On Weighing Matrices with Square Weights

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We give a new construction for a known family of weighing matrices using the 2-adjugate method of Vartak and Patwardhan. We review the existence of W(n, k^2), k = 1, ... ,12, giving new results for k = 8,...12.

1. Introduction

A *weighing matrix*, W = W(n,k), is a square matrix of order n, with entries 0,+1, or -1, with the property that the scalar product of distinct rows is zero. Thus

$$WW^T = kI_n.$$

It has been conjectured that

Conjecture 1. (Seberry). *When* $n \equiv 0 (mod 4)$ *there exists a W(n,k) for all* $k \leq n$.

We discuss the conjecture

Conjecture 2. *For given k there exists an integer* n_0, *so that for every* $n > n_0$, *there exists a W(n,k).*

It is known that

THEOREM 1 (Geramita, Geramita and Seberry[1975/76]). *If a W(n,k) exists and n is odd, then k is a square and*

$$(n-k)^2 - (n-k) + 2 > n.$$

We know that

(a) a W(m^2 + m + 1, m^2) exists whenever m is a prime power and only if m is the order of a projective plane;
(b) given any integer k^2 there exists an n dependent on k^2 so that for every m > n a $W(m,k^2)$ exists;
(c) there exists a

(i) W(m,4), where m > 3 (except m=5,9, which do not exist);

(ii) W(m,9), where m > 21 (m = 11 does not exist and m = 17, 21 are undecided; m = 15 exists, Gibbons and Mathon[1986] found a W(19,9));

(iii) W(m,16), where m > 35 (m = 17, 19 do not exist and m = 23, 25, 27, 29, 33, 35 are undecided; m = 31 exists);

iv) W(m,25), where m > 81 (m = 27,29 do not exist-- there are many undecided cases);

(v) W(m,36), where m > 137; (m = 37, ... ,41 do not exist)

(vi) W(m,49), where m > 199; (m = 49, ... ,58 do not exist).

The proof of these results is in Geramita and Seberry [1979, p324-325].

We present a construction for weighing matrices using the 2-adjugation method of Vartak and Patwardhan and present results for W(n,64),W(n,81),W(n,100),W(n,121),W(n,144). This allows us to say

(vii) W(n,64) exist for n > 135 and W(2n,64) exist for 2n > 31 (Lemma 8); W(n,64) for n = 65, ... ,71 do not exist;

(viii) W(n,81) for n > 377, W(4n,81) for n > 39, W(130,81) and W(91,81) exist (Section 1 (a)); W(n,81) for n = 81, ... ,89 do not exist;

(ix)W(n,100) exist for all n > 336 and W(2n,100) exist for all n > 59. (Lemma 10); W(n,100) for n = 101, ... ,109 do not exist;

(x) W(n,121) exist for all n > 513, W(2n,121) exist for 2n > 381. W(4n,121) exist for 4n > 259, W(8n,121) exist for 8n > 127. (Lemma 11); W(n,121) do not exist for n = 121, ... ,131;

(xi) W(n,144) exists for all n > 649, W(2n,144) exists for all 2n > 377; (Lemma 12); W(n,144) do not exist for n = 145, ... ,155.

2. Known results

We use the following results

THEOREM 2. (Dandawate). *If 4t is the order of an Hadamard matrix then there exists a* $W(2t(4t-1), 4t^2)$.

THEOREM 3. (Geramita and Seberry) *Let r be any number of the form* $2^a \cdot 10^b \cdot 26^c \cdot 5^d \cdot 13^e$ *, and let n be any integer* $> 2^a \cdot 10^b \cdot 26^c \cdot 6^d \cdot 14^e$ *, a,b,c,d,e nonnegative integers. Then there exist (a) orthogonal designs of order 4n and types (1,1,r,r) and (1,4,r,r); (b) an orthogonal design of order 2n and type (r,r) or a W(2n,2r).*

COROLLARY 4. *Let 2n be any integer > 119 then there exists a W(2n,100).*

We use the fact that if there is an orthogonal design of type (1,k) in order n there is an orthogonal design of type (1,1,k,k) in order 2n. Hence a W(2n,2k+1) exists.

By the results of Geramita and Seberry[1979, Chapter 8] and Seberry[1982] we have

THEOREM 5. *Every orthogonal design of type (1,k), k < n-1, exists in order* $n = 2^t, 2^{t+1}\cdot 3, 2^{t+1}\cdot 5, 2^{t+1}\cdot 7, 2^{t+1}\cdot 9, 2^{t+4}\cdot 15, 2^{t+4}\cdot 21$*, t a positive integer. Hence W(2n,81) and W(2m,121) exist for those m,n provided n > 40, m > 60. Thus W(16a,81) exist for a > 5 and W(16b,121) exist for all b > 7 (since by Theorem 1.149 of Geramita and Seberry an OD(1,30) exists in every order 4n, n ≥ 11 we have OD(1,1,2,60,60) and hence W(16n,121) for n ≥ 11).*

Also from Theorem 1.149 of Geramita and Seberry we have

LEMMA 6. *Orthogonal designs of type (1,40) exist in all orders 4n, n > 10. Thus W(8n,81) exist n > 10.*

Since there are Golay sequences of length 40 we have

LEMMA 7. *There is an orthogonal design of type (1,1,40,40) in all orders 4n, n ≥ 40. Thus W(4n,81) exist for n ≥ 40.*

3. Some W(n, k^2)

3.1. Case W(n,64)

The following exist:

W(2n,64), n > 31 (Corollary 4),
W(120,64) (Dandawate),
W(73,64) (Section 1, (i)).

Thus we have

LEMMA 8. *There exist W(n,64) for n > 135.*

3.2. Case W(n,81)

The following exist:

W(4n,81), n > 39 (Lemma 7)
W(91,81), (complement of PG(9,2))
W(130,81), (from W(10,9) and W(13,9))
W(169,81), (from W(13,9)).

Thus we have

LEMMA 9. *There exist W(n,81) for n > 337.*

3.3. Case W(n,100)

W(2n,100) exist for all n > 59, (Corollary 4). W(m,25) and W(n,4) give a W(mn,100) so since W(m,25) exist for m ε {26,28,31,32,36,40,42,44,48, and all even orders > 51} and W(n,4) exist for n ε {4,6,7,8,10,11,...} we have W(mn,100) for mn ε {104,112,217}. Thus we have

LEMMA 10. *There exist W(n,100) for all n > 336. W(2n,100) exist for 2n = 104,112 and all 2n > 119.*

3.4. Case W(n,121)

W(133,121) exists ((i), section 1).
W($16t^2$,121) exist for all t > 2 since W(4t,11) exists.
W(16b,121) exists for all b > 7 (from Theorem 5).
W(122,121) exists since 121 is a prime power.

We construct a W(132,121). Let A be a (1,11) design in order 12, replace the first variable by J_{11}, the matrix of ones, and the second variable by B where B = (b_{ij}) = (χ(j-i)), χ the quadratic character with χ(0)=0. Then the new matrix is the required W(132,121).

Thus we have

LEMMA 11. *There exist W(n,121) for all n > 513. W(2n,121) exist for 2n > 381. W(4n,121) exist for 4n*

> 259. W(16n,121) exist for 16n > 127.

3.5 Case W(n,144)

Since there are 4-complementary sequences of length 9 and weight 36 and Golay sequences of length 4, we have 4-complementary sequences of length 36 and weight 144. Thus we have

W(4n,144) exist for all n > 35.
W(273,144) exists as it is W(13,9) × W(21,16).
W(234,144) exists as it is W(18,16) × W(13,9).

Hence

LEMMA 12. *W(n,144) exists for all n > 649. W(2n,144) exists for all 2n > 377.*

4. A new construction for weighing matrices using 2-adjugates

Vartak and Patwardhan[1971] and Patwardhan and Vartak[1980] constructed the 2-adjugate mod 2 class of designs for unreduced BIBDs and symmetrical BIBDs with $\lambda = 1$. Patwardhan, Dandawate and Vartak[1984] applied the technique of 2-adjugation to a (0,1,-1) matrix and obtained a class of generalized PBIBDs with two associate classes and with triangular association scheme with unequal block sizes.

In this section we use the concept of 2-adjugates to obtain a construction for weighing matrices.

The 2-adjugate of a (0,1,-1) matrix of N of order (v x b) is defined to be the matrix whose elements are the determinants of all possible 2 x 2 submatrices of N, arranged in lexicographic order. The 2-adjugate N* of a matrix N whose entries come from a group is defined to be the matrix whose entries are formed by taking formal determinants of all possible 2 x 2 submatrices of N. Here by formal determinant we mean the following:

If the submatrix is

$$\begin{matrix} x, y \\ z, w \end{matrix}$$

then the corresponding entry of N* is xw-yz.

Notice that in the special case when N = GH(p,EA(p)), the generalized Hadamard matrix of order p, p a prime power, the $(i,j)^{th}$ entry of N is $T^{(i-1)(j-1)}$ and hence by replacing the group elements by the corresponding matrix representation the 2-adjugate of any arbitrary two by two submatrix

$$\begin{matrix} T^{(i-1)(s-1)} & T^{(i-1)(t-1)} \\ T^{(j-1)(s-1)} & T^{(j-1)(t-1)} \end{matrix}$$

with i < j is $T^{(i-1)(s-1)+(j-1)(t-1)} - T^{(i-1)(t-1)+(j-1)(s-1)}$.

An arbitrary row of H* obtained from the i^{th} and j^{th} rows of H is given by

$$T^{(i-1)(t-1)+(j-1)(s-1)} - T^{(i-1)(s-1)+(j-1)(t-1)}\ ;\ t=1,\ldots,p-1;\ s=t+1,\ldots,p.$$

We prove the following

LEMMA 13. *Let H^* be the 2-adjugate formed from H = GH(p,EA(p)), the generalized Hadamard matrix* *of size p, p a prime power, then the inner product of any distinct rows of H^* obtained from the i^{th} and j^{th} rows and k and l^{th} rows of H respectively is zero and the inner product of a row obtained from the i^{th} and j^{th} rows of H with itself is p^2I - pJ.*

Proof:
The inner product of two rows of H^* is given by

$$\sum_{t=1}^{t=(p-1)} \sum_{s=(t+1)}^{s=p} [T^{(i-1)(t-1)+(j-1)(s-1)} - T^{(i-1)(s-1)+(j-1)(t-1)}]$$

$$\cdot [T^{p-(k-1)(t-1)-(r-1)(s-1)} - T^{p-(k-1)(s-1)-(r-1)(t-1)}]$$

$$= \sum_{t=1}^{t=(p-1)} \sum_{s=(t+1)}^{s=p} T^{(i-k)(t-1)+(j-r)(s-1)} + \sum_{t=1}^{t=(p-1)} \sum_{s=(t+1)}^{s=p} T^{(i-k)(s-1)+(j-r)(t-1)}$$

$$- \sum_{t=1}^{t=(p-1)} \sum_{s=(t+1)}^{s=p} T^{(i-r)(s-1)+(j-k)(t-1)} - \sum_{t=1}^{t=(p-1)} \sum_{s=(t+1)}^{s=p} T^{(i-r)(t-1)+(j-k)(s-1)}.$$

Notice that the first and second terms give

$$\sum_{t=1}^{t=(p-1)} \sum_{s=(t+1)}^{s=p} T^{(i-k)(t-1)} \cdot T^{(j-r)(s-1)} + \sum_{s=1}^{s=(p-1)} \sum_{t=(s+1)}^{t=p} T^{(i-k)(t-1)} \cdot T^{(j-r)(s-1)}$$

Which gives us

$$\sum_{t=1}^{t=p} [T^{(i-k)(t-1)} \cdot \sum_{s=1}^{s=p} T^{(j-r)(s-1)}] \ (s \neq t).$$

Now we add and subtract

$$\sum_{t=1}^{t=p} T^{(i-k)(t-1)+(j-r)(t-1)}.$$

Similarly we evaluate third and fourth terms and get:

$$\sum_{t=1}^{t=p} T^{(i-k)(t-1)} \cdot \sum_{s=1}^{s=p} T^{(j-r)(s-1)} - \sum_{t=1}^{t=p} T^{(t-1)(i+j-k-r)} - \sum_{t=1}^{t=p} T^{(i-r)(s-1)} \cdot \sum_{s=1}^{s=p} + \sum_{t=1}^{t=p} T^{(t-1)(i+j-k-r)}.$$

Which is p^2I - pJ when i=k and j=r, and zero otherwise.

THEOREM 14. *Weighing matrices $W(p^2(p-1), p^2)$ exist for all prime powers p.*

Proof: Form H* from H = GH(p,EA(p)) the generalized Hadamard matrix.Obtain G from H* by replacing the entries of H* by the corresponding (0,1,-1) matrices.Form a matrix $S = I \times J_p$. Then

$$\begin{bmatrix} G & S \\ S & G \end{bmatrix}$$

is the required weighing matrix.

EXAMPLE. Let H = GH(3,EA(3)) given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

Replace ω^i by T^i and 1 by the identity matrix of order 3, where T is given by

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Now form H*:

$$\begin{bmatrix} 1-\omega & 1-\omega^2 & \omega^2-\omega \\ 1-\omega^2 & 1-\omega & \omega-\omega^2 \\ \omega^2-\omega & \omega-\omega^2 & \omega^2-\omega \end{bmatrix}$$

which gives G:

$$\begin{array}{ccccccccc}
- & 1 & 0 & - & 0 & 1 & 0 & - & 1 \\
0 & - & 1 & 1 & - & 0 & 1 & 0 & - \\
1 & 0 & - & 0 & 1 & - & - & 1 & 0 \\
 & & & & & & & & \\
- & 0 & 1 & - & 1 & 0 & 0 & 1 & - \\
1 & - & 0 & 0 & - & 1 & - & 0 & 1 \\
0 & 1 & - & 1 & 0 & - & 1 & - & 0 \\
 & & & & & & & & \\
0 & - & 1 & 0 & 1 & - & 0 & - & 1 \\
1 & 0 & - & - & 0 & 1 & 1 & 0 & - \\
- & 1 & 0 & 1 & - & 0 & - & 1 & 0
\end{array}$$

Now form S:

$$\begin{array}{ccccccccc}
1 & 1 & 1 & & & & & & \\
1 & 1 & 1 & & & & & & \\
1 & 1 & 1 & & & & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & & & & & 1 & 1 & 1 \\
 & & & & & & 1 & 1 & 1 \\
 & & & & & & 1 & 1 & 1
\end{array}$$

Now form W:

$$\begin{bmatrix} G & S \\ S & G \end{bmatrix}$$

Note that in this particular case we have $W(18, 3^2)$. We also note that in the case of order 3 we have using

$$GW(4,3,2) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & - \\ 1 & - & 0 & 1 \\ 1 & 1 & - & 0 \end{bmatrix}$$

that

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0 0 0 0   1 1 1  1 1 1  1 1 1
0 1 1 1   0 0 0  - - -  1 1 1
0 1 1 1   1 1 1  0 0 0  - - -
0 1 1 1   - - -  1 1 1  0 0 0

1 0 1 -
1 0 1 -
1 0 1 -

1 - 0 1
1 - 0 1          G
1 - 0 1

1 1 - 0
1 1 - 0
1 1 - 0
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is a W(13,9).

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