Some Results on the Excesses of Hadamard Matrices

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ABSTRACT

We give some results on the excess of Hadamard matrices. We give a list for Hadamard matrices of order ≤ 1000 of the smallest upper bounds known for the excess for each order. A construction is indicated for the maximal known excess.

1. Introduction

An *Hadamard matrix* of order n is an $n \times n$ matrix H with elements $+1, -1$, satisfying $H^T H = HH^T = nI_n$. The sum of the elements of H, denoted by $\sigma(H)$, is called *excess* of H. The maximum excess of H, over all Hadamard matrices of order n, is denoted by $\sigma(n)$, i.e.

$$\sigma(n) = \max \sigma(H) \text{ for all Hadamard matrices of order } n \qquad (1)$$

An equivalent notion is the *weight* $w(H)$ which is the number of 1's in H, then $\sigma(H) = 2w(H) - n^2$ and $\sigma(n) = 2w(n) - n^2$, see [7,13,22,23,36,41].

Kounias and Farmakis [18] proved that $\sigma(n) = n\sqrt{n}$ when $n = 4(2m+1)^2$ thus satisfying the equality of Best's inequality:

$$\sigma(n) \leq n\sqrt{n}$$

Further they showed that for $n = 4(2m+1)^2$ a regular Hadamard matrix exists.

Infinite families of Hadamard matrices satisfying this bound have been found by Seberry [26] and Yamada [41]; more have been found by Koukouvinos and Kounias [16], Koukouvinos, Kounias and Seberry [17]. Also, Kounias and Farmakis [18] proved that $\sigma(n) = n\sqrt{n-3}$ can be attained when $n = (2m+1)^2 + 3$ thus satisfying the equality of the Hammer - Levingston - Seberry bound

$$\sigma(n) \leq n\sqrt{n-3}$$

for this bound. This is discussed further in Section 4. Koukouvinos, Kounias and Seberry [17] have found an infinite family of Hadamard matrices

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of order $n = 4v$ with maximal excess $\sigma(n) = n\sqrt{n-3}$, where q is a prime power and $v = q^2 + q + 1$ is a prime.

A *regular Hadamard matrix* has constant row and column sum. These are discussed by Seberry Wallis [24,pp341–346].

A *symmetric balanced incomplete block design* or $SBIBD(v, k, \lambda)$ can be defined as a square matrix of order v with entries 0 or 1, with k 1's in row and column and the inner product of an pair of distinct rows is λ. For more details see Street and Street [29].

An *orthogonal design* $D = x_1 A_1 + x_2 A_2 + \ldots + x_u A_u$ of order n and type $(s_1, \ldots, s_u)$, written $OD(n; s_1, s_2, \ldots, s_u)$, on the commuting variables $x_1, \ldots, x_u$ is a square matrix with entries $0, \pm x_1, \ldots, \pm x_u$ where x_i or $-x_i$ occurs s_i times in each row and column and distinct rows are formally orthogonal. That is

$$DD^T = \sum_{j=1}^{u} s_j x_j^2.$$

Each A_j is a $(0, 1, -1)$-matrix satisfying $A_j A_j^T = s_j I_n$ and is called a *weighing matrix* of weight s_j. This confusion over the use of 'weight' has made the concept of excess preferable. A weighing matrix of order n and weight n is called an *Hadamard matrix*.

We define the *excess of the orthogonal design* D as

$$\sigma(D) = \sigma(A_1) + \ldots + \sigma(A_u),$$

where $\sigma(Y_i)$ is the sum of the entries of A_i, this is equivalent to putting all the variables equal to $+1$.

Suitable matrices are matrices with elements +1 and -1 which can be used to replace the variables of ODs to form Hadamard matrices. Of special interest are *Williamson type matrices*, which are 4 matrices, W_1, W_2, W_3, W_4 with elements +1 or -1 of order w which satisfy

$$\begin{aligned} \sum_{i=1}^{4} W_i W_j^T &= 4wI_w \\ W_i W_j^T &= W_j W_i^T \end{aligned}$$

Some of our constructions follow that of Hammer, Levingston and Seberry [13], and Seberry [26], who formed orthogonal designs $OD(4t; t, t, t, t)$ and then replaced the variables by suitable matrices.

This practice for constructing Hadamard matrices derived from extensions due to Baumert–Hall [2] who found the first $OD(12; 3, 3, 3, 3)$

and Cooper and (Seberry) Wallis [5] who first introduced T-matrices to form $OD(4t; t, t, t, t)$. The variables of these ODs are then replaced by Williamson type matrices of order w to form Hadamard matrices of order $4wt$. These are discussed extensively by Geramita and Seberry [9,pp120–125]. Cohen et al [4] survey the most recent results. This method was also used by Koukouvinos and Kounias [16] to find Hadamard matrices with maximal excess. It is discussed in more detail later in this section.

Since we are concerned with orthogonal designs, we shall consider sequences of commmuting variables.

Let $X = \{\{a_{11}, \ldots, a_{1n}\}, \{a_{21}, \ldots, a_{2n}\} \ldots \{a_{m1}, \ldots, a_{mn}\}\}$ be m sequences of commuting variables of length n.

The *nonperiodic auto-correlation function of the family of sequences X* (denoted N_X) is a function defined by

$$N_X(j) = \sum_{i=1}^{n-j} (a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \ldots + a_{m,i}a_{m,i+j}).$$

. Note that if the following collection of m matrices of order n is formed,

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ & a_{11} & & a_{1,n-1} \\ & & \ddots & \\ \bigcirc & & & a_{11} \end{bmatrix},$$

$$\begin{bmatrix} a_{21} & a_{22} & \ldots & a_{2n} \\ & a_{21} & & a_{2,n-1} \\ & & \ddots & \\ \bigcirc & & & a_{21} \end{bmatrix}, \ldots, \begin{bmatrix} a_{m1} & a_{m2} & \ldots & a_{mn} \\ & a_{m1} & & a_{n,n-1} \\ & & \ddots & \\ \bigcirc & & & a_{m1} \end{bmatrix}$$

then $N_X(j)$ is simply the sum of the inner products of rows 1 and $j+1$ of these matrices.

The *periodic auto-correlation function of the family of sequences X* (denoted P_X) is a function defined by

$$P_X(j) = \sum_{i=1}^{n} (a_{1,i}a_{1,j} + a_{x,i}a_{j,i+j} + \ldots + a_{m,i}a_{m,i+j}),$$

where we assume the second subscript is actually chosen from the complete set of residues (mod n).

for Golay sequences of order ≤ 68 which exist are 2, 4, 8, 10, 16, 20, 26, 32, 40, 52 and 64. Malcolm Griffin[10] has shown no Golay sequences can exist for lengths $n = 2.9^t$. The value $n = 18$ previously excluded by a complete search but is now theoretically excluded by Griffin's theorem and independently by a result of Kruskal[19] and C.H.Yang. Recent work of Andres[1] and James[14] has led to a greatly improved computer algorithms for studying these sequences. James has found that the Golay sequences of lengths 2,4 and 26 are unique, there are 4 inequivalent Golay sequences of length 8, 2 of length 10, 36 of length 16, 25 of length 20 and 336 of length 32. For longer lengths the number of inequivalent sequences increases.

4-complementary disjoint sequences A, B, C, D all of length t, in which $-A+B+C+D$, $A-B+C+D$, $A+B-C+D$, $A+B+C-D$ all have entries +1 or -1 only, are call *T-sequences*. If A, B, C, D are 4-complementary sequences, A, B are of length $m+p$ and C, D are of length m and $\{\frac{1}{2}(A+B), \frac{1}{2}(A-B)\}, \{\frac{1}{2}(C+D), \frac{1}{2}(C-D)\}$ are disjoint they will be called *suitable sequences of lengths* $m+p, m+p, m, m$. If A, B, C, D are 4-complementary sequences of lengths $m+p, m+p, m, m$ they are called *base sequences*.

Notation: We sometimes use - for -1, and $\bar{x}$ for $-x$, and A' to mean the order of the entries in the sequence A are reversed.

One more piece of notation is in order. If g_r denotes a sequence of integers of length r, then by xg_r we mean the sequence of integers of length r obtained from g_r by multiplying each member of g_r by x.

The Hammer – Levingston – Seberry construction

Hammer, Levingston and Seberry[13] suggested (following Cooper and (Seberry) Wallis[5]) using 4 circulant (or type 1) matrices of order t, X_1, X_2, X_3, X_4, with entries 0,+1,-1 row sums x_1, x_2, x_3, x_4 respectively satisfying

$$\begin{cases} \text{(i) } \sum_{i=1}^{4} X_i X_i^T = tI_t, \\ \text{(ii) } X_i J = x_i J, \\ \text{(iii) } X_i * X_j = 0, \quad i \neq j. \\ \text{(iv) } \sum_{i=1}^{4} X_i \text{ is a (1,-1)-matrix,} \\ \text{(v) } x_1^2 + x_2^2 + x_3^2 + x_4^2 = t. \end{cases}$$

These matrices are called T-matrices. The essential difference between T-matrices and T-sequences is that the former have zero periodic autocorrelation function and the latter have zero non periodic autocorrelation fuction.

This means $\sigma(X_i)$, the excess of X_i is tx_i, $i = 1, 2, 3, 4$, because each X_i is circulant (or type 1 = block circulant).

Let y_1, y_2, y_3, y_4 be commuting variables and

$$U = \begin{bmatrix} -y_1 & y_2 & y_3 & y_4 \\ y_2 & y_1 & y_4 & -y_3 \\ y_3 & -y_4 & y_1 & y_2 \\ y_4 & y_3 & -y_2 & y_1 \end{bmatrix} = (u_{ij}),$$

$$V = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & -y_1 & -y_4 & y_3 \\ y_3 & y_4 & -y_1 & -y_2 \\ y_4 & -y_3 & y_2 & -y_1 \end{bmatrix} = (v_{ij}).$$

Now we can form A_i by either choosing

$$A_i = \sum_{k=1}^{4} u_{ik} X_k, \quad i = 1, 2, 3, 4,$$

or

$$A_i = \sum_{k=1}^{4} v_{ik} X_k, \quad i = 1, 2, 3, 4.$$

A_i will be circulant (or type 1) according as X_i is circulant (or type 1).

Now the elements of A_i are variables, so the excess is a linear expression in x_i (constants) and y_i (variables). Depending on which coefficients are used (the u_{ik} or v_{ik}) the excesses of the A_i will change. In [26] Seberry distinguishes four cases using the results in [13].

Case 1G where:

$$\sigma(OD) = 2t(y_1 y_2 y_3 y_4) \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Case 1H where:

$$\sigma(OD) = 4\sigma(A_1) = 4t(y_1 y_2 y_3 y_4) \begin{pmatrix} -x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Case 2G where:

$$\sigma(OD) = 2t(y_1 y_2 y_3 y_4) \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Case 2H where:

$$\sigma(OD) = 4t(y_1 y_2 y_3 y_4) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Case 1H is never used as for positive x_i and y_i (which can always be assumed as a row or matrix with negative row sum or excess can be just negated to get a row or matrix with positive row sum or excess).

Now if instead we replace y_1, y_2, y_3, y_4 by suitable matrices (for example Williamson matrices) W_1, W_2, W_3, W_4 of order w with row and column sums a, b, c, d respectively where

$$e(\sum_{i=1}^{4} W_i W_i^T)e^T = w(a^2 + b^2 + c^2 + d^2) = 4wee^T = 4w^2$$

e being the $1 \times w$ matrix of 1s, we have

$$\sigma(W_1) = aw, \sigma(W_2) = bw, \sigma(W_3) = cw, \sigma(W_4) = dw$$

and

Thus

$$\sigma(4tw) = 2tw \begin{pmatrix} a & b & c & d \end{pmatrix} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \text{(case } 1G\text{)}$$

$$\sigma(4tw) = 4tw \begin{pmatrix} a & b & c & d \end{pmatrix} \begin{pmatrix} -x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \text{(case } 1H\text{)}$$

$$\sigma(4tw) = 2tw \begin{pmatrix} a & b & c & d \end{pmatrix} \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \text{(case } 2G\text{)}$$

$$\sigma(4tw) = 4tw \begin{pmatrix} a & b & c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \text{(case } 2H\text{)}$$

For convenience these results were stated in [26] as a theorem.

Theorem 1 *Suppose there are Williamson type matrices of order w and row sums a, b, c, d. Suppose there are T-matrices of order t and row sums x_1, x_2, x_3, x_4 then the excess of the Hadamard matrix of order $4wt$ formed from these matrices satisfies (writing* $\mathbf{A}$ *for* $(abcd)$ *and* $\mathbf{X}$ *for* $(x_1 x_2 x_3 x_4)^T$*.)*

$$\sigma(4wt) \geq \max\left(4wt\ \mathbf{AX},\ 2wt\ \mathbf{A} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \mathbf{X},\right.$$

$$\left. 2wt\ \mathbf{A} \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \mathbf{X}\right)$$

This theorem has two useful corollaries:

Corollary 2 *Suppose there are Williamson type matrices of order w and row sums a, b, c, d. Then the excess of the Hadamard matrix of order $4w$ formed from these matrices satisfies*

$$\sigma(4w) \geq 2w\ max\ (2a,\ a + b + c + d).$$

Proof: Set $t = 1, x_1 = 1$ and $x_2 = x_3 = x_4 = 0$ in the Theorem. □

Table 1 gives excesses that arise in this way.

Corollary 3 *Suppose there are T-matrices of order t and row sums x_1, x_2, x_3, x_4. Then the excess of the Hadamard matrix of order $4t$ formed from these matrices satisfies*

$$\sigma(4t) \geq 4t\ max\ (2x_1, x_1 + x_2 + x_3 + x_4).$$

Proof: Set $w = a = b = v = d = 1$ in the Theorem. □.

Table 2 gives excesses that arise in this way.

2. Excesses arising from known Williamson and T-matrices

We use the results of Baumert and Williamson, as quoted in Seberry Wallis [24, p388-389], for the Williamson matrices of odd orders $w = 1, 3, \ldots, 29, 37, 43$, of order 33 found by Koukouvinos and Kounias [15] for the decompositions

$$132 = 4 \times 33 = 9^2 + 7^2 + 1^2 + 1^2 = 9^2 + 5^2 + 5^2 + 1^2 = 7^2 + 7^2 + 5^2 + 3^2$$

and from Seberry (Wallis)[25] and Whiteman[38],[39] for other Williamson-type matrices.

In [26] it was shown that Turyn's construction for Williamson matrices of order $(q+1)/2$, $q \equiv 1 (\mathrm{mod}\ 4)$ will never give maximal excess in Corollary 2. This is case WII below. Matrices whose excesses are constructed by Corollary 2 are given in Table 1. Table 1 gives a result for the excess for each odd number less than 200 (if one is known) and the highest excess if more than one is known.

We use the results of many authors described in the survey by Cohen et al [4] to give the maximal excess obtained using Corollary 3. Matrices whose excesses are constructed by Corollary 3 are given in Table 2. Table 2 gives a result for the excess for each odd number less than 100 (if one is known) and the highest excess if more than one is known. A list for all decompositions into squares will appear in [4].

Finally the results are combined in Table 3 to make Hadamard matrices from orthogonal designs with their excesses determined by the method of Hammer, Levingston and Seberry [13] for many Hadamard matrices up to order 1000 in many cases the excess found meets the theoretical bound.

Table 4 gives the maximum excess that can be constructed using orthogonal designs with Williamson matrices of order 1 for Hadamard matrices of order $4t$, $t \leq 209$, or 225 or 243.

All the known results for Hadamard matrices of order $4t$, $t \leq 250$ are given in Appendix A.

We give a few remarks about how the excess of the Williamson matrices were calculated.

Case WII: If $p \equiv 1 (\mathrm{mod}\ 4)$ is a prime power then there are Williamson matrices $I + S, \pm(I - S), R_1 R$ of order $\frac{1}{2}(p+1)$. So if $p = x^2 + y^2$, x even, y odd, is a unique decomposition into squares we know the row sums are $|x+1|, |x-1|, y, y$.

Case WIII: We use the four matrices described in [41, section 8]. The Williamson–type matrices of order 3^{2k} have row sum 3^k. So for the Williamson–type matrices of order 81 the row sums are 9,9,9,9.

Case WIV: If $p \equiv 1 \pmod 4$ is a prime power, and S and R are as in case WII with row sums $|x|$ even, and $|y|$ odd, respectively then

$$I \times J_p + S \times (X + I),\ I \times J_p + S(-I + X),\ R \times (X + I),\ R \times (I - X)$$

are Williamson-type matrices of order $\frac{1}{2}p(p+1)$ [25] where $X = (X(j-i))$ is formed from the quadratic residues [24, p283-289], then the row sum of X is zero. Hence the row sums of the Williamson-type matrices are $|p+x|, |p-x|, y, y$.

We note Whiteman [38] has shown how to make these matrices both circulant and symmetric.

Case WV: These matrices use Williamson-type matrices $A + I, B, C, D$ where $A^T = -A, B^T = B, C^T = C, D^T = D$, called *good* matrices of order m. Suppose these have row sums $1, b, c, d$ where $1 + b^2 + c^2 + d^2 = 4m$. These can be used with an Hadamard matrix of order $v + 1 = 4m$ in the form

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ -1 & & & \\ -1 & & & \\ \vdots & & E & \\ -1 & & & \end{bmatrix}$$

so E always has a row sum $+1$. Then the required Williamson-type matrices are

$$I \times J_v + A \times E,\ B \times E,\ C \times E,\ D \times E$$

(see Seberry Wallis [25]). These have row sums v, b, c, d and $v^2 + b^2 + c^2 + d^2 = 4mv$.

Case WVII: This proceeds similarly to Case WV but requires two cyclic difference sets on v treatments which satisfy constraints spelt out in Seberry Wallis [25]. For the purposes of this paper we have used (31,25,20) and (31,16,8) difference sets with row sums 19 and 1 respectively as in the construction of Case WV with $m = 3, b = 3, c = d = 1$ so the Williamson-type matrices of order 93 have row sums 19,3,1,1.

Case WVIII: Good matrices can be used as Williamson-type matrices. They exist for $3 \leq n \leq 25$ [24] and $(q+1)/4$ where $q \equiv 3 \pmod 4$ is a prime power, Whiteman [39].

1 Order $n = (2m+1)^2 + 3$

First we show that the Hammer - Levingston - Seberry [13, p246] bound for these n is the same as that found by Kounias and Farmakis [18, section 4].

Hammer et al [13, p247] show that for weighing matrices of order n and weight $k = n$, i.e. an Hadamard matrix, writing x for the greatest even integer $< \sqrt{n}$, $t = x$ if $|n - x^2| < |(x+2)^2 - n|$ and $t = x - 2$ otherwise, i the integer part of $n((t+4)^2 - n)/8(t+2)$, the excess of the matrix is bounded by

$$\sigma(n) = n(t+4) - 4i.$$

Write $n = (2m+1)^2 + 3 = 4(m^2+m+1)$. Now x, even, is the greatest even integer $< \sqrt{n}$. Let $x = 2a$, then

$$2a < \sqrt{n}$$

and

$$4m^2 \leq 4a^2 < 4(m^2+m+1) < 4(m+1)^2.$$

Hence

$$m \leq a < m+1.$$

Thus we can write

$$x = 2a = 2m, \;\; t = x - 2 = 2m - 2 \text{ and } i = m^2 + m + 1.$$

Hence

$$\sigma(n) \leq n(2m+2) - 4i = 2nm + 2n - n = n(2m+1).$$

This was the result given in Kounias and Farmakis [18] and discussed further in Koukouvinos and Kounias [16].

We summarize this as a Lemma:

Lemma 4 *The Hammer-Levingston-Seberry bound is equivalent to*

$$\sigma(n) \leq n(2m+1)$$

when

$$n = (2m+1)^2 + 3.$$

2 Some Excess Results

We now give two results where the excess can be generally calculated. These results can only be maximal for small p.

Theorem 5 *Let H be any skew - Hadamard matrix of order $p+1$. Then there is an Hadamard matrix of order $p(p+1)$ with excess $p^2(p+1)$.*

Proof: Let $H = I+S$. Since H is skew - Hadamard S is skew - symmetric and so the sum of its elements is zero. Let J be the matrix of order p consisting of all ones, I the identity matrix, and B the core of H, so that

$$H = \begin{bmatrix} 1 & 1 & \dots & 1 \\ -1 & & & \\ \vdots & & B & \\ -1 & & & \end{bmatrix}$$

Then B, of order p satisfies

$$\begin{aligned} BB^T &= (p+1)I - J \\ BJ &= J \end{aligned}$$

Now $G = I \times J + S \times B$ is the required matrix (this is a standard construction for this matrix first given by Paley [21]. G has a contribution to its excess only from $I \times J$, which is $p+1$ copies of J each of which has excess p^2. Hence G has excess $p^2(p+1)$. □

Theorem 6 *Let H be any skew Hadamard matrix of order $p-3$ and a symmetric Hadamard matrix, F, of order $p+1$. Then there is an Hadamard matrix of order $p(p-3)$ with excess $p(p-2)(p-3)$.*

Proof: Let $H = I+S$ be a skew - Hadamard matrix as in Theorem 1. Let J be the matrix of all ones, I the identity matrix and A the core of F, so that

$$F = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & & & \\ \vdots & & A & \\ 1 & & & \end{bmatrix}.$$

Then, A, of order p satisfies

$$AA^T = (p+1)I - J$$

$$AJ = -J, \ A^T = A.$$

Now $G = I \times (J - 2I) + S \times A$ is the required matrix (as found by Paley [21]). G has a contribution to its excess only from $I \times (J - 2I)$, which is $p - 3$ copies of $J - 2I$ each of which has excess $p(p-2)$. Hence G has excess $p(p-2)(p-3)$. □

p	Theorem 4		Theorem 5	
	order	excess	order	excess
3	12	36*		
7	56	392	28	140*
11	132	1452	88	792
15	240	3600	180	2340
19	380	7220	304	5168
23	552	12696	460	9660
27	756	20412	648	16200
31	992	30752	868	25172

Golay Sequences and their excesses

The Golay sequences of lengths 2, 10 and 26 [12] have row sums $\{2,0\}$, $\{4,2\}$ and $\{6,4\}$ respectively.

Let X, Y be Golay sequences of length n and W, Z be Golay sequences of length m. Write T' for any of these sequences with its elements reversed.

Then Turyn found

$$\begin{aligned} P &= X \times \frac{1}{2}(W + Z) + Y \times \frac{1}{2}(W' - Z') \\ Q &= X \times \frac{1}{2}(W - Z) - Y \times \frac{1}{2}(W' + Z') \end{aligned}$$

or

$$\begin{aligned} P &= \frac{1}{2}(X + Y) \times W + \frac{1}{2}(X - Y) \times Z' \\ Q &= \frac{1}{2}(X + Y) \times Z - \frac{1}{2}(X - Y) \times W' \end{aligned}$$

are Golay sequences of length mn. Furthermore if the row sums of X, Y, Z, W are x, y, z, w respectively the row sums of P, Q are $\frac{1}{2}(x(w+z)+y(w-z))$ and $\frac{1}{2}(x(w-z)-y(w+z))$ or $\frac{1}{2}(x(w+z)+y(w-z))$ and $\frac{1}{2}(x(z-w))+y(z+w))$.

Lemma 7 *Let W, Z be Golay sequences of length m with row sums w and z. Then the Golay sequences of length*

(I) $2^{2i-1}m$ *have row sums* $2^{i-1}(w+z)$ *and* $2^{i-1}(w+z)$.

(II) $2^{2i}m$ *have row sums* $2^i w$ *and* $2^i z$.

In particular Golay sequences of length

(a) 2^{2i} *have row sums* 2^i *and* 2^i

(b) 2^{2i-1} *have row sums* 2^i *and 0.*

(c) $2^{2i}.5$ *have row sums* $2^i.3$ *and* 2^i.

(d) $2^{2i+1}.5$ *have row sums* 2^{i+2} *and* 2^{i+1}.

(e) $2^{2i}.13$ *have row sums* $2^i.5$ *and* 2^i.

(f) 2^{2i+1} *have row sums* $2^{i+1}.3$ *and* 2^{i+2}.

Let $2m = w^2 + z^2$ *where* m *is the length of Golay sequences with row sums* w *and* z. *Then up to equivalence there are Golay sequences of lengths with the indicated row sums:*

(g) 10^{2i} *have row sums* $(10^i, 10^i)$, $(10^{i-1}.14, 10^{i-1}.2$ *plus others for* $i \geq 2$.

(h) 10^{2i+1} *have row sums* $(10^i.4, 10^i.2)$, $(10^{i-1}.44, 10^{i-1}.8)$ *plus others for* $i \geq 2$.

(i) $10m$ *have row sums* $(3w \pm z, w \mp 3z)$.

(j) $100m$ *have row sums* $(10w, 10z)$ *or* $(8w \mp 6z, 6w \pm 8z)$.

(k) $1000m$ *have row sums* $(10(3w \pm z), 10(w \mp 3z))$ *or* $(2(9w \mp 3z, 13w \pm 9z)$.

(l) $2^{2i-1}.100$ *have row sums* $(2^{i-1}.10, 2^{i-1}.10)$ *and* $(2^i.7, 2^i)$.

(m) $2^{2i}.100$ *have row sums* $(2^i.10, 0)$ *and* $(2^{i+3}, 2^{i+1}.3)$

(n) 26^2 *have row sums* $(26, 26)$ *or* $(34, 14)$.
26^4 *have row sums* $(26^2, 26^2)$, $(26.34, 26.14)$, *or* $956, 4)$.
26^{2i} *have row sums* $(26^i, 26^i)$, $(26^{i-1}.34, 26^{i-1}.14)$ *plus others for* $i \geq 1$.

(o) 26^3 *have row sums* $(6.26, 4.26)$, *or* $(5.36, 1.36)$.
26^{2i+1} *have row sums* $(26^i.6, 26^i.4)$ *plus others for* $i \geq 1$.

(p) $26m$ *have row sums* $(5w \pm z, w \mp 5z)$.

(q) $676m$ *have row sums* $(26w, 26z)$ *or* $(24w \pm 10z, 10w \mp 24w)$.

(r) $2^{2i-1}.26^2$ *have row sums* $(2^i.26, 0)$ *and* $(2^{i+1}.12, 2^{i+1}.5)$

(s) $2^{2i}.26^2$ *have row sums* $(2^i.26, 2^i.26)$ *and* $(2^i.34, 2^i.14)$.

(t) *260 have row sums (22,6) or (18,4).*

(u) $2^{2i-1}.260$ *have row sums* $(2^{i+2}.7, 2^{i+4})$ *and* $(2^{i+5}, 2^{i+2})$.

(v) $2^{2i}.260$ *have row sums* $(2^{i+1}.11, 2^{i+1}.3)$ *and* $(2^{i+1}.9, 2^{i+1}.7)$.

Because of the power of the results we use some theorems of Yang to obtain powerful theorems about excesses:

Lemma 8 *Let a, b, c, d be the row sums of suitable sequences of length $m+1, m+1, m, m$, so that $2m+1 = a^2+b^2+c^2+d^2$. Then using Yang's method to multiply by 3 we get disjoint T-sequences of lengths $3(2m+1)$ corresponding to one of the decompositions:*

$$\begin{aligned}
3(2m+1) &= (a+b+c)^2+(a-b-d)^2+(a-c+d)^2+(b-c-d)^2\\
3(2m+1) &= (a+b+c)^2+(b-a-d)^2+(b-c+d)^2+(a-c-d)^2\\
3(2m+1) &= (a+b+d)^2+(a-b-c)^2+(a-d+c)^2+(b-d-c)^2\\
3(2m+1) &= (a+b+d)^2+(b-a-c)^2+(b-d+c)^2+(a-d-c)^2
\end{aligned}$$

.

Example: There are suitable sequences for $m = 20$ corresponding to $a = 6, b = 2, c = 1$ and $d = 0$. These give sequences of lengths 123 corresponding to:

$$\begin{aligned}
123 &= 9^2+5^2+4^2+1^2 \text{ giving excess } \sigma(H^1_{492}) = 8856.\\
123 &= 9^2+5^2+4^2+1^2 \text{ giving excess } \sigma(H^2_{492}) = 8856.\\
123 &= 8^2+7^2+3^2+1^2 \text{ giving excess } \sigma(H^3_{492}) = 9348.\\
123 &= 8^2+5^2+5^2+3^2 \text{ giving excess } \sigma(H^4_{492}) = 10332.
\end{aligned}$$

Comment: Yang has other theorems which allow suitable sequences of lengths $m+p, m+p, m, m$, to be multiplied by 7 and 13 to get disjoint T-sequences of lengths $7(2m+p)$ and $13(2m+p)$ and to be multiplied by 11 to get four complementary sequences of lengths $11(2m+p)$. His most powerful theorem [30] can be used as follows:

Theorem 9 (Yang) *Let a, b, c, d be the row sums of suitable sequences of length $m+1, m+1, m, m$, so that $2m+1 = a^2+b^2+c^2+d^2$. Let f and g be the row sums of two Golay sequences of length s so that $2s = f^2+g^2$. Then using Yang's method to multiply by $2s+1$ we get disjoint T-sequences of lengths $(2s+1)(2m+1)$ corresponding to one of the decompositions:*

$$
\begin{aligned}
(2s+1)(2m+1) &= (af+cg-b)^2+(bf+dg+a)^2+\\
&\quad (ag-cf-d)^2+(bg-df+c)^2\\
(2s+1)(2m+1) &= (bf+cg-a)^2+(af+dg+b)^2+\\
&\quad (bg-cf-d)^2+(ag-df+c)^2\\
(2s+1)(2m+1) &= (af+dg-b)^2+(bf+cg+a)^2+\\
&\quad (ag-df-c)^2+(bg-cf+d)^2\\
(2s+1)(2m+1) &= (bf+dg-a)^2+(af+cg+b)^2+\\
&\quad (bg-df-c)^2+(ag-cf+d)^2.
\end{aligned}
$$

Example: There are Golay sequences of lengths $s = 2$, with row sums 2, 0, $s = 4$, with row sums 2, 2, $s = 10$, with row sums 4, 2 and $s = 26$, with row sums 6, 4 which are unique up to equivalence (James, [14]), and $s = 8$, which even though there are four inequivalent pairs of sequences they all have row sums 4, 0. These Golay sequences give the following decompositions into squares when used in Yang's Theorem:

$$
\begin{aligned}
5(2m+1) &= (2a-b)^2+(a+2b)^2+(2c+d)^2+(2d-c)^2\\
5(2m+1) &= (2b-a)^2+(b+2a)^2+(2c+d)^2+(2d-c)^2\\
5(2m+1) &= (2a-b)^2+(a+2b)^2+(2d+c)^2+(2c-d)^2\\
5(2m+1) &= (2b-a)^2+(b+2a)^2+(2d+c)^2+(2c-d)^2
\end{aligned}
$$

Thus we are in the position to combine the results and obtain results about the excess of Hadamard matrices made by composing Golay sequences using Yang's method.

Theorem 10 *Let a, b, c, d be the row sums of suitable sequences of length $m+1, m+1, m, m$, so that $2m+1 = a^2+b^2+c^2+d^2$. Let f and g be the row sums of two Golay sequences of length s so that $2s = f^2+g^2$. Then using Yang's method to multiply by $2s+1$ we get disjoint T-sequences of lengths $(2s+1)(2m+1)$ we obtain Hadamard matrices with excess:*

$$
\begin{aligned}
4(2s+1)(2m+1)\ max\ [&2(af+cg-b), 2(bf+dg+a),\\
&2(ag-cf-d), 2(bg-df+c), x]
\end{aligned}
$$

with $x = a(f+g+1)+b(f+g-1)+c(g-f+1)+d(g-f-1)$;

$$
4(2s+1)(2m+1)\ max\ [2(bf+cg-a), 2(af+dg+b),
$$

$$2(bg - cf - d), 2(ag - df + c), x]$$
$$\textit{with } x = a(f + g - 1) + b(f + g + 1) + c(g - f + 1) + d(g - f - 1);$$

$$4(2s + 1)(2m + 1) \; max \; [2(af + dg - b), 2(bf + cg + a),$$
$$2(ag - df - c), 2(bg - cf + d), x]$$
$$\textit{with } x = a(f + g + 1) + b(f + g - 1) + c(g - f - 1) + d(g - f + 1);$$

$$4(2s + 1)(2m + 1) max[2(bf + dg - a), 2(af + cg + b),$$
$$2(bg - df - c), 2(ag - cf + d), x]$$
$$\textit{with } x = a(f + g - 1) + b(f + g + 1) + c(g - f - 1) + d(g - f + 1);$$

Appendix A

The "Best bound" is from [3], the "HLS" and "Lower bound" from [13] and the "FK bound" is found from [14, relation 16, Theorem 3] by noting we can always write n in the form:

$$\frac{n}{4} = 4m^2 + 2mq + u$$

where

$$m = \left[\frac{\sqrt{n}}{4}\right], \; q = \left[\frac{2mq + u}{2m}\right], \; u = \frac{n}{4} - 4m^2 - 2mq,$$

and $[x]$ denotes the integral part of x.

Kounias and his colleagues have found Hadamard matrices with $\sigma(H_{60}) = 452$ and $\sigma(H_{68}) = 548$. These are indicated in the table.

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w	Order	Sum of Squares	Excess
3	12	$3^2+1^2+1^2+1^2$	36*
5	20	$3^2+3^2+1^2+1^2$	80*
7	28	$3^2+3^2+3^2+1^2$	140*
7	28	$5^2+1^2+1^2+1^2$	140*
9	36	$3^2+3^2+3^2+3^2$	216*
9	36	$5^2+3^2+1^2+1^2$	180
11	44	$5^2+3^2+3^2+1^2$	264
13	52	$7^2+1^2+1^2+1^2$	364*
13	52	$5^2+3^2+3^2+3^2$	364*
13	52	$5^2+5^2+1^2+1^2$	312
15	60	$7^2+3^2+1^2+1^2$	420
15	60	$5^2+5^2+3^2+1^2$	420
17	68	$7^2+3^2+3^2+1^2$	476
17	68	$5^2+5^2+3^2+3^2$	544
19	76	$7^2+5^2+1^2+1^2$	532
19	76	$7^2+3^2+3^2+3^2$	608
19	76	$5^2+5^2+5^2+1^2$	608 good
21	84	$9^2+1^2+1^2+1^2$	756*
21	84	$7^2+5^2+3^2+1^2$	672
21	84	$5^2+5^2+5^2+3^2$	756*
23	92	$9^2+3^2+1^2+1^2$	828 good
23	92	$7^2+5^2+3^2+3^2$	828 #
25	100	$9^2+3^2+3^2+1^2$	900
25	100	$7^2+7^2+1^2+1^2$	800
25	100	$7^2+5^2+5^2+5^2$	900 #
25	100	$5^2+5^2+5^2+5^2$	1000*
27	108	$9^2+5^2+1^2+1^2$	972
27	108	$7^2+5^2+5^2+3^2$	1080 #
27	108	$3^2+1^2+7^2+7^2$	972 WII
29	116	$9^2+5^2+3^2+1^2$	1044
31	124	$7^2+5^2+5^2+5^2$	1364 WII*
33	132	$11^2+3^2+1^2+1^2$	1452 WV
33	132	$9^2+7^2+1^2+1^2$	1188
33	132	$9^2+5^2+5^2+1^2$	1320
33	132	$7^2+7^2+5^2+3^2$	1452
37	148	$11^2+3^2+3^2+3^2$	1628 #
37	148	$9^2+7^2+3^2+3^2$	1628 WII
37	148	$7^2+7^2+7^2+1^2$	1628
41	164	$1^2+1^2+9^2+9^2$	1640 WII
43	172	$13^2+1^2+1^2+1^2$	2236
43	172	$7^2+7^2+7^2+5^2$	2236
45	180	$9^2+7^2+5^2+5^2$	2340 WII
45	180	$9^2+9^2+3^2+3^2$	2160 WIV
49	196	$5^2+3^2+9^2+9^2$	2548 WII
51	204	$11^2+9^2+1^2+1^2$	2244 WII
55	220	$11^2+9^2+3^2+3^2$	2860 WII

Table 1a: Excesses made from Williamson-type matrices
*** maximum excess**
 # Koukouvinos and Kounias private communication

w	Order	Sum of Squares	Excess
57	228	$9^2 + 7^2 + 7^2 + 7^2$	3420 WII*
61	244	$1^2 + 1^2 + 11^2 + 11^2$	2928 WII
63	252	don't know which decomposition	WII
69	276	$5^2 + 3^2 + 11^2 + 11^2$	4140 WII
75	300	$11^2 + 9^2 + 7^2 + 7^2$	5100 WII
79	316	$7^2 + 5^2 + 11^2 + 11^2$	5372 WII
81	324	$9^2 + 9^2 + 9^2 + 9^2$	5832 WIII*
85	332	don't know which decomposition	WII
87	348	$3^2 + 1^2 + 13^2 + 13^2$	5220 WII
91	364	$15^2 + 11^2 + 3^2 + 3^2$	5824 WIV
91	364	$11^2 + 9^2 + 9^2 + 9^2$	6916 WII*
93	372	$19^2 + 3^2 + 1^2 + 1^2$	7068 WVII
95	380	$19^2 + 3^2 + 3^2 + 1^2$	7220 WV
97	388	$13^2 + 11^2 + 7^2 + 7^2$	7372 WII
99	396	$15^2 + 13^2 + 1^2 + 1^2$	5940 WII
115	460	$3^2 + 1^2 + 15^2 + 15^2$	7820 WII
117	468	$9^2 + 7^2 + 13^2 + 13^2$	9828 WII
121	484	$5^2 + 3^2 + 15^2 + 15^2$	9196 WII
129	516	$17^2 + 15^2 + 1^2 + 1^2$	8772 WII
135	540	$11^2 + 9^2 + 13^2 + 13^2$	12420 WII
139	556	$15^2 + 13^2 + 9^2 + 9^2$	12788 WII
141	564	$17^2 + 15^2 + 5^2 + 5^2$	11844 WII
147	588	$3^2 + 1^2 + 17^2 + 17^2$	11172 WII
153	612	$21^2 + 13^2 + 1^2 + 1^2$	11016 WIV
157	628	$13^2 + 11^2 + 13^2 + 13^2$	15700 WII*
159	636	$15^2 + 13^2 + 11^2 + 11^2$	15900 WII
169	676	$17^2 + 15^2 + 9^2 + 9^2$	16900 WII
175	700	$19^2 + 17^2 + 5^2 + 5^2$	16100 WII
177	708	$9^2 + 7^2 + 17^2 + 17^2$	17700 WII
181	724	$19^2 + 19^2 + 1^2 + 1^2$	14480 WII
187	748	$19^2 + 17^2 + 7^2 + 7^2$	18700 WII
189	756	$27^2 + 3^2 + 3^2 + 3^2$	13608 WV
189	756	$27^2 + 5^2 + 1^2 + 1^2$	13608 WV
195	780	$11^2 + 9^2 + 17^2 + 17^2$	21060 WII
199	796	$7^2 + 5^2 + 19^2 + 19^2$	19900 WII
315	1260	$35^2 + 5^2 + 3^2 + 1^2$	44100 WV

Table 1b: Excesses made from Williamson-type matrices
*** maximum excess**

t	Order	Sum of Squares	Excess
3	12	$1^2+1^2+1^2$	36
5	20	2^2+1^2	80
7	28	$2^2+1^2+1^2+1^2$	140
9	36	3^2	216
11	44	$3^2+1^2+1^2$	264
13	52	$2^2+2^2+2^2+1^2$	364 #
15	60	$3^2+2^2+1^2+1^2$	420
17	68	4^2+1^2	544
19	76	$4^2+1^2+1^2+1^2$	608
21	84	$3^2+2^2+2^2+2^2$	756
23	92	$3^2+3^2+2^2+1^2$	828
25	100	5^2	1000
27	108	$5^2+1^2+1^2$	1080
29	116	5^2+2^2	1160
31	124	$5^2+2^2+1^2+1^2$	1240
31	124	$3^2+3^2+3^2+2^2$	1364 T-matrices
33	132	$4^2+3^2+2^2+2^2$	1452
35	140	$4^2+3^2+3^2+1^2$	1540
37	148	6^2+1^2	1776 T-matrices
39	156	$6^2+1^2+1^2+1^2$	1872 #
41	164	$6^2+2^2+1^2$	1968
43	172	$5^2+4^2+1^2+1^2$	not known (1892)
43	172	$5^2+3^2+3^2$	not known (1892)
43	172	$4^2+3^2+3^2+3^2$	not known (2236)
45	180	$4^2+4^2+3^2+2^2$	2340
47	188	$5^2+3^2+3^2+2^2$	2444
49	196	7^2	2744 #
51	204	$7^2+1^2+1^2$	2856
53	212	7^2+2^2	not known (2968)
53	212	$6^2+4^2+1^2$	2544
53	212	$6^2+3^2+2^2+2^2$	not known (2968)
55	220	$7^2+2^2+1^2+1^2$	3080 #
57	228	$4^2+4^2+4^2+3^2$	3420 #
59	236	$5^2+4^2+3^2+3^2$	not known (3540)
59	236	$7^2+3^2+1^2$	3304
61	244	6^2+5^2	2928 T-matrices
61	244	$5^2+4^2+4^2+2^2$	not known (3660)
63	252	$6^2+3^2+3^2+3^2$	3780
63	252	$5^2+5^2+3^2+2^2$	3780
65	260	8^2+1^2	4160
67	268	$8^2+1^2+1^2+1^2$	4288 *
69	276	$8^2+2^2+1^2$	4416 #
71	284	$7^2+3^2+3^2+2^2$	not known (4260)
71	284	$6^2+5^2+3^2+1^2$	not known (4260)
73	292	$5^2+4^2+4^2+4^2$	not known (4260)
75	300	$5^2+5^2+4^2+3^2$	5100 #

Table 2a: Excesses made from 4-disjoint T-sequences or T-matrices
 # Koukouvinos and Kounias private communication
*** Sawade**

t	Order	Sum of Squares	Excess
77	308	$8^2 + 3^2 + 2^2$	4928
79	316	$5^2 + 5^2 + 5^2 + 2^2$	not known (5372)
81	324	9^2	5832 #
83	332	$9^2 + 1^2 + 1^2$	not known (5976)
85	340	$9^2 + 2^2$	6120
87	348	$9^2 + 2^2 + 1^2 + 1^2$	6264 #
87	348	$7^2 + 5^2 + 3^2 + 2^2$	5916
89	356	$9^2 + 2^2 + 2^2$	not known (6408)
91	364	$5^2 + 5^2 + 5^2 + 4^2$	6916 #
93	372	$8^2 + 4^2 + 3^2 + 2^2$	6324
93	372	$6^2 + 5^2 + 4^2 + 4^2$	7068 #
95	380	$6^2 + 5^2 + 5^2 + 3^2$	7220
97	388	$7^2 + 4^2 + 4^2 + 4^2$	not known (7372)
97	388	$6^2 + 6^2 + 5^2$	not known (7372)
97	388	$6^2 + 6^2 + 4^2 + 3^2$	not known (7372)
99	396	$7^2 + 5^2 + 4^2 + 3^2$	7524

Table 2b: Excesses made from 4-disjoint T-sequences or T-matrices
 # Koukouvinos and Kounias private communication

4tw	Excess	t	Sum of Squares	w	Sum of Squares	Method	Ref[w]
156	1872	13	$2^2 + 2^2 + 2^2 + 1^2$	3	$3^2 + 1^2 + 1^2 + 1^2$	1G	
196	2744*	7	$2^2 + 1^2 + 1^2 + 1^2$	7	$3^2 + 3^2 + 3^2 + 1^2$	1G	
220	3080	11	$3^2 + 1^2 + 1^2$	5	$3^2 + 3^2 + 1^2 + 1^2$	2G	
228	3420*	19	$4^2 + 1^2 + 1^2 + 1^2$	3	$3^2 + 1^2 + 1^2 + 1^2$	2H	
252	3780	21	$3^2 + 2^2 + 2^2 + 2^2$	3	$3^2 + 1^2 + 1^2 + 1^2$	2H	
260	4160	13	$3^2 + 2^2$	5	$3^2 + 3^2 + 1^2 + 1^2$	2G	
276	4416	23	$3^2 + 3^2 + 2^2 + 1^2$	3	$3^2 + 1^2 + 1^2 + 1^2$	1G	
300	5100	15	$3^2 + 2^2 + 1^2 + 1^2$	5	$3^2 + 3^2 + 1^2 + 1^2$	1G,2H	
308	5236	11	$3^2 + 1^2 + 1^2$	7	$3^2 + 3^2 + 3^2 + 1^2$	2G	
324	5832*	9	3^2	9	$3^2 + 3^2 + 3^2 + 3^2$	2G	
340	6120	17	$4^2 + 1^2$	5	$3^2 + 3^2 + 1^2 + 1^2$	2G	
348	6264	3	$1^2 + 1^2 + 1^2$	29	$9^2 + 5^2 + 3^2 + 1^2$	1G	
364	6916*	13	$2^2 + 2^2 + 2^2 + 1^2$	7	$3^2 + 3^2 + 3^2 + 1^2$	2H	
380	7220	19	$3^2 + 3^2 + 1^2$	5	$3^2 + 3^2 + 1^2 + 1^2$	1G,2H	
396	7524	11	$3^2 + 1^2 + 1^2$	9	$5^2 + 3^2 + 1^2 + 1^2$	2G	
420	8400	15	$3^2 + 2^2 + 1^2 + 1^2$	7	$3^2 + 3^2 + 3^2 + 1^2$	1G	
444	9324*	3	$1^2 + 1^2 + 1^2$	37	$7^2 + 7^2 + 7^2 + 1^2$	2H	
460	9660	23	$3^2 + 3^2 + 2^2 + 1^2$	5	$3^2 + 3^2 + 1^2 + 1^2$	1G,2H	
468	9828	13	$3^2 + 2^2$	9	$5^2 + 3^2 + 1^2 + 1^2$	2G	
476	9996	17	$4^2 + 1^2$	7	$5^2 + 1^2 + 1^2 + 1^2$	2H	
484	10648*	11	$3^2 + 1^2 + 1^2$	11	$5^2 + 3^2 + 3^2 + 1^2$	2G	
492	9348	3	$1^2 + 1^2 + 1^2$	41	$1^2 + 1^2 + 9^2 + 9^2$	2H	WII
492	9348	3	$1^2 + 1^2 + 1^2$	41	$9^2 + 9^2 + 1^2 + 1^2$	2H	
500	11000	25	$4^2 + 3^2$	5	$3^2 + 3^2 + 1^2 + 1^2$	2G	
516	10836	3	$1^2 + 1^2 + 1^2$	43	$13^2 + 1^2 + 1^2 + 1^2$	1G	
532	12236*	19	$4^2 + 1^2 + 1^2 + 1^2$	7	$3^2 + 3^2 + 3^2 + 1^2$	2G	
540	11340	3	$1^2 + 1^2 + 1^2$	45	$9^2 + 7^2 + 5^2 + 5^2$	2H	WII
540	11340	3	$1^2 + 1^2 + 1^2$	45	$9^2 + 9^2 + 3^2 + 3^2$	2H	WIV
540	12420	15	$3^2 + 2^2 + 1^2 + 1^2$	9	$5^2 + 3^2 + 1^2 + 1^2$	2H	
572	13156	13	$2^2 + 2^2 + 2^2 + 1^2$	11	$5^2 + 3^2 + 3^2 + 1^2$	1G	
580	13340	5	$2^2 + 1^2$	29	$9^2 + 5^2 + 3^2 + 1^2$	2H	
588	13524	3	$1^2 + 1^2 + 1^2$	49	$5^2 + 3^2 + 9^2 + 9^2$	2H	WII
588	14112	7	$2^2 + 1^2 + 1^2 + 1^2$	21	$5^2 + 5^2 + 5^2 + 3^2$	1G	
612	14688	17	$4^2 + 1^2$	9	$3^2 + 3^2 + 3^2 + 3^2$	2G	
612	13464	3	$1^2 + 1^2 + 1^2$	51	$11^2 + 9^2 + 1^2 + 1^2$	1G	WII
644	16100	23	$3^2 + 3^2 + 2^2 + 1^2$	7	$3^2 + 3^2 + 3^2 + 1^2$	2H	
660	16500	15	$3^2 + 2^2 + 1^2 + 1^2$	11	$5^2 + 3^2 + 3^2 + 1^2$	2H	
660	15840	3	$1^2 + 1^2 + 1^2$	55	$11^2 + 9^2 + 3^2 + 3^2$	1G	WII
676	17576*	13	$2^2 + 2^2 + 2^2 + 1^2$	13	$5^2 + 3^2 + 3^2 + 3^2$	1G	
684	16416	3	$1^2 + 1^2 + 1^2$	57	$9^2 + 7^2 + 7^2 + 7^2$	1G	WII
684	17784	9	$2^2 + 2^2 + 1^2$	19	$7^2 + 5^2 + 1^2 + 1^2$	1G	

Table 3a: Excesses made from OD(4wt;wt,wt,wt,wt) where t is the order of 4 distinct T-sequences and w is the order of Williamson-type matrices
*** means the excess meets the upper bound where no indication is given for the construction of the Williamson-type matrices it is WI the T-matrices are given in Cohen et al [4]**

4tw	Excess	t	Sum of Squares	w	Sum of Squares	Method	Ref[w]
700	17500	25	5^2	7	$3^2+3^2+3^2+1^2$	2G	
732	16836	3	$1^2+1^2+1^2$	61	$1^2+1^2+11^2+11^2$	1G,2H	WII
740	18500	5	2^2+1^2	37	$7^2+7^2+7^2+1^2$	2G	
748	20196	11	$3^2+1^2+1^2$	17	$7^2+3^2+3^2+1^2$	2H	
756	20412	21	$3^2+2^2+2^2+2^2$	9	$3^2+3^2+3^2+3^2$	1G,2H	
780	21060	13	$2^2+2^2+2^2+1^2$	15	$5^2+5^2+3^2+1^2$	2H	
812	21112	7	$2^2+1^2+1^2+1^2$	29	$9^2+5^2+3^2+1^2$	1G	
820	22960	5	2^2+1^2	41	$1^2+1^2+9^2+9^2$	2G	WII
820	22960	5	2^2+1^2	41	$9^2+9^2+1^2+1^2$	2G	
828	22356	3	$1^2+1^2+1^2$	69	$5^2+3^2+11^2+11^2$	2H	WII
828	23184	23	$3^2+3^2+2^2+1^2$	9	$5^2+3^2+1^2+1^2$	1G	
836	23408	19	$4^2+1^2+1^2+1^2$	11	$5^2+3^2+3^2+1^2$	2G	
884	25636	13	$2^2+2^2+2^2+1^2$	17	$5^2+5^2+3^2+3^2$	1G,2H	
900	25200	3	$1^2+1^2+1^2$	75	$11^2+9^2+7^2+7^2$	2H	WII
900	26100	5	2^2+1^2	45	$9^2+7^2+5^2+5^2$	2G	WII
900	27000*	5	2^2+1^2	45	$9^2+9^2+3^2+3^2$	2G	WIV
900	27000*	15	$3^2+2^2+1^2+1^2$	15	$5^2+5^2+3^2+1^2$	1G	
924	27720	11	$3^2+1^2+1^2$	21	$7^2+5^2+3^2+1^2$	2G	
948	27492	3	$1^2+1^2+1^2$	79	$7^2+5^2+11^2+11^2$	2H	WII
972	26244	3	$1^2+1^2+1^2$	81	$9^2+9^2+9^2+9^2$	1G,2H	WIII
972	30132	9	$2^2+2^2+1^2$	27	$9^2+5^2+1^2+1^2$	1G	
980	30380	5	2^2+1^2	49	$5^2+3^2+9^2+9^2$	2G	WII
980	30380	5	2^2+1^2	41	$9^2+9^2+5^2+3^2$	2G	
988	30628	13	3^2+2^2	19	$7^2+5^2+1^2+1^2$	1G	
1020	31620	5	2^2+1^2	51	$11^2+9^2+1^2+1^2$	2G,2H	WII
1044	30276	3	$1^2+1^2+1^2$	87	$3^2+1^2+13^2+13^2$	2H	WII
1148	35588	7	$2^2+1^2+1^2+1^2$	41	$1^2+1^2+9^2+9^2$	1G	WII

Table 3b: Excesses made from OD(4wt;wt,wt,wt,wt) where
t is the order of 4 distinct T-sequences and
w is the order of Williamson-type matrices
*** the excess meets the bound**
where no indication is given for the construction
of the Williamson-type matrices it is WI
the T-matrices are given in Cohen et al [4]

t		Sum of Squares	Comment	Excess
101	1 + 100	$10^2 + 1^2$	Turyn-Golay	8080
101	1 + 100	$8^2 + 6^2 + 1^2$	Turyn-Golay	6464
103				
105	1+4×26	$10^2 + 2^2 + 1^2$	Turyn-Golay	8400
105	3 × 35	$9^2 + 4^2 + 2^2 + 2^2$	y = 3, m = 17	7560
105	3 × 35	$8^2 + 4^2 + 4^2 + 3^2$	y = 3, m = 17	7980
105	3 × 35	$7^2 + 6^2 + 4^2 + 2^2$	y = 3, m = 17	7980
105	5 × 21		y = 5, m = 10	
105	21 × 5	$8^2 + 4^2 + 4^2 + 3^2$	y = 21, m = 2	7980
105	21 × 5	$8^2 + 5^2 + 4^2$	y = 21, m = 2	7140
105	7 × 15	$8^2 + 4^2 + 4^2 + 3^2$	y = 7, m = 7	7980
105	7 × 15	$7^2 + 6^2 + 4^2 + 2^2$	y = 7, m = 7	7980
107				
109				
111	3 × 37			
113				
115	5 × 23	$7^2 + 5^2 + 5^2 + 4^2$	y = 5, m = 11	9660
115	5 × 23	$7^2 + 7^2 + 4^2 + 1^2$	y = 5, m = 11	8740
115	5 × 23	$8^2 + 5^2 + 5^2 + 1^2$	y = 5, m = 11	8740
115	5 × 23	$8^2 + 7^2 + 1^2 + 1^2$	y = 5, m = 11	7820
117	9 × 13	$8^2 + 6^2 + 4^2 + 1^2$	y = 9, m = 6	8892
117	9 × 13	$7^2 + 6^2 + 4^2 + 4^2$	y = 9, m = 6	9828
117	13 × 9	$10^2 + 3^2 + 2^2 + 2^2$	y = 13, m = 4	9360
117	13 × 9	$9^2 + 6^2$	y = 13, m = 4	8424
117	13 × 9	$8^2 + 6^2 + 4^2 + 1^2$	y = 13, m = 4	8892
117	13 × 9	$7^2 + 6^2 + 4^2 + 4^2$	y = 13, m = 4	9828
119	7 × 17	$9^2 + 5^2 + 3^2 + 2^2$	y = 7, m = 8	9044
119	7 × 17	$7^2 + 6^2 + 5^2 + 3^2$	y = 7, m = 8	9996
119	17 × 7	$7^2 + 6^2 + 5^2 + 3^2$	y = 17, m = 3	9996
119	17 × 7	$9^2 + 5^2 + 3^2 + 2^2$	y = 17, m = 3	9044
121	11 × 11		OD(484;121,121,121,121)exists	
123	3 × 41	$9^2 + 5^2 + 4^2 + 1^2$	y = 3, m = 20	8856
123	3 × 41	$8^2 + 7^2 + 3^2 + 1^2$	y = 3, m = 20	9348
123	3 × 41	$8^2 + 5^2 + 5^2 + 3^2$	y = 3, m = 20	10332
123	41 × 3	$7^2 + 7^2 + 4^2 + 3^2$	y = 41, m = 1	10332
123	41 × 3	$9^2 + 5^2 + 4^2 + 1^2$	y = 41, m = 1	9348
125	5 × 25	$10^2 + 5^2$	y = 5, m = 12	10000
125	5 × 25	$11^2 + 2^2$	y = 5, m = 12	11000
127				
129	$1 + 2^7$	$8^2 + 8^2 + 1^2$	Turyn-Golay	8772
131				
133	7 × 19	$10^2 + 5^2 + 2^2 + 2^2$	y = 7, m = 9	10640
133	7 × 19	$10^2 + 4^2 + 4^2 + 1^2$	y = 7, m = 9	10640
133	7 × 19	$8^2 + 8^2 + 2^2 + 1^2$	y = 7, m = 9	10108
135	3 × 45		y = 3, m = 22	
135	5 × 27	$9^2 + 7^2 + 2^2 + 1^2$	y = 5, m = 13	10260
135	5 × 27	$11^2 + 3^2 + 2^2 + 1^2$	y = 5, m = 13	11880

Table 4a: 4-disjoint T-sequences

t		Sum of Squares	Comment	Excess
137				
139				
141	3 × 47	$10^2 + 3^2 + 2^2 + 1^2$	y = 3, m = 23	11280
141	3 × 47	$9^2 + 5^2 + 2^2 + 2^2$	y = 3, m = 23	10152
141	3 × 47	$9^2 + 4^2 + 4^2 + 1^2$	y = 3, m = 23	10152
143	13 × 11	$11^2 + 3^2 + 3^2 + 2^2$	y = 13, m = 5	12584
143	13 × 11	$9^2 + 7^2 + 3^2 + 2^2$	y = 13, m = 5	12012
145	5 × 29	$9^2 + 8^2$	y = 5, m = 14	10440
145	5 × 29	$12^2 + 1^2$	y = 5, m = 14	13920
147	7 × 21		y = 7, m = 10	
147	21 × 7	$9^2 + 7^2 + 4^2 + 1^2$	y = 21, m = 3	12348
147	21 × 7	$11^2 + 5^2 + 1^2$	y = 21, m = 3	12936
149				
151				
153	3 × 51			
153	9 × 17	$9^2 + 8^2 + 2^2 + 2^2$	y = 9, m = 8	12852
153	9 × 17	$8^2 + 7^2 + 6^2 + 2^2$	y = 9, m = 8	14076
153	17 × 9	$12^2 + 3^2$	y = 17, m = 4	14688
155	5 × 31	$9^2 + 7^2 + 5^2$	y = 5, m = 15	13020
155	5 × 31	$9^2 + 7^2 + 4^2 + 3^2$	y = 5, m = 15	14260
155	5 × 31	$11^2 + 5^2 + 3^2$	y = 5, m = 15	13640
155	5 × 31	$11^2 + 4^2 + 3^2 + 3^2$	y = 5, m = 15	13640
157				
159	53 × 3	$9^2 + 7^2 + 5^2 + 2^2$	y = 53, m = 1	14628
159	53 × 3	$11^2 + 5^2 + 3^2 + 2^2$	y = 53, m = 1	13992
161	1+16×10	$12^2 + 4^2 + 1^2$	Turyn-Golay	15456
161	7 × 23	$10^2 + 6^2 + 5^2$	y = 7, m = 11	13524
161	7 × 23	$9^2 + 8^2 + 4^2$	y = 7, m = 11	13524
161	7 × 23	$8^2 + 6^2 + 6^2 + 5^2$	y = 7, m = 11	16100
163				
165	5 × 33		y = 5, m = 16	
165	11 × 15			
165	33 × 5	$8^2 + 7^2 + 6^2 + 4^2$	y = 33, m = 2	16500
165	33 × 5	$9^2 + 8^2 + 4^2 + 2^2$	y = 33, m = 2	15180
167				
169	13 × 13	$9^2 + 6^2 + 6^2 + 4^2$	y = 13, m = 6	16900
171	3 × 57		y = 33, m = 28	
171	9 × 19	$9^2 + 7^2 + 5^2 + 4^2$	y = 9, m = 9	17100
171	9 × 19	$11^2 + 5^2 + 5^2$	y = 9, m = 9	15048
171	9 × 19	$11^2 + 5^2 + 4^2 + 3^2$	y = 9, m = 9	17784
173				
175	5 × 35	$10^2 + 5^2 + 5^2 + 5^2$	y = 7, m = 12	17500
175	5 × 35	$10^2 + 7^2 + 5^2 + 1^2$	y = 7, m = 12	16100
175	7 × 25	$11^2 + 7^2 + 2^2 + 1^2$	y = 7, m = 12	15400
175	7 × 25	$11^2 + 6^2 + 3^2 + 3^2$	y = 7, m = 12	15400
175	7 × 25	$10^2 + 7^2 + 5^2 + 1^2$	y = 7, m = 12	16100
175	7 × 25	$10^2 + 5^2 + 5^2 + 5^2$	y = 7, m = 12	17500
175	7 × 25	$9^2 + 9^2 + 3^2 + 2^2$	y = 7, m = 12	16100
175	7 × 25	$9^2 + 7^2 + 6^2 + 3^2$	y = 7, m = 12	17500

Table 4b: 4-disjoint T-sequences

t		Sum of Squares	Comment	Excess
177	3×59	$11^2 + 6^2 + 4^2 + 2^2$	y = 3, m = 29	16284
177	3×59	$10^2 + 8^2 + 3^2 + 2^2$	y = 3, m = 29	16284
177	3×59	$9^2 + 8^2 + 4^2 + 4^2$	y = 3, m = 29	17700
179				
181				
183	3×61			
185	5×37		OD(740;185,185,185,185) exists	
187	17×11	$12^2 + 5^2 + 3^2 + 3^2$	y = 17, m = 5	17952
189	7×27	$12^2 + 5^2 + 4^2 + 2^2$	y = 7, m = 13	18144
189	7×27	$10^2 + 8^2 + 4^2 + 3^2$	y = 7, m = 13	18900
189	7×27	$8^2 + 8^2 + 6^2 + 5^2$	y = 7, m = 13	20412
189	9×21		y = 9, m = 10	
189	21×9	$12^2 + 6^2 + 3^2$	y = 21, m = 4	18144
191				
193				
195	5×39		y = 5, m = 19	
195	13×15	$11^2 + 7^2 + 4^2 + 3^2$	y = 13, m = 7	19500
195	13×15	$9^2 + 8^2 + 7^2 + 1^2$	y = 13, m = 7	19500
197				
199				
201	$1+100\times2$	$14^2 + 2^2 + 1^2$	Turyn-Golay	22512
201	$1+100\times2$	$10^2 + 2^2 + 1^2$	Turyn-Golay	16884
203	7×29	$12^2 + 7^2 + 3^2 + 1^2$	y = 7, m = 14	19488
203	7×29	$9^2 + 8^2 + 7^2 + 3^2$	y = 7, m = 14	21924
205	5×41		y = 5, m = 20	
205	41×5	$11^2 + 8^2 + 4^2 + 2^2$	y = 41, m = 2	20500
205	41×5	$13^2 + 4^2 + 4^2 + 2^2$	y = 41, m = 2	21320
207	9×23	$11^2 + 6^2 + 5^2 + 5^2$	y = 9, m = 11	22356
207	9×23	$11^2 + 9^2 + 2^2 + 1^2$	y = 9, m = 11	19044
207	9×23	$13^2 + 5^2 + 3^2 + 2^2$	y = 9, m = 11	21528
209	$1+8\times26$	$12^2 + 8^2 + 1^2$	Turyn-Golay	20064
225	9×25	$10^2 + 8^2 + 6^2 + 5^2$	y = 9, m = 12	26100
225	9×25	$11^2 + 8^2 + 6^2 + 2^2$	y = 9, m = 12	24300
243	9×27	$9^2 + 9^2 + 9^2$	y = 9, m = 13	26244
243	9×27	$11^2 + 8^2 + 7^2 + 3^2$	y = 9, m = 13	28188
243	9×27	$11^2 + 11^2 + 1^2$	y = 9, m = 13	22356
243	9×27	$13^2 + 8^2 + 3^2 + 1^2$	y = 9, m = 13	25272

Table 4c: 4-disjoint T-sequences

Appendix A

n	Lower Bound	Best Bound	HLS Bound	KF Found	Known Excess	Reference and Comment
4	5	8	8	8	8*	
8	18	22	20	20	20*	W D Wallis [36]
12	33	41	36	36	36*	Best [3]
16	51	64	64	64	64*	Best [3]
20	71	89	88	84	80	W D Wallis [36]
24	93	117	112	112	112*	Best [3]
28	118	148	140	140	140*	HLS [13], Table 1
32	144	181	176	172	172*	HLS [13]
36	172	216	216	216	216*	HLS [13], Table 1
40	201	252	252	244	244*	F & K [18]
44	232	291	288	280	280*	F & K [18]
48	265	332	324	324	324*	S & S [22], F & K [18]
52	299	374	364	364	364*	HLS [13], Table 1
56	334	419	412	408	392	Theorem 1, [14] says ≥ 400
60	370	464	460	460	452# 420	HLS [13], Table 1, [14] says ≥ 440
64	408	512	512	512	512*	HLS [13]
68	447	560	560	556	548# 544	HLS [13], Table 2
72	487	610	608	600	576	F & K [18]
76	528	662	656	652	620	F & K [18]
80	570	715	704	704	704*	S & S [22], F & K [18]
84	614	769	756	756	756*	HLS [13], Table 1
88	658	825	816	812	792	Theorem 2
92	703	882	876	872	828	HLS [13], Table 1
96	750	940	936	932		S & S [22] say $920 < \sigma(96) < 928$
100	797	1000	1000	1000	1000*	Seberry [26] 10^2
104	846	1060	1060	1052		
108	895	1122	1116	1112	1080	HLS [13], Table 2
112	945	1185	1176	1172		
116	996	1249	1240	1232	1160	HLS [13], Table 2
120	1048	1314	1300	1300		
124	1101	1380	1364	1364	1364*	HLS [13], Table 1, Table 2
128	1155	1448	1436	1432		
132	1209	1516	1508	1500	1452	HLS [13], Table 1, Table 2
136	1265	1586	1580	1576		
140	1321	1656	1652	1652	1540	HLS [13], Table 2
144	1378	1728	1728	1728	1728*	Seberry [26] 12^2
148	1436	1800	1800	1796	1776	Table 2
152	1494	1873	1868	1864		
156	1554	1948	1940	1932	1872	Table 3
160	1614	2023	2012	2008		

n	Lower Bound	Best Bound	HLS Bound	KF Found	Known Excess	Reference and Comment
164	1675	2100	2088	2084	1640	Table 1
168	1737	2177	2160	2160		
172	1799	2255	2236	2236	2236*	Table 1
176	1862	2334	2320	2316		
180	1926	2414	2400	2400	2340	Table 1
184	1991	2495	2484	2484		
188	2056	2577	2572	2568	2444	Table 2
192	2122	2660	2656	2652		
196	2188	2744	2744	2744	2744*	Seberry [26] 14^2
200	2256	2828	2828	2820		
204	2324	2913	2908	2904	2856	Table 2
208	2393	2999	2992	2988		
212	2462	3086	3076	3072	2544	Table 2
216	2532	3174	3160	3156		
220	2603	3263	3248	3240	3080	Table 3
224	2674	3352	3332	3332		
228	2746	3442	3420	3420	3420*	Table 1
232	2818	3533	3516	3512		
236	2892	3625	3608	3604		
240	2965	3718	3704	3696	3600	Theorem 1
244	3040	3811	3800	3796	2928	Table 1
248	3115	3905	3900	3896		
252	3191	4000	3996	3996	3780	Table 3
256	3267	4096	4096	4096	4096*	Seberry [26] 16^2
260	3344	4192	4192	4188	4160	HLS [13], Table 3, Table 4
264	3421	4289	4284	4280		
268	3499	4387	4380	4372	4288	Table 2
272	3578	4485	4476	4472		
276	3657	4585	4572	4564	4416	Table 3
280	3737	4685	4668	4664		
284	3817	4786	4768	4764		
288	3898	4887	4864	4864		
292	3980	4989	4964	4964	4964*	K, K & S [17]
296	4062	5092	5072	5068		
300	4145	5196	5176	5168	5100	Table 1, Table 3
304	4228	5300	5284	5276		
308	4311	5405	5392	5384	5236	Table 3
312	4396	5511	5500	5492		
316	4481	5617	5612	5608	5372	Table 1
320	4566	5724	5720	5716		

n	Lower Bound	Best Bound	HLS Bound	KF Found	Known Excess	Reference and Comment
324	4652	5832	5832	5832	5832*	Seberry [26] 18^2
328	4738	5940	5940	5932		
332	4825	6049	6044	6040		
336	4913	6158	6152	6148		
340	5001	6269	6256	6256	6120	Table 2
344	5089	6380	6364	6364		
348	5178	6491	6476	6472	5916	Table 2
352	5268	6604	6584	6580		
356	5358	6716	6696	6688		
360	5448	6830	6804	6804		
364	5539	6944	6916	6916	6916*	Table 1
368	5631	7059	7036	7032		
372	5723	7174	7152	7148	7068	Table 1
376	5816	7290	7272	7264		
380	5909	7407	7392	7388	7220	Theorem 1, Table 2
384	6002	7524	7512	7504		
388	6096	7642	7632	7628	7372	Table 1
392	6191	7761	7756	7752		
396	6286	7880	7876	7876	7524	HLS [13], Table 3
400	6381	8000	8000	8000	8000*	Seberry [26] 20^2
404	6477	8120	8120	8116	8080	Table 4
408	6574	8241	8236	8232		
412	6671	8362	8356	8348		
416	6768	8484	8472	8464		
420	6866	8607	8592	8588	8400	Table 3, Table 4
424	6964	8730	8712	8704		
428	7063	8854	8836	8828		
432	7162	8978	8956	8952		
436	7262	9103	9080	9076		
440	7362	9229	9200	9200		
444	7463	9355	9324	9324	9324*	Table 3
448	7564	9482	9456	9452		
452	7665	9609	9584	9576		
456	7767	9737	9716	9708		
460	7870	9865	9844	9840	9660	Table 3, Table 4
464	7973	9994	9976	9972		
468	8076	10124	10112	10104	9828	Table 1, Table 3
472	8180	10254	10244	10236		
476	8284	10385	10380	10376	9996	Table 3, Table 4
480	8389	10516	10512	10508		
484	8494	10648	10648	10648	10648*	Seberry [26] 22^2
488	8599	10780	10780	10772		
492	8705	10913	10908	10904	10332	Table 3, Table 4
496	8812	11046	11036	11036		
500	8918	11180	11168	11160	11000	Table 4

n	Lower Bound	Best Bound	HLS Bound	KF Found	Known Excess	Reference and Comment
504	9026	11314	11300	11292		
508	9133	11449	11432	11424		
512	9241	11585	11564	11556		
516	9350	11721	11696	11696	10836	Table 3
520	9459	11857	11832	11828		
524	9568	11994	11968	11960		
528	9678	12132	12100	12100		
532	9788	12270	12236	12236	10640	Table 4
536	9899	12409	12380	12376		
540	10010	12548	12520	12516	12420	Table 1
544	10121	12688	12664	12656		
548	10233	12828	12804	12796		
552	10345	12969	12948	12944	12696	Theorem 1
556	10458	13110	13092	13084	12788	Table 1
560	10571	13252	13240	13232		
564	10684	13394	13384	13380	11844	Table 1
568	10798	13537	13532	13528		
572	10913	13680	13676	13676	13156	Table 3
576	11027	13824	13824	13824	13824*	Seberry [26] 24^2
580	11142	13968	13968	13964	13920	Table 4
584	11258	14112	14108	14104		
588	11374	14258	14248	14244	14112	Table 3
592	11490	14403	14392	14384		
596	11607	14550	14536	14532		
600	11724	14696	14680	14672		
604	11841	14844	14824	14820		
608	11959	14991	14968	14960		
612	12077	15140	15112	15108	14688	Table 3, Table 4
616	12196	15288	15260	15256		
620	12315	15437	15408	15404	14260	Table 4
624	12434	15587	15552	15552		
628	12554	15737	15700	15700	15700*	Table 1
632	12674	15888	15856	15852		
636	12794	16039	16008	16000	15900	Table 1
640	12915	16190	16160	16156		
644	13037	16342	16316	16312	16100	Table 4
648	13158	16495	16472	16468	16200	Theorem 2
652	13280	16648	16628	16624		
656	13403	16801	16784	16780		
660	13525	16955	16940	16936	16500	Table 3, Table 4
664	13649	17110	17100	17092		
668	13772	17264	17260	17256		
672	13896	17420	17416	17412		
676	14020	17576	17576	17576	17576*	Seberry [26] 26^2

n	Lower Bound	Best Bound	HLS Bound	KF Found	Known Excess	Reference and Comment
680	14145	17732	17732	17724		
684	14270	17888	17884	17880	17784	Table 3, Table 4
688	14395	18046	18036	18028		
692	14521	18203	18192	18184		
696	14647	18361	18348	18340		
700	14774	18520	18500	18496	17500	Table 3, Table 4
704	14900	18679	18656	18652		
708	15028	18838	18816	18808	17700	Table 1
712	15155	18998	18972	18964		
716	15283	19158	19128	19120		
720	15411	19319	19288	19284		
724	15540	19480	19448	19440	14480	Table 1
728	15669	19642	19604	19604		
732	15798	19804	19764	19764		
736	15928	19967	19932	19928		
740	16058	20130	20096	20092		
744	16188	20293	20260	20256		
748	16319	20457	20428	20420	20196	Table 3
752	16450	20621	20596	20592		
756	16581	20786	20764	20756	20412	Table 3
760	16713	20951	20932	20928		
764	16845	21117	21100	21092		
768	16978	21283	21268	20264		
772	17111	21449	21440	21436		
776	17244	21616	21612	21608		
780	17377	21784	21780	21780	21060	Table 1, Table 3
784	17511	21952	21952	21952	21952*	Seberry [26] 28^2
788	17645	22120	22120	22116		
792	17780	22288	22284	22280		
796	17915	22457	22448	22444	19900	Table 1
800	18050	22627	22616	22608		
804	18185	22797	22780	22780	22512	Table 4
808	18321	22976	22948	22944		
812	18458	23138	23116	23108	21924	Table 4
816	18594	23309	23284	23280		
820	18731	23481	23452	23452	21320	Table 4
824	18868	23653	23624	23616		
828	19006	23825	23792	23788	23184	Table 3
832	19144	23998	23964	23960		
836	19282	24171	24136	24132	23408	Table 3

n	Lower Bound	Best Bound	HLS Bound	KF Found	Known Excess	Reference and Comment
840	19421	24345	24304	24304		
844	19559	24519	24476	24476		
848	19699	24694	24656	24652		
852	19838	24869	24832	24824		
856	19978	25044	25008	25004		
860	20118	25220	25188	25184		
864	20259	25396	25368	25364		
868	20400	25572	25544	25544	25172	Table 4
872	20541	25749	25724	25724		
876	20682	25927	25908	25904		
880	20824	26105	26088	26084		
884	20966	26283	26268	26264	25636	Table 3
888	21109	26461	26452	26444		
892	21252	26640	26636	26632		
896	21395	26820	26816	26812		
900	21538	27000	27000	27000	27000*	Seberry [26] 30^2
904	21682	27180	27180	27172		
908	21826	27360	27356	27352		
912	21970	27541	27532	27524		
916	22115	27723	27712	27704		
920	22260	27904	27888	27884		
924	22405	28087	28068	28064	27720	Table 3
928	22551	28269	28248	28244		
932	22697	28452	28428	28424		
936	22843	28636	28608	28604		
940	22990	28819	28788	28784		
944	23137	29004	28972	28964		
948	23284	29188	29152	29144		
952	23431	29373	29336	29332		
956	23579	29558	29520	29512		
960	23727	29744	29700	29700		
964	23876	29930	29884	29884		
968	24025	30117	30076	30072		
972	24174	30303	30264	30260	30132	Table 3
976	24323	30491	30452	30448		
980	24473	30678	30644	30636	27440	Table 4
984	24623	30866	30832	30832		
988	24773	31055	31024	31020	30628	Table 3
992	24924	31244	31216	31208	30952	Theorem 1
996	25075	31433	31408	31404		
1000	25226	31622	31600	31600		