BHASKAR RAO DESIGNS OVER SMALL GROUPS

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Abstract:

We show that for each of the groups S_3, D_4, Q_4, $Z_4 \times Z_2$ and D_6 the necessary conditions are sufficient for the existence of a generalized Bhaskar Rao design.

That is, we show that:

(i) a GBRD $(v, 3, \lambda; S_3)$ exists if and only if $\lambda \equiv 0 \pmod 6$ and $\lambda v(v-1) \equiv 0 \pmod{24}$;

(ii) if G is one of the groups D_4, Q_4, and $Z_4 \times Z_2$, a GBRD$(v, 3, \lambda; G)$ exists if and only if $\lambda \equiv 0 \pmod 8$ and $\lambda v(v-1) \equiv 0 \pmod 6$;

(iii) a GBRD $(v, 3, \lambda; D_6)$ exists if and only if $\lambda \equiv 0 \pmod{12}$.

From these designs, families of regular group divisible designs are constructed.

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1. Introduction

A *design* is a pair (X,B) where X is a finite set of *elements* and B is a collection of (not necessarily distinct) subsets B_i (called *blocks*) of X.

A *balanced incomplete block design*, BIBD(v,b,r,k,λ), is an arrangement of v elements into b blocks such that:

(i) each element appears in exactly r blocks;
(ii) each block contains exactly $k(<v)$ elements; and
(iii) each pair of distinct elements appear together in exactly λ blocks.

As $r(k-1) = \lambda(v-1)$ and $vr = bk$ are well known necessary conditions for the existence of a BIBD(v,b,r, k,λ) we denote this design by BIBD(v,k,λ).

Let v and λ be positive integers and K a set of positive integers. An arrangement of the elements of a set X into blocks is a *pairwise balanced design*, PBD(v;K;λ), if:

(i) X contains exactly v elements;
(ii) if a block contains k elements then k belongs to K;
(iii) each pair of distinct elements appear together in exactly λ blocks.

A pairwise balanced design PBD(v;{k};λ), where K = {k} consists of exactly one integer, is a BIBD(v,k,λ). It is well known that a PBD(v-1;{k,k-1};λ) can be obtained from the BIBD(v,b,r,k,λ).

A *generalized Bhaskar Rao* design, W is defined as follows. Let W be a v x b matrix with entries from $G \cup \{o\}$ where $G = \{h_1 = e, h_2, \ldots, h_g\}$ is a finite group of order g. W is then expressed as a sum $W = h_1A_1 + \ldots + h_gA_g$, where $A_1, \ldots, A_g$ are v x b (0,1) - matrices such that the Hadamard product $A_i * A_j = 0$ for any $i \neq j$. Denote by W^+ the transpose of $h_1^{-1} A_1 + \ldots + h_g^{-1}A_g$ and let $N = A_1 + \ldots + A_g$. In this paper we are concerned with the special case where W, denoted by GBRD(v,b,r,k,λ;G), satisfies

(i) $WW^+ = reI + \frac{\lambda}{g}(h_1 + \ldots + h_g)(J-I)$, and
(ii) $NN^T = (r-\lambda) I + \lambda J$.

It can be seen that the second condition requires that N be the incidence matrix of a BIBD(v,b,r,k,λ) and thus we can use the shorter notation GBRD(v,k,λ;G) for a generalized Bhaskar Rao design. A GBRD(v,k,λ;Z_2) will be sometimes referred to as a BRD(v,k,λ).

A GBRD(v,k,λ;G) with v = b is a symmetric GBRD or *generalized weighing matrix*. If W has no 0 entries the GBRD is also known as a *generalized Hadamard matrix*.

A *group divisible design*, GDD, on v points is a triple (X,S,A) where

(i) X is a set (of points),
(ii) S is a class of non-empty subsets of X (called *groups*) which partition X,
(iii) A is a class of subsets of X (called *blocks*), each containing at least two points,
(iv) no block meets a group in more than one point,
(v) each pair {x,y} of points not contained in a group is contained in precisely λ blocks.

Bhaskar Rao designs with elements $0, \pm 1$ have been studied by a number of authors including Bhaskar Rao (1966, 1970), Seberry (1982), 1984), Singh (1982), Sinha (1978), Street (1981), Street and Rodger (1980) and Vyas(1982). Bhaskar Rao (1966) used these designs to construct partially balanced designs and this was improved by Street and Rodger (1980) and Seberry (1984). Another technique for studying partially balanced designs has involved looking at generalized orthogonal matrices which have elements from elementary abelian groups and the element 0. Matrices with group elements as entries have been studied by Berman (1977, 1978), Butson (1962, 1963), Delsarte and Goethals (1969), Drake (1979), Rajkundlia (1978), Seberry (1979, 1980), Shrikhande (1964) and Street (1979).

Generalized Hadamard matrices has been studied by Street (1979), Seberry (1979, 1980), Dawson (1985), and de Launey (1984, 1986).

Bhaskar Rao designs over elementary abelian groups other than Z_2 have been studied recently by Lam and Seberry (1984) and Seberry (1985) .de Launey, Sarvate and Seberry (1985) considered Bhaskar Rao designs over Z_4 which is an abelian (but not elementary) group. Some Bhaskar Rao designs over the non-abelian groups S_3 and Q_4 have been studied by Gibbons and Mathon (1987).

In this paper we are concerned with generalized Bhaskar Rao designs over the non-abelian groups S_3, D_4, Q_4, D_6 and over the small abelian group $Z_2 \times Z_4$.

We use the following notation for initial blocks of a GBRD. We say $(a_\alpha, b_\beta, \ldots, c_\gamma)$ is an initial block, when the latin letters are developed mod n and the greek subscripts are the elements of the group, which will be placed in the incidence matrix in the position indicated by the latin letter. That is in the (i,b-1+i)th position we place β and so on.

We form the difference table of an initial block $(a_\alpha, b_\beta, \ldots, c_\gamma)$ by placing in the position headed by x_δ and by row y_η the element $(x-y)_{\delta\eta^{-1}}$ where $(x-y)$ is mod n and $\delta\eta^{-1}$ is in the group.

A set of initial blocks will be said to form a *GBR difference set* (if there is one initial block) or *GBR supplementary difference sets* (if more than one) if in the totality of elements

$$(x-y)_{\delta\eta^{-1}} \pmod{n, G}$$

each non-zero element a_g, a(mod n), $g \in G$, occurs $\lambda/|G|$ times.

Examples of the use of these GBR supplementary difference sets are given in Seberry (1982).

2. Construction Theorems

The following theorem which was established in Lam and Seberry (1984) and the associated corollaries will be extensively used in the remaining sections of the paper.

Theorem 2.1

Suppose we have a $GBRD(v,k,\lambda;H)$ and a $GBRD(k,j,\mu;G)$. Then there exists a $GBRD(v,j,\lambda\mu;\ H \times G)$.

Corollary 2.2 (Lam and Seberry (1984))

Suppose we have a BIBD(v,k,λ) and a GBRD$(k,j,\mu;G)$. Then there exists a GBRD$(v,j,\lambda\mu;G)$.

Corollary 2.3 (de Launey and Seberry (1982, 1984))

Suppose there exists a pairwise balanced design PBD$(v;K;\lambda)$ where $K = \{k_1,\ldots,k_b\}$ and a GBRD $(k_i,j,\mu;G)$ for each $k_i \in K$, then there exists a GBRD$(v,j,\lambda\mu;G)$.

3. Existence of GBRD in S_3

The *symmetric group*, S_3, can be generated by means of the defining relations

$$p^3 = a^2 = 1,\ ap = p^2a. \tag{3.1}$$

We firstly establish the necessary conditions for the existence of GBRD over S_3 with block size 3.

For a GBRD$(v,3,\lambda;S_3)$ to exist

$$\lambda \equiv 0 \pmod 6 \tag{3.2}$$

and there must exist a BIBD$(v,3,\lambda)$. We, therefore, have the necessary conditions for the existence of a GBRD$(v,3,\lambda;S_3)$

$$v \geq 3 \tag{3.3}$$

$$\lambda(v-1) \equiv 0 \pmod 2 \tag{3.4}$$

$$\lambda v(v-1) \equiv 0 \pmod 6 \tag{3.5}$$

Gibbons and Mathon (1987) have proved the following

Theorem 3.1

Let W be a GBRD$(v,k,\lambda;G)$ and suppose that G contains a normal subgroup T. Then there exists a GBRD$(v,k,\lambda;H)$, where $H = G/T$ is the factor group of G with respect to T.

It should be noted that this theorem is a generalisation of Proposition 4.1 appearing in Lam and Seberry (1984).

We now derive an extra necessary condition for the existence of a GBRD($v,3,\lambda; S_3$).

Theorem 3.2

A necessary condition for the existence of a GBRD($v,k,\lambda;S_3$) is the existence of a BRD(v,k,λ).

Proof: As Z_2 is isomorphic to the factor group $S_3/\{1,p,p^2\}$ of $\{1,p,p^2\}$ in S_3, we see that a GBRD($v,k,\lambda;S_3$) exists only if a GBRD ($v,k,\lambda;Z_2$) - which is merely a BRD(v,k,λ) - exists.

Corollary 3.3

A GBRD($v,3,\lambda; S_3$) exists only when $\lambda v(v-1) \equiv 0$ (mod 24) (3.6)

Proof: By Theorem 1 of Seberry (1984) the BRD($v,3,\lambda$) can only exist when $\lambda v(v-1) \equiv 0 (\text{mod } 24)$.

Remark 3.4

As an immediate application we see that a GBRD (3,3,6; S_3) does not exist; and hence, for $h \geq 3$, a GBRD($h,h,6,S_3$) does not exist because if it did exist any three, distinct rows could be taken to be a GBRD($3,3,6;S_3$).

Theorem 3.5

A GBRD$(v,3,\lambda;S_3)$ exists if and only if

$$\lambda \equiv 0 (mod\ 6),\ v \geqslant 3 \qquad (3.7)$$

and

$$\lambda v(v-1) \equiv 0 (mod\ 24),\ v \geqslant 3. \qquad (3.8)$$

Proof: It is clear that (3.7) and (3.8) are necessary conditions for the existence of a GBRD$(v,3,\lambda;S_3)$. To prove that (3.7) and (3.8) imply that a GBRD $(v,3,\lambda;S_3)$ exists, we put $\lambda = 6t$ and observe that (3.7) and (3.8) are equivalent to one of the following conditions

(a) $t \equiv 0(\text{mod } 2),\ v \geqslant 3$

(b) $t \equiv 1(\text{mod } 2),\ v \equiv 0$ or $1(\text{mod } 4)$.

Case 1: $t \equiv 1(\text{mod } 2)$. By applying Corollary 2.2 with Hanani's theorem (Hall(1967), Lemma 15.5.1) we only have to establish the existence of GBRD$(v,3,6;S_3)$ for $v \in K_4' = \{4,5,8,9,12\}$ to demonstrate the existence of all GBRD$(u,3,6;S_3)$ for $u \equiv 0$ or $1(\text{mod } 4)$, $u \geqslant 4$. The GBRD $(v,3,6;S_3)$ for $v \in K_4'$ are given in the Appendix.

Now by taking t copies (t odd) of the GBRD $(u,3,6;S_3)$ whose existence has been established we have shown the existence of GBRD$(u,3,6t;S_3)$ for $t \equiv 1(\text{mod } 2)$ and $u \equiv 0$ or $1(\text{mod } 4)$.

Case 2: $t = 2s \equiv 0(\text{mod})$. By Corollary 2.2 and Hanani's theorem (Hall(1967), Lemma 15.4.2) we need to establish the existence of GBRD$(v,3,12;S_3)$ for $v \in \{3,4,5,6,8,11,14\}$ to show that GBRD $(v,3,12;S_3)$ exist for all $v \geqslant 3$. The designs GBRD$(v,3,12;S_3)$ for $v \in \{3,6,14\}$ are given in the Appendix and the designs GBRD$(v,3,12;S_3)$, where $v \in \{4,5,8,11\}$ exist:

v	Construction
4	2 copies of the design with $\lambda = 6$
5	2 copies of the design with $\lambda = 6$
8	2 copies of the design with $\lambda = 6$
11	Use a BIBD(11,5,2) with the GBRD $(5,3,6;S_3)$

Thus we can see that a GBRD$(v,3,12;S_3)$ exists for all $v \geq 3$. Taking s copies of each of these designs we obtain a GBRD$(v,3,6(2s);S_3)$ for all $v \geq 3$.

Thus, the theorem is established.

4. Existence of GBRD in $Q_4, D_4, Z_2 x Z_4$.

The *quaternion group*, Q_4, consists of elements 1,-1,i,-i,j,-j,k,-k subject to the relations

$$i^2 = j^2 = k^2 = -1,$$
$$ij = -ji = k,$$
$$jk = -kj = i,$$
$$ki = -ik = j.$$

The *dihedral group*, D_4, can be generated by means of the defining relations

$$r^4 = a^2 = 1, \; ar = r^3a$$

For the *direct product*, $Z_2 \times Z_4$, of groups Z_2 and Z_4 we represent the elements by 0,1,2,3,-0,-1,-2, and -3.

We establish the necessary conditions for the existence of a GBRD$(v,3,\lambda;G)$ where G is one of the groups Q_4, D_4, or $Z_2 \times Z_4$.

For a GBRD$(v,3,\lambda;G)$ to exist

$$\lambda \equiv 0 \pmod 8 \qquad (4.1)$$

and there must exist a BIBD$(v,3,\lambda)$. We, therefore, have the necessary conditions for the existence of a GBRD$(v,3,\lambda;G)$

$$v \geqslant 3 \qquad (4.2)$$

$$\lambda(v-1) \equiv 0 \pmod 2 \qquad (4.3)$$

$$\lambda v(v-1) \equiv 0 \pmod 6 \qquad (4.4)$$

Theorem 3.1 reveals no further necessary conditions.

We consider $\lambda = 8t$. We see that:

(a) for $t \equiv 1,2,4$ or $5 \pmod 6$ that $v \equiv 0$ or $1 \pmod 3$

and (b) for $t \equiv 0 \pmod 3$ that $v \geqslant 3$ are necessary conditions for the existence of a GBRD $(v,3,8t;G)$.

Theorem 4.1

The necessary conditions:

$$\lambda \equiv 0 \pmod 8$$

$$\lambda v(v-1) \equiv 0 \pmod 6$$

are sufficient for the existence of a GBRD$(v,3,8t;G)$, $v \geqslant 3$, $t \equiv 1,2,4,5 \pmod 6$.

Proof: GBRD$(v,3,8;G)$ for $v \in \{3,4,6\}$ are given in the Appendix. Hence by Corollary 2.2 and Hanani's theorem (Hall(1967), Lemma 15.4 GBRD$(v,3,8;G)$ exist if $v \equiv 0$ or $1 \pmod 3$).

Theorem 4.2

The necessary conditions:

$t \equiv 0 \pmod 3$, $v \geqslant 3$

are sufficient for the existence of a GBRD(v,3,8t;G).

Proof: By Hanani's theorem (Hall(1967), Lemma 15.4.2) and Corollary 2.2 we have to establish the existence of GBRD(v,3,24;G) for v ε {3,4,5,6, 8,11,14}, and these designs exist for the following reasons:

v	Construction
3	3 copies of the design with λ = 8
4	3 copies of the design with λ = 8
5	Use a BIBD(5,3,3) with the GBRD (3,3,8;G)
6	3 copies of the design with λ = 8
7	Use a BIBD(7,3,3) with the GBRD (3,3,8;G)
8	Use a BIBD(8,4,3) with the GBRD (4,3,8;G)
11	Use a BIBD(11,3,3) with the GBRD (3,3,8;G)
14	Use the SBIBD(15,7,3) to obtain a PBD(14;{7,6};3) Now the GBRD(v,3,8;G), v ε {6,7} are used.

Multiple copies of the GBRD(v,3,24;G) v $\geqslant$ 3 give the complete result.

Thus we have established the following theorem:

Theorem 4.3

The necessary conditions:

$$\lambda \equiv 0 \pmod 8$$
$$\lambda v(v-1) \equiv 0 \pmod 6$$

are sufficient for the existence of a GBRD(v,3,λ;G).

5. Existence of GBRD in D_6

The *dihedral group*, D_6, is given by the defining relations

$$r^6 = a^2 = 1, \; ar = r^5 a$$

For a GBRD(v,3,λ;D_6) to exist

$$\lambda \equiv 0 \pmod{12} \qquad (5.1)$$

and there must exist a BIBD(v,3,λ). We therefore, have the necessary conditions

$$v \geqslant 3 \qquad (5.2)$$
$$\lambda(v-1) \equiv 0 \pmod 2 \qquad (5.3)$$
$$\lambda v(v-1) \equiv 0 \pmod 6 \qquad (5.4)$$

Theorem 3.1 reveals no further necessary conditions. It is clear that the conditions (5.3) and (5.4) are redundant so that a GBRD(v,3,λ;D_6) exists only if $v \geqslant 3$ and $\lambda \equiv 0 \pmod{12}$.

Lemma 5.1

A GBRD(5,3,12;D_6) exists.

Proof: The development of the initial blocks $(0_1, 1_1, 3_p, 4_a)$, $(0_1, 1_{p^2a}, 3_1, 4_{p^2})$ (mod 5,S_3) is a GBRD(5,4,6;S_3). Applying Theorem 2.1 with this design and a GBRD(4,3,2;Z_2)(which is known to exist) we can construct a GBRD $(5,3,12;S_3 \times Z_2)$. As $S_3 \times Z_2$ is isomorphic to D_6 we have shown that a GBRD$(5,3,12;D_6)$ exists.

Theorem 5.2

The necessary conditions:

$$\lambda \equiv 0 \pmod{12}$$
$$v \geq 3$$

are sufficient for the existence of a GBRD$(v,3,\lambda;D_6)$.

Proof: By Lemma 5.1 GBRD$(5,3,12;D_6)$ exists and the designs GBRD$(v,3,12;D_6)$, $v \in \{3,4,6,8,11,14\}$ are exhibited in the Appendix. Hence, by Hanani's theorem (Hall(1967), Lemma 15.4.2), a GBRD$(v,3,12;D_6)$ exists for all $v \geq 3$; and by taking multiples of these designs we can construct all GBRD$(v,3,\lambda;D_6)$ for $\lambda \equiv 0 \pmod{12}$, $v \geq 3$.

6. Application

Proceeding as in Street and Rodger(1980), Lam and Seberry (1984), Seberry (1982, 1984, 1985) and Gibbons and Mathon (1987), we now have the results:

Theorem 6.1

Whenever $\lambda \equiv 0 \pmod 6$ and $\lambda v(v-1) \equiv 0 \pmod{24}$ there exists a regular group divisible design with parameters

$$\left(6v,\ \lambda v(v-1),\ \frac{\lambda(v-1)}{2},\ 3,\ \lambda_1 = 0,\ \lambda_2 = \frac{\lambda}{6},\ m = v,\ n = 6\right).$$

Theorem 6.2

Whenever $\lambda \equiv 0 \pmod 8$ and $\lambda v(v-1) \equiv 0 \pmod 6$ there exists a regular group divisible design with parameters

$$\left(8v,\ \frac{4\lambda v(v-1)}{3}\ \frac{\lambda(v-1)}{2},\ 3,\ \lambda_1 = 0,\ \lambda_2 = \frac{\lambda}{8},\ m = v,\ n = 8\right)$$

Theorem 6.3

Whenever $\lambda \equiv 0 (mod\ 12)$ there exists a regular group divisible design with parameters

$$\left(12v,\ 2\lambda v(v-1),\ \frac{\lambda(v-1)}{2},\ 3,\ \lambda_1 = 0,\ \lambda_2 = \frac{\lambda}{12},\ m = v,\ n = 12\right)$$

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Appendix

The designs used in the proofs of our theorems are exhibited here. In many cases only the initial blocks are shown:

1. The Group S_3

GBRD $(4,3,6;S_3)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & p & p^2 & a & pa & p^2a & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & p^2 & p & 0 & 0 & 0 & a & pa & p^2a & a & pa & p^2a \\ 0 & 0 & 0 & 1 & p & p^2 & a & p^2a & pa & 1 & p & p^2 \end{bmatrix}$$

It should be noted that this design was constructed independently by Mathon and Gibbons (1987).

GBRD $(5,3,6;S_3)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & p & p^2 & a & pa & p^2a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & p^2 & 0 & 0 & 0 & 0 & 0 & 0 & p & a & pa & p^2a & 1 & 1 & 0 & 0 & a & pa & p^2 & p^2a \\ 0 & 0 & 0 & 0 & 1 & p & p^2 & a & 0 & 0 & p^2a & pa & 1 & a & p & p^2 & 0 & 0 & p^2a & 1 \\ 0 & 0 & 1 & p & 0 & 0 & p^2a & pa & p^2 & a & 0 & 0 & p^2 & a & p^2a & a & p^2 & 1 & 0 & 0 \end{bmatrix}$$

GBRD $(8,3,6;S_3)$

$(\infty_1, 0_{p^2}, 1_p)$, $(\infty_1, 3_{p^2a}, 0_{pa})$, $(\infty_1, 1_1, 3_a)$, $(0_1, 1_p, 3_a)$
$(0_1, 1_a, 3_{p^2})$, $(0_1, 1_{p^2a}, 3_{pa})$, $(0_1, 1_1, 3_1)$, $(0_1, 1_{pa}, 3_{p^2a})$
(mod 7, S_3).

GBRD $(9,3,6;S_3)$

$$
\left[\begin{array}{ccccc|ccccc|ccccc|ccccc|c}
e & e & e & e & e & & & & & & & & & & & & & & & & \\
& & & & & e & e & e & e & e & & & & & & & & & & & D \\
& & & & & & & & & & e & e & e & e & e & & & & & & \\
& & & & & & & & & & & & & & & e & e & e & e & e & \\
\hline
x & s & & & & x & s & & & & x & s & & & & x & s & & & & \\
y & & s & & & & y & s & & & & & x & s & & & & x & s & & \\
& y & & s & & & & y & s & & y & & & & s & & & & t & x & 0 \\
& & t & & x & & & & t & x & & t & & y & & & y & & & t & \\
& & & y & t & t & & & & y & & & y & & t & y & & t & & &
\end{array}\right]
$$

where D is a GBRD $(4,3,6;S_3)$; and

$e = (1,1,1)$, $\quad x = (1,p,p^2)$ $\quad s = (a,p^2a,pa)$

$y = (1,p^2,p)$, $\quad t = (p^2a,a,pa)$

so that

$x.y^{-1} = (1,p^2,p)$, $\quad xt^{-1} = (p^2a,pa,a)$

$s.t^{-1} = (p,p^2,1)$, $\quad sy^{-1} = (a,pa,p^2a)$.

This design contains a subdesign on 4 points.

GBRD $(12,3,6;S_3)$

$(\infty_1,0_p,2_a),(\infty_1,0_{p^2},5_{pa}),(\infty_1,0_1,5_{p^2a}),(0_1,3_1,5_p),$
$(0_1,1_1,5_{pa}),(0_1,3_p,8_p),(0_1,4_1,5_{p^2}),(0_1,4_p,7_a),$
$(0_1,2_{p^2a},9_1),(0_1,3_a,4_{p^2}),(0_1,1_p,3_{p^2a}),(0_1,1_a,10_{pa})$
$(\text{mod } 11,S_3)$

GBRD $(3,3,12;S_3)$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & p & p^2 & a & pa & p^2a & 1 & p & p^2 & a & pa & p^2a \\
1 & p^2 & p & 1 & p & p^2 & a & pa & p^2a & a & p^2a & pa
\end{bmatrix}
$$

GBRD $(6,3,12;S_3)$

$(0_1,3_1,2_a),(0_1,1_p,4_p),(0_1,4_1,1_{pa}),(0_1,2_p,3_p),(0_1,1_{p^2a},4_p)$
$(0_1,2_{pa},3_{p^2a}),(\infty_1,0_p,3_{pa}),(\infty_1,0_1,1_a),(\infty_1,0_{p^2},4_{p^2a}),$
$(\infty_1,0_{p^2},1_a),(\infty_1,0_{pa},2_{p^2a}),\ (\infty_1,\ 0_1,\ 3_p),\ (\mathrm{mod}\ 5,\ S_3).$

GBRD $(14,3,12;S_3)$

$(0_1,i_a,(13-i)_{p^2a})\ ,\quad i = 3,\ldots,12$
$(0_1,j_{p^2},(13-j)_{p^2}),\quad j = 1,\ldots,6$
$(0_1,j_{pa},(13-j)_{pa}),\quad j = 1,\ldots,6$
$(\infty_1,0_p,1_{pa}),(\infty_1,0_{p^2},2_a),(\infty_1,0_1,12_a),(\infty_1,0_{p^2a},11_p)$
$(\infty_1,0_{p^2a},11_{pa}),(\infty_1,0_1,9_{p^2})\quad (\mathrm{mod}\ 13,S_3)$

2. The Group $Z_2 \times Z_4$

We use 0,1,2,3,-0,-1,-2,-3 for the group elements of $Z_2 \times Z_4$ and * in the other positions in the design. The following designs exist:

GBRD $(3,3,8;Z_2 \times Z_4)$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & -0 & -1 & -2 & -3 \\ 0 & 3 & -3 & -2 & -1 & -0 & 2 & 1 \end{bmatrix}$$

GBRD $(4,3,8;Z_2 \times Z_4)$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 1 & * & -0 & -2 & * & -1 & 2 & * & 3 & -3 & * & 0 & 3 & -2 & 0 \\ * & 0 & 1 & * & -0 & -2 & * & -1 & 2 & * & 3 & -3 & -2 & 0 & 3 & 0 \\ 1 & * & 0 & -2 & * & -0 & 2 & * & -1 & -3 & * & 3 & 3 & -2 & 0 & 0 \end{bmatrix}$$

GBRD $(6,3,8;Z_2 \times Z_4)$

$(\infty_0,0_0,1_1),(\infty_0,0_{-1},1_{-3}),(\infty_0,0_{-0},2_{-2}),(\infty_0,0_3,2_2),$

$(0_0,2_0,3_{-0}),(0_0,2_{-2},3_{-1}),(0_0,1_{-1},4_0),(0_0,1_{-3},4_{-2})$

$(\mathrm{mod}\ 5,\ Z_2 \times Z_4)$

3. The Group D_4

GBRD $(3,3,8;\ D_4)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & r & r^2 & r^3 & a & ra & r^3a & r^2a \\ 1 & r^2 & a & r^2a & r^3 & r^3a & r & ra \end{bmatrix}$$

GBRD $(4,3,8;D_4)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & r & r^2 & r^3 & a & ra & r^3a & r^2a & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & r^2 & a & r^2a & 0 & 0 & 0 & 0 & r^3 & r^3a & r & ra & r^3a & r^2 & a & r^3 \\ 0 & 0 & 0 & 0 & a & r^2a & ra & r^2 & 1 & r^3 & r & r^3a & r^2a & r^3a & r^3 & ra \end{bmatrix}$$

GBRD $(6,3,8;\ D_4)$

$(\infty_1, 0_1, 1_{r^2}), (\infty_1, 0_r, 1_a), (\infty_1, 0_{r^3}, 2_{ra}), (\infty_1, 0_{r^2a}, 2_{r^3a}),$

$(0_1, 2_{r^3a}, 3_r), (0_1, 2_{r^2}, 3_{ra}), (0_1, 1_{r^3}, 4_{r^3}),$

$(0_1, 1_{r^2a}, 4_1)$ (mod $5, D_4$)

4. The Group Q_4

GBRD $(3,3,8;Q_4)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & i & -k & j & -1 & -j & k & -i \\ 1 & j & k & -1 & i & -i & -j & -k \end{bmatrix}$$

GBRD $(4,3,8;Q_4)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & j & k & -k & -i & -1 & -j & i & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & -i & -j & i & 0 & 0 & 0 & 0 & j & -k & -1 & k & j & -k & -1 & -i \\ 0 & 0 & 0 & 0 & i & -i & -1 & -k & -j & 1 & j & k & -k & -i & k & 1 \end{bmatrix}$$

GBRD $(6,3,8;Q_4)$

$(\infty_1,1_i,4_j),(\infty_1,1_{-k},4_{-i}),(\infty_1,2_i,3_k),(\infty_1,2_{-1},3_{-j})$

$(0_1,1_{-1},4_j),(0_1,1_1,4_{-i}),(0_1,2_i,3_{-k}),(0_1,2_{-i},3_{-1})$

$(\mathrm{mod}\ 5,Q_4)$.

5. The Group D_6

GBRD $(3,3,12;D_6)$

$(0_1,1_1,2_r),(0_1,1_a,2_{r^2}),(0_1,1_{r^3},2_{ra}),(0_1,1_{r^2},2_{r^3a})$

$(\mathrm{mod}\ 3,D_6)$

GBRD $(4,3,12;D_6)$

$(\infty_1,1_1,2_r),(\infty_1,1_{r^2},2_{r^4a}),(\infty_1,1_{r^3},2_{r^5a}),(\infty_1,1_{r^4},2_{ra})$

$(\infty_1,1_{r^2a},2_{r^5}),(\infty_1,1_{r^3a},2_a)$

$(0_1,1_1,2_{r^2}),(0_1,1_{r^5},2_{r^4a})$ $(\mathrm{mod}\ 3,\ D_6)$

GBRD $(6,3,12;D_6)$

$(\infty_1,0_1,3_r),(\infty_1,0_{r^3},3_{ra}),(\infty_1,0_{r^5},3_{r^2a})$

$(\infty_1,0_a,4_{r^2}),(\infty_1,0_{r^4},4_{r^3a}),(\infty_1,0_{r^4a},4_{r^5a})$

$(0_1,2_1,3_{r^2}),(0_1,2_{r^2a},3_{r^3a}),(0_1,2_{r^3},3_a)$

$(0_1,1_{r^3},4_{r^2}),(0_1,1_a,4_{r^4a}),(0_1,1_1,4_{r^5a})$ $(\mathrm{mod}\ 5,D_6)$

(GBRD $(8,3,12;D_6)$

$(\infty_1,0_1,5_r),(\infty_1,0_{r^2},3_{r^4a}),(\infty_1,0_{r^3},6_{r^5a}),(\infty_1,0_{r^4},5_{ra}),$

$(\infty_1,0_{r^2a},1_{r^5}),(\infty_1,0_{r^3a},2_a)$

$(0_1,3_1,4_{r^2}),(0_1,4_{r^3},3_{r^2}),(0_1,5_{ra},2_r),(0_1,6_{r^5a},1_{r^3a})$
$(0_1,1_1,6_a),(0_1,2_1,5_{r^4a}),(0_1,3_r,4_{r^3a})$
$(0_1,1_{r^3},6_r),(0_1,2_{r^3a},5_{r^2a}),(0_1,3_{ra},4_{r^5a})(\mathrm{mod}\ 7,D_6)$

GDD (14,3,12,D_6)

$(\infty_1,0_{r^2a},1_{r^5}),(\infty_1,0_{r^4},5_{ra}),(\infty_1,0_{r^3},7_a),(\infty_1,0_{r^3a},6_{r^5a})$

$(\infty_1,0_1,8_r),\ (\infty_1,0_{r^2},10_{r^4a})$

$(0_1,1_1,12_r),(0_1,1_r,12_{r^5a}),(0_1,1_{r^2},12_{r^3a}),(0_1,1_{r^3},12_{r^4a})$

$(0_1,2_1,11_{r^2}),(0_1,2_r,11_{r^4a}),(0_1,2_{r^3},11_{r^3a}),(0_1,2_{r^2},11_{r^2a}),$

$(0_1,3_1,10_{r^5}),(0_1,3_{r^2},10_{r^5a}),(0_1,3_{r^3},10_{r^3a}),(0_1,3_{r^4},10_{ra}),$

$(0_1,4_1,9_{r^3}),(0_1,4_r,9_{r^3a}),(0_1,4_{r^2},9_{ra}),(0_1,4_{r^5},9_{r^2a}),$

$(0_1,5_1,8_{r^5}),(0_1,5_{r^2},8_a)(0_1,5_{r^2a},8_{r^2}),$

$(0_1,6_{r^3},7_r),(0_1,6_{r^4a},7_{r^2}),(0_1,6_1,7_{r^2a})$ (mod 13,D_6)