $4t(-x_1 + x_2 + x_3 + x_4)$ ,  $8tx_1$ ,  $4t(x_1 + x_2 + x_3 + x_4)$ , respectively. So the excess of the corresponding Hadamard matrix of order  $4t$  is

$$\sigma(4t) \geq 4t \max(2x_1, x_1 + x_2 + x_3 + x_4).$$

Where  $x_i$  is the row sum of the  $T$ -matrices.

*Example 1.* Suppose that  $X_1, X_2, X_3, X_4$  are the circulant matrices of order 9 with first rows

$$(1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0), (0\ 0\ 1\ 0\ -1\ 0\ 0\ 0\ 0), (0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ -1), (0\ 0\ 0\ 0\ 0\ 0\ 1\ -1\ 0)$$

Then  $x_1 = 3, x_2 = 0, x_3 = 0, x_4 = 0$  and

$$\sigma(36) \geq 36 \max(6, 3 + 0 + 0 + 0) = 216 = 36\sqrt{36}$$

So we in fact have the matrix of order 36 with maximal excess.

Now instead of replacing  $y_1, y_2, y_3, y_4$  by 1 we replace them by suitable matrices (for example Williamson matrices)  $W_1, W_2, W_3, W_4$  of order  $w$  with row and column sums  $a, b, c, d$  respectively where

$$e \left( \sum_{i=1}^4 W_i W_i^T \right) e^T = w(a^2 + b^2 + c^2 + d^2) = 4wee^T = 4w^2$$

$e$  being the  $1 \times w$  matrix of 1s.

So

$$\sigma(W_1) = aw, \quad \sigma(W_2) = bw, \quad \sigma(W_3) = cw, \quad \sigma(W_4) = dw$$

and

$$\sigma(4tw) = 2tw(a\ b\ c\ d) \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad (\text{case 1G})$$

$$\sigma(4tw) = 4tw(a\ b\ c\ d) \begin{bmatrix} -x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad (\text{case 1H})$$

$$\sigma(4tw) = 2tw(a\ b\ c\ d) \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad (\text{case 2G})$$

$$\sigma(4tw) = 4tw(a\ b\ c\ d) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad (\text{case 2H})$$

*Example 2.* Suppose that  $X_1, X_2, X_3, X_4$  are as in Example 1. Then  $x_1 = 3, x_2 = x_3 = x_4 = 0$ . Thus



$$\sigma(36w) = 54w(-a + b + c + d) \tag{case 1G}$$

$$\sigma(36w) = -108wa \tag{case 1H}$$

$$\sigma(36w) = 54w(a + b + c + d) \tag{case 2G}$$

$$\sigma(36w) = 108wa \tag{case 2H}$$

We now observe that if

$$J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

then

$$W_1 = W_2 = W_3 = \begin{bmatrix} J & B & -B \\ -B & J & B \\ B & -B & J \end{bmatrix} \quad W_4 = \begin{bmatrix} B & B & B \\ B & B & B \\ B & B & B \end{bmatrix}$$

where  $\sum_{i=1}^4 W_i W_i^T = 36I$ ,  $W_i W_j^T = W_j W_i^T$ , and the row sums are  $a = b = c = d = 3$ .

Thus  $\sigma(36 \cdot 9) \geq \max(54 \cdot 9 \cdot 6, 54 \cdot 9 \cdot 12, 108 \cdot 9 \cdot 3)$  from cases 1G, 2G and 2H respectively i.e.

$$\sigma(4 \cdot 9 \cdot 9) \geq 8 \cdot 3^6 = 9 \cdot 36 \sqrt{9 \cdot 36}$$

So we have the Hadamard matrix with maximal excess.

This method is that used by Koukouvinos and Kounias [12] (but not quite in this form) to construct their maximal excess Hadamard matrices. For convenience we state these results as a theorem.

**Theorem 5.** *Suppose there are Williamson type matrices of order  $w$  and row sums  $a, b, c, d$ . Suppose there are  $T$ -matrices of order  $t$  and row sums  $x_1, x_2, x_3, x_4$  then the excess of the Hadamard matrix of order  $4wt$  formed from these matrices satisfies (writing  $A$  for  $(a \ b \ c \ d)$  and  $X$  for  $(x_1 x_2 x_3 x_4)^T$ .)*

$$\sigma(4wt) \geq \max(4wtAX, 2wtA \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix} X, 2wtA \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} X)$$

### 5. Some Numerical Results

We have seen that we can establish the existence of SBIBDs and regular Hadamard matrices by looking for Hadamard matrices with maximal excess.

First we note that Koukouvinos and Kounias [12] have shown:

**Theorem 6.** *Hadamard matrices of order  $4k^2$  with maximal excess exist for  $k = 1, 3, \dots, 13, \dots, 17, 19, 21, \dots, 25, 33, 37$ .*

Combining the results of Remark 2 with this theorem and using the Equivalence Theorem we get many more matrices with maximal excess.

Yamada [26, Section 3] has shown there are Hadamard matrices

- (i) of order  $4 \cdot 3^{2m}$  and excess  $8 \cdot 3^{3m}$
- (ii) of order  $2^{2t} \cdot 5^2$  and excess  $2^{3t} \cdot 5^3$ .

This means there are Hadamard matrices with maximal excess for orders  $4 \cdot 27^2$  and  $4 \cdot 81^2$ .

In Geramita and Seberry [8, p. 175] the  $T$ -matrices to construct an  $OD(4 \cdot 61; 61, 61, 61, 61)$  with row sums 6, 5, 0, 0 are given. This gives an excess of either  $4 \cdot 61(6y_3 + 5y_4)$  or  $2 \cdot 61(y_1 + 11y_2 + y_3 + 11y_4)$ . Now  $121 = 11^2 + 0^2$  is a prime power so there are Williamson matrices of order  $n = \frac{121 + 1}{2} = 61$ , with row sums 1, 11, 1, 11. Thus there is an Hadamard matrix of order  $4 \cdot 61^2$  with excess  $2 \cdot 61^2 \cdot 4 \cdot 61$ .

Now the sequences  $\{1 0_{11}\}$ ,  $\{0 1 1 1 1 1 -1 -1 1 -1 1 -1\}$  can be used for  $A$  or  $B$  and  $\{1 0 1 0 -1 0 1 0 -1 0 1\}$ ,  $\{0 1 0 1 0 -1 0 1 0 1 0\}$  can be used for  $C$  or  $D$  in Yang's construction [27] to form  $T$ -matrices of order 69. Depending on the order the matrices are used we can get  $T$ -matrices of row sums 6, 5, 2, 2 or 7, 4, 0, 2 or 6, 1, 4, 4 and order 69. We use the  $T$ -matrices with row sums 6, 5, 2, 2.

For  $t = 25$  use the  $T$ -matrices given in Geramita and Seberry [8, p. 175] which give an  $OD(100; 25, 25, 25, 25)$  and which have row sums 5, 0, 0, 0. This gives an excess of  $50(5y_1 + 5y_2 + 5y_3 + 5y_4) = 2 \cdot 5^3(y_1 + y_2 + y_3 + y_4)$ . Now as Yamada remarks [26, section 3] there are Williamson matrices of order  $n = 3^{2m}$  with row sums  $3^m, 3^m, 3^m, 3^m$ . So there are Hadamard matrices of order  $4 \cdot 5^2 \cdot 3^{2m}$  with maximal excess  $8 \cdot 5^3 \cdot 3^{3m}$ . There are also (see [24, p. 389] Williamson matrices of order  $n = 25$  with row sums 5, 5, 5, 5. So there are Hadamard matrices of order  $4 \cdot 5^4$  with maximal excess  $8 \cdot 5^6$ .

**Lemma 7.** *There are Hadamard matrices of order  $100 \cdot 3^{2m}$ ,  $m \geq 0$  and maximal excess  $1000 \cdot 3^{3m}$ . There are Hadamard matrices of order  $4 \cdot 5^4$  and maximal excess  $8 \cdot 5^6$ .*

## 6. Summarizing

**Theorem 8.** *Hadamard matrices of order  $4k^2$  with maximal excess exist for*

- (i)  $k$  even  $k \leq 210$ , or an Hadamard matrix of order  $2k$  exists,
- (ii)  $k \in \{1, 3, 5, \dots, 29, 33, \dots, 41, 45, 51, 53, 61, \dots, 69, 75, 81, 83, 89, 95, 99, 625, 3^{2m}, 5^2 \cdot 3^{2m}, m \geq 0\}$ .

*This means that regular Hadamard matrices of order  $4k^2$  and  $SBIBD(4k^2, 2k^2 \pm k, k^2 \pm k)$  also exist for these  $k$  values.*

**Remark.** Koukouvinos, Kounias and Seberry have subsequently, in "Further Hadamard matrices with maximal excess and new  $SBIBD(4k^2, 2k^2 + k, k^2 + k)$ " *Utilitas Math.* (to appear) extended (ii) to include

$$k \in \{31, 43, 49, 55, 57, 85, 87, 91, 93, 115, 117\}$$

## References

1. Baumert, L.D., Hall, Jr., M.: A new construction for Hadamard matrices. *Bull. Amer. Math. Soc.* **71**, 169–170 (1965)
2. Best, M.R.: The excess of a Hadamard matrix. *Ind. Math.* **39**, 357–361 (1977)
3. Cohen, G., Rubie, D., Koukouvinos, C., Seberry, J., Yamada, M.: A survey of base sequences, disjoint complementary sequences and  $OD(4t; t, t, t, t)$ . *J. Comb. Math. Comb. Comp.* **5**, 69–104 (1989)
4. Cooper, J., Wallis, J.: A construction for Hadamard arrays. *Bull. Austr. Math. Soc.* **7**, 269–277 (1972)
5. Enomoto, H., Miyamoto, M.: On maximal weights of Hadamard matrices. *J. Comb. Theory (A)* **29**, 94–100 (1980)
6. Farmakis, N., Kounias, S.: The excess of Hadamard matrices and optimal designs. *Discrete Math.* **67**, 165–176 (1987)
7. Fisher, R.A., Yates, F.: *Statistical Tables for Biological, Agricultural and Medical Research*, 2nd ed. London: Oliver & Boyd Ltd. 1943
8. Geramita, A.V., Seberry, J.: *Orthogonal designs: Quadratic forms and Hadamard Matrices*. New York-Basel: Marcel Dekker 1979
9. Goethals, J.M., Seidel, J.J.: Strongly regular graphs derived from combinatorial designs. *Canad. J. Math.* **22**, 597–614 (1970)
10. Hammer, J., Levinston, R., Seberry, J.: A remark on the excess of Hadamard matrices and orthogonal designs. *Ars Comb.* **5**, 237–254 (1978)
11. Koukouvinos, C., Kounias, S.: Hadamard matrices of the Williamson type of order  $4 \cdot m$ ,  $m = p \cdot q$ . An exhaustive search for  $m = 33$ . *Discrete Math.* **68**, 45–57 (1988)
12. Koukouvinos, C., Kounias, S.: Construction of some Hadamard matrices with maximum excess. *Discrete Math.* (to appear)
13. Kounias, S., Farmakis, N.: On the excess of Hadamard matrices. *Discrete Math.* **68**, 59–69 (1988)
14. McFarland, R.L.: On  $(v, k, \lambda)$ -configurations with  $v = 4p^e$ . *Glasg. Math. J.* **15**, 180–183 (1974)
15. Sathé, Y.S., Shenoy, R.G.: Construction of maximal weight Hadamard matrices of order 48 and 80. *Ars Comb.* **19**, 25–35 (1985)
16. Schmidt, K.W., Wang, E.T.H.: The weights of Hadamard matrices. *J. Comb. Theory (A)* **23**, 257–263 (1977)
17. Street, A.P., Street, D.J.: *Combinatorics of Experimental Design*. Oxford: Oxford University Press 1987
18. Shrikhande, S.S.: On a two parameter family of balanced incomplete block designs. *Sankhya, Ser. A* **24**, 33–40 (1962)
19. Shrikhande, S.S., Singh, N.K.: On a method of constructing symmetrical balanced incomplete block designs. *Sankhya, Ser. A* **24**, 25–32 (1962)
20. Szekeres, G.: Cyclotomy and complementary difference sets. *Acta Arith.* **18**, 349–353 (1971)
21. Vajda, S.: *The Mathematics of Experimental Design, Incomplete Block Designs and Latin Squares*. No. 23, Griffins Statistical Monographs and Courses. London: Griffin & Co. 1967
22. Wallis, J.: Hadamard matrices of order  $28m$ ,  $36m$  and  $44m$ . *J. Comb. Theory (A)* **15**, 323–328 (1973)
23. Wallis, J.S., Whiteman, A.L.: Some classes of Hadamard matrices with constant diagonal. *Bull. Aust. Math. Soc.* **7**, 233–249 (1972)
24. Wallis, J.S.: Hadamard matrices. In: W.D. Wallis, A.P. Street, J.S. Wallis, *Combinatorics: Room Squares, Sum-free Sets, Hadamard Matrices*. *Lect. Notes Math.* **292**, 273–489 (1972)
25. Wallis, W.D.: On the weights of Hadamard matrices. *Ars Comb.* **3**, 287–292 (1977)
26. Yamada, M.: On a series of Hadamard matrices of order  $2^t$  and the maximal excess of Hadamard matrices of order  $2^{2t}$ . *Graphs and Combinatorics*, **4**, 297–301 (1988)
27. Yang, C.H.: Hadamard matrices and  $\delta$ -codes of length  $3n$ . *Proc. Amer. Math. Soc.* **85**, 480–482 (1982)

Received: March 18, 1988