

# VANSTONE'S CONSTRUCTION APPLIED TO BHASKAR RAO DESIGNS

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## ABSTRACT

We show how Vanstone's construction, given in his paper "A note on a construction for BIBD's", *Utilitas Mathematica*, 7(1975), 321-322, can be applied to symmetric GBRD( $v, k, \lambda; |G|$ ),  $|G|$  odd, can be used to obtain GBRD( $v, \begin{bmatrix} v \\ 2 \end{bmatrix}, \begin{bmatrix} k \\ 2 \end{bmatrix}, \lambda, \begin{bmatrix} \lambda \\ 2 \end{bmatrix}; G$ ) and hence many families of BIBD.

## 1. INTRODUCTION

Definitions of SBIBD and BIBD are standard.

Let  $A = [a_{ij}]$  be a matrix of order  $n$  with  $a_{ij} \in \{0, 1, -1\}$ .  $A$  is called a *weighing matrix* of weight  $p$  and order  $n$ , if  $AA^T = A^T A = pI_n$ , where  $I_n$  denotes the identity matrix of order  $n$ . Such a matrix is denoted by  $W(n, p)$ . If squaring all its entries gives an incidence matrix of a SBIBD then  $W$  is called a balanced weighing matrix.

An *Hadamard matrix*,  $A = [a_{ij}]$ , is a  $W(n, n)$ , that is, it is a square matrix of order  $n$  with entries  $a_{ij} \in \{1, -1\}$  which satisfies

$$AA^T = A^T A = nI_n.$$

A *generalized Hadamard matrix*  $GH(gh, G) = (g_{ij}) = H$  over the group  $G$  of order  $g$  is a  $gh \times gh$  matrix such that

(i)  $g_{ij} \in G$  for all  $1 \leq i, j \leq gh$ , and

(ii)  $\sum_{k=1}^{gh} g_{ik} g_{jk}^{-1} = \sum_{a \in G} ha$  whenever  $i \neq j$  where the summation is in the group ring

$R(G)$ . We also write this as

$$HH^* = hG.$$

Suppose we have a matrix  $W$  with elements from an elementary abelian group  $G = [h_1, h_2, \dots, h_g]$ , where  $W = h_1 A_1 + h_2 A_2 + \dots + h_g A_g$ ; here  $A_1, \dots, A_g$  are

$v \times b$  (0,1) matrices, and the Hadamard product  $A_i * A_j$  ( $i \neq j$ ) is zero. Suppose  $(a_{i1}, \dots, a_{ib})$  and  $(b_{j1}, \dots, b_{jb})$  are the  $i$ th and  $j$ th rows of  $W$ ; then we define  $WW^*$  by

$$(WW^*)_{ij} = (a_{i1}, \dots, a_{ib}) \cdot (b_{j1}^{-1}, \dots, b_{jb}^{-1})$$

with  $\cdot$  designating the scalar product. Then  $W$  is a *generalized Bhaskar Rao design* or *GBRD* if

$$(i) \quad WW^* = rI + \sum_{i=1}^m (c_i G) B_i$$

$$(ii) \quad N = A_1 + \dots + A_g \text{ satisfies } NN^T = rI + \sum_{i=1}^m \lambda_i B_i,$$

that is,  $N$  is the incidence matrix of a *PBIBD* ( $m$ ), and  $(c_i G)$  gives the number of times a complete copy of the group  $G$  occurs.

Such a matrix will be denoted by  $GBRD_G(v, b, r, k; \lambda_1, \dots, \lambda_m; c_1, \dots, c_m)$ . In this paper we shall only be concerned with  $m = 1, c = \lambda/g$ , and  $B_1 = J - I$ . In this case  $N$  is the incidence matrix of a *PBIBD* (1), that is a *BIBD*. Hence, the equations become:

$$(i) \quad WW^* = rI + \lambda G/g (J - I)$$

$$(ii) \quad NN^T = (r - \lambda)I + \lambda J.$$

Thus  $W$  is a  $GBRD_G(v, b, r, k, \lambda)$ . Since  $\lambda(v-1) = r(k-1)$  and  $bk = vr$ , we sometimes use the notation  $GBRD(v, k, \lambda; G)$ .

## 2. THE CONSTRUCTION

In his 1975 paper, Vanstone gave a powerful method for constructing BIBD from SBIBDs. We show his method applies to symmetric GBRD over groups which have no elements of order 2.

**THEOREM 1.** *Suppose there is a symmetric  $GBRD(v, k, \lambda; G)$ ,  $|G|$  odd, then there is a  $GBRD(v, \begin{bmatrix} v \\ 2 \end{bmatrix}, \begin{bmatrix} k \\ 2 \end{bmatrix}, \lambda, \begin{bmatrix} \lambda \\ 2 \end{bmatrix}; G$ .*

*Proof:* We modify the construction Vanstone used to show that an SBIBD( $v, k, \lambda$ ) yields a BIBD( $v, \begin{bmatrix} v \\ 2 \end{bmatrix}, \begin{bmatrix} k \\ 2 \end{bmatrix}, \lambda, \begin{bmatrix} \lambda \\ 2 \end{bmatrix}$ ).

Let  $A = (a_{ij})$  be the incidence matrix of the  $GBRD(v, k, \lambda; G)$ . Label the columns of a  $v \times \begin{bmatrix} v \\ 2 \end{bmatrix}$  matrix  $B = (b_{ij})$ , with the  $n = \begin{bmatrix} v \\ 2 \end{bmatrix}$  pairs from the set  $\{1, \dots, v\}$ .

Consider the column labelled  $xy, (b_{1k}, \dots, b_{vk})^T$ , choose

$$b_{ik} = a_{ix}a_{iy}, i = 1, \dots, v.$$

Clearly, every element of  $B$  is zero or a group element, as that was true of  $A$ .

To establish the inner product property, we consider the inner product of two distinct rows

$$\sum_{k=1}^n b_{ik}b_{jk}^{-1} = \sum_{1 \leq x < y \leq n} a_{ix}a_{iy}a_{jy}^{-1}a_{ix}^{-1}, i \neq j.$$

We first note that, for any group  $G$  of order  $g$  with elements  $g_1, g_2, \dots, g_g$

$$G^2 = (g_1 + g_2 + \dots + g_g)^2 = gG.$$

With  $(G + \dots + G)$  denoting  $t$  copies of  $G$ ,

$$(G + G + \dots + G)^2 = tG^2 + 2 \binom{t}{2} G^2 = t^2 gG.$$

Since  $g$  is odd and  $n = v = tg$ , if  $g_1, \dots, g_v$  are the elements of a row of the GBRD,  $g_1^2, \dots, g_v^2 = tG$ .

Hence, noting

$$(\sum x_i)^2 = \sum x_i^2 + 2 \sum_{i \neq j} x_i x_j,$$

$$\begin{aligned} \sum_{1 \leq x < y \leq n} a_{ix}a_{iy}a_{jx}^{-1}a_{jy}^{-1} &= \frac{1}{2} \left[ \sum_{k=1}^n a_{ik}a_{jk}^{-1} \right]^2 - \frac{1}{2} \sum_{k=1}^n (a_{ik}a_{jk}^{-1})^2 \\ &= \frac{1}{2}(G + G + \dots + G)^2 - \frac{1}{2}tG \quad (t \text{ copies}) \\ &= \frac{1}{2}(t^2 g - t)G. \end{aligned}$$

Now, we know from Vanstone's result that a BIBD( $v, k, \lambda$ ) gives a BIBD( $v, \binom{v}{2}, \binom{k}{2}, \lambda, \binom{\lambda}{2}$ ). Thus, we wish to show a GBRD( $v, k, \lambda; G$ ) gives a GBRD( $v, \binom{v}{2}, \binom{k}{2}, \lambda, \binom{\lambda}{2}; G$ ). Certainly, the underlying BIBD has these parameters. The GBRD( $v, k, \lambda; G$ ) has  $t = \lambda / g$  copies of the group as the inner product of each pair of rows and the constructed GBRD needs to have  $\binom{\lambda}{2} / g$  copies of the group as the inner product of each pair of rows. But

$$\binom{\lambda}{2} / g = \frac{1}{2}\lambda(\lambda-1) / g = \frac{1}{2}t(tg-1)$$

as required. □

**Example 1.** Let the group of order 3,  $Z_3$ , have generator  $\omega$ . Represent  $\omega$  by 1,  $\omega^2$  by 2 and  $\omega^3$  by 0. Then, the GH(6,  $Z_3$ ) or GBRD(6, 6, 6;  $Z_3$ ) is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix},$$

yielding

12	13	14	15	16	23	24	25	26	34	35	36	45	46	56
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	2	1	1	2	2	1	0	0	2	1	0	0
1	0	1	2	2	1	2	0	0	1	2	2	0	0	1
2	1	0	1	2	0	2	0	1	1	2	0	1	2	0
2	2	1	0	1	1	0	2	0	0	2	0	1	2	1
1	2	2	1	0	0	0	2	1	1	0	2	0	2	1

a GBRD(6,15,15,6,15; $Z_3$ ).

**Example 2.** Proceed as in Example 1, but represent the zero element by \*. Then the GBRD(5,4,3; $Z_3$ )

*	0	0	0	0
0	*	0	1	2
0	0	*	2	1
0	1	2	*	0
0	2	1	0	*

yields the GBRD(5,10,6,3,3; $Z_3$ ):

12	13	14	15	23	24	25	34	35	45
*	*	*	*	0	0	0	0	0	0
*	0	1	2	*	*	*	1	2	0
0	*	2	1	*	2	1	*	*	0
1	2	*	0	0	*	1	*	2	*
2	1	0	*	0	2	*	1	*	*

This method is so powerful when applied to generalized Hadamard matrices that we give it as a theorem in its own right.

### 3. USING GENERALIZED HADAMARD MATRICES IN THE CONSTRUCTION TO FORM BIBDS

**THEOREM 2.** Suppose there is a GH( $tg;G$ ),  $|G| = g$  odd. Then there is a GBRD( $tg, \begin{bmatrix} tg \\ 2 \end{bmatrix}, \begin{bmatrix} tg \\ 2 \end{bmatrix}, tg, \begin{bmatrix} tg \\ 2 \end{bmatrix}; G$ ). This can be used to form a

$$\text{GDD}(g(tg+1), g \begin{bmatrix} tg \\ 2 \end{bmatrix}, \begin{bmatrix} tg \\ 2 \end{bmatrix}, tg+1, \lambda_1 = 0, \lambda_2 = \frac{1}{2}t(tg-1), m = g, n = tg+1).$$

**COMMENT.** The following construction is valid for any  $\text{GH}(2 | G | G)$  but these are presently only known for prime power orders  $|G|$ . The BIBD's constructed would be multiples of biplanes  $\text{SBIBD}(2p^2 + p + 1, 2p + 1, 2)$  but these are not generally known as yet.

**THEOREM 3.** Let  $p$  be any prime power. Then there exists a  $\text{BIBD}(2p^2 + p + 1, p(2p^2 + p + 1), p(2p + 1), 2p + 1, 2p)$ .

*Proof.* We note a  $\text{GH}(2p, \text{EA}(p))$  exists for every prime power (Jungnickle (1979), D.J.Street (1979)). Use Theorem 2 to form a  $\text{GBRD}(2p, p(2p-1), p(2p-1), 2p, p(2p-1); \text{EA}(p))$ . We replace each element of the GBRD by its  $p \times p$  permutation matrix representation to obtain a  $(0,1)$  matrix  $B$ . Let  $e$  be the  $1 \times p(2p-1)$  matrix of ones. Then

$$A = \begin{bmatrix} I_p \times e \\ B \end{bmatrix}$$

is a  $\text{GDD}(2p^2 + p, p^2(2p - 1), p(2p - 1), 2p + 1, \lambda_1 = 0, \lambda_2 = (2p - 1))$ .

Now a  $\text{BIBD}(2p+1, p(2p+1), 2p, 2, 1)$  exists. Let  $C$  be the matrix obtained from this BIBD by replacing each 1 and 0 in its incidence matrix by the  $p \times 1$  matrices of ones and zeros respectively. Then the matrix

$$[C:A]$$

has  $2p^2 + p$  rows,  $2p^3 + p^2 + p$  columns,  $2p^2 + p$  ones per row,  $2p$  or  $2p + 1$  ones per column and inner product  $2p$ . So if we let  $f$  be a  $1 \times p(2p + 1)$  matrix of ones

$$\begin{bmatrix} f & 0 \\ C & A \end{bmatrix}$$

is a  $\text{BIBD}(2p^2 + p + 1, p(2p^2 + p + 1), p(2p + 1), 2p + 1, 2p)$ . □

**COROLLARY 4.** Let  $p$  be any prime power and  $q$  any integer. Then there exists a  $\text{PBIBD}(2p^2, p(p+q)(2p-1), (p+q)(2p-1), 2p, \lambda_1 = q(2p-1), \lambda_2 = 2p-1+q)$ .

*Proof:* As in the proof of Theorem 3, we use the  $\text{GH}(2p, \text{EA}(p))$  to first form a  $\text{GBRD}(2p, p(2p-1), p(2p-1), 2p, p(2p-1); \text{EA}(p))$ .

This then yields a

$$\text{GDD}(2p^2, p^2(2p - 1), p(2p - 1), 2p, \lambda_1 = 0, \lambda_2 = 2p-1),$$

A. Form  $C$  as before from a

$$\text{BIBD}(2p, qp(2p-1), q(2p-1), 2, q).$$

Then  $[C:A]$  is a

$$\text{PBIBD}(2p^2, p(p+q)(2p-1), (p+q)(2p-1), 2p, \lambda_1 = q(2p-1), \lambda_2 = 2p-1+q). \quad \square$$

**Example 3.** A GH(6,EA(3)) exists so there is a GBRD(6,10,10,6,10;EA(3)). This can be used with a BIBD(7,21,6,2,1) to form a BIBD(22,66,21,7,6).

**Example 4.** A GH(18,EA(9)) exists, so there is a GBRD(18, 153, 153, 18, 153; EA(9)). This is used with a BIBD(19, 171, 18, 2, 1) to form a BIBD(172, 9 · 172, 171, 19, 18).

All the following constructions can be obtained by a similar, slightly modified, technique.

**THEOREM 5.** Suppose there exists a GH( $tg, G$ ),  $g = |G|$  odd. Further suppose that there exists a BIBD( $tg + 1, s(tg + 1), ts, t, \lambda$ ). Then there exists a BIBD( $tg^2 + g + 1, \alpha s(tg^2 + g + 1), \alpha s(tg + 1), tg + 1, \alpha st$ ) where  $s = \lambda g / (t - 1)$  is an integer and  $2\alpha\lambda(tg - t + 1) = \beta t(tg - 1)(t - 1)$  for some  $\alpha$  and  $\beta$ . In particular, if  $\alpha = \binom{tg}{2}$  and  $\beta = tg - t + 1$ , there is a

$$\text{BIBD}(tg^2 + g + 1, s(tg^2 + g + 1) \binom{tg}{2}, s(tg + 1) \binom{tg}{2}, tg + 1, st \binom{tg}{2}).$$

*Proof:* From theorem 1, there exists a GBRD( $tg, \binom{tg}{2}, \binom{tg}{2}, tg, \binom{tg}{2}; G$ ). We replace each element of  $G$  by its  $p \times p$  permutation matrix to form a (0,1) matrix  $E$ . Further, let  $e$  be the  $1 \times \binom{tg}{2}$  matrix of ones. Then,

$$B = \begin{bmatrix} I_g \times e \\ E \end{bmatrix}$$

is a GDD( $g(tg + 1), g \binom{tg}{2}, \binom{tg}{2}, tg + 1, \lambda_1 = 0, \lambda_2 = \frac{1}{2}t(tg - 1), m = g, n = tg + 1$ ).

We now replace each 0 and 1 of the

$$\text{BIBD}(tg + 1, \lambda g(tg + 1) / (t - 1), \lambda tg / (t - 1), t, \lambda)$$

by the  $g \times 1$  matrix of zeros and ones respectively to form a GDD( $g(tg + 1), \lambda g(tg + 1) / (t - 1), \lambda tg / (t - 1), tg, \lambda_1 = \lambda tg / (t - 1), \lambda_2 = \lambda, m = g, n = tg + 1$ ),  $A$ .

We now form the following (0,1) matrix:

$$C = \begin{bmatrix} 11 \cdots 11 & | & 00 \cdots 00 \\ \alpha \text{copies } A & | & \beta \text{copies } B \end{bmatrix}$$

The first row of  $C$  has  $\alpha\lambda g(tg + 1) / (t - 1)$  ones and has intersection  $\alpha\lambda tg / (t - 1)$  with the other rows of  $C$ .

Every other row of  $C$  has  $\alpha\lambda tg / (t - 1) + \beta\lambda tg(tg - 1)$  ones. So we require

$$\alpha\lambda g(tg + 1) / (t - 1) = \alpha\lambda tg / (t - 1) + \beta\lambda tg(tg - 1)$$

or

$$\alpha\lambda = \beta\lambda t(tg - 1)(t - 1) / (tg - t + 1) \quad (1)$$

The intersection numbers for the rows are required to be equal, so we need  

$$\alpha\lambda t g / (t - 1) = \alpha\lambda t g / (t - 1) + \beta \cdot 0 = \alpha\lambda + \beta/2t(tg - 1)$$
or, as in (1)

$$\alpha\lambda = \beta/2t(tg - 1)(t - 1) / (tg - t + 1).$$

Thus C is a  
 $\text{BIBD}(tg^2 + g + 1, \alpha\lambda g(tg^2 + g + 1)/(t - 1), \alpha\lambda g(tg + 1)/(t - 1), tg + 1, \alpha\lambda t g / (t - 1))$ .  
Where  $\lambda g / (t - 1) = s$  an integer, and a possible solution for  $\alpha$  and  $\beta$  is  
 $\alpha = \binom{tg}{2}, \beta = s(tg - t + 1)$ . That is, C is a

$$\text{BIBD}(tg^2 + g + 1, s(tg^2 + g + 1) \binom{tg}{2}, s(tg + 1) \binom{tg}{2}, tg + 1, st \binom{tg}{2}). \quad \square$$

**COROLLARY 6.** Let  $g$  and  $g-1$  be prime powers,  $g$  odd. If there exists a  $\text{BIBD}(g^2 - g + 1, g(g^2 - g + 1), g(g - 1), g - 1, g - 2)$  then there exists a  $\text{BIBD}(g^3 - g^2 + g + 1, \alpha g(g^3 - g^2 + g + 1), \alpha g(g^2 - g + 1), g^2 - g + 1, \alpha g(g - 1))$ , where  $2\alpha(g^2 - 2g + 2) = \beta(g - 1)(g^2 - g - 1)$  has an integer solution.

*Proof:* By a Theorem of Rajkundlia (1978) and Seberry (1981), a  $\text{GH}(g(g-1); \text{EA}(g))$  always exists in these cases.  $\square$

**Remark.** The  $\text{BIBD}$  obtained would be a multiple of an  $\text{SBIBD}(g^3 - g^2 + g + 1, g^2 - g + 1, g - 1)$  which theoretically, can never exist, as  $g^3 - g^2 + g + 1$  is even and  $k - \lambda = g^2 - 2g + 2$  is not a square.

**Example 5.** Let  $g = 5$ . There exists a  $\text{BIBD}(21, 105, 20, 4, 3)$ . Hence, there exists a  $\text{BIBD}(106, 38 \cdot 5 \cdot 106, 38 \cdot 5 \cdot 21, 21, 38 \cdot 5 \cdot 4)$ ,  $\alpha = 38$ . This is a multiple of the  $\text{SBIBD}(106, 21, 4)$  which is non-existent.

**COROLLARY 7.** Let  $g$  be an odd prime power. Let  $\alpha = 2(4g - 1)$ . Then there is a  $\text{BIBD}(4g^2 + g + 1, 2g^2(4g^2 + g + 1)(4g - 1), 2g^2(4g + 1)(4g - 1), 4g + 1, 8g^2(4g - 1))$ .

*Proof:* Dawson (1985) has shown a  $\text{GH}(4g, \text{EA}(g))$  always exists. Also, the required  $\text{BIBD}(4g + 1, g(4g + 1), 4g, 4, 3)$  always exists and so, with  $\alpha = 2(4g - 1), \beta = 4g - 3$  in Theorem 5, we get the result.  $\square$

**Remark.** This would be a multiple of the  $\text{SBIBD}(4g^2 + g + 1, 4g + 1, 4)$  but this can only exist ( since  $4g^2 + g + 1$  is even ) if  $k - \lambda = 4g - 3$  is a square.

**Example 6.** Let  $g = 9$ . Then  $\alpha = 70$  and a  $\text{BIBD}(334, 70 \cdot 9 \cdot 334, 70 \cdot 9 \cdot 37, 37, 36 \cdot 70)$  exists.

**COROLLARY 8.** Let  $g = 3^h$ . Then there exists

$\text{BIBD}(4g^2 + g + 1, \alpha\lambda g(4g^2 + g + 1)/3, \alpha\lambda g(4g + 1)/3, 4g + 1, 4\alpha\lambda g/3)$   
where  $2\alpha\lambda(4g - 3) = 12\beta(4g - 1)$  for some  $\alpha$  and  $\beta$ . In particular, if  $\alpha\lambda = 2(4g - 1)$

and  $\beta = (4g-3)/3$ , there is a  
 $\text{BIBD}(4g^2+g+1, 2g(4g-1)(4g^2+g+1)/3, 2g(4g+1)(4g-1)/3, 4g+1, 8g(4g-1)/3)$ .

*Proof:* We again use the  $\text{GH}(4g, \text{EA}(g))$  found by Dawson (1985). We note that a  $\text{BIBD}(4g+1, \lambda g(4g+1)/3, 4\lambda g/3, 4, \lambda)$  exists for all  $\lambda$ . We use these in Theorem 5 to get the result.  $\square$

**Remark.** The constructed designs are also multiples of an  $\text{SBIBD}(4g^2 + g + 1, 4g + 1, 4)$  which never exists as  $4g^2 + g + 1$  is even and  $k - \lambda = 4g - 3$  is never a square for  $g = 3^h, h > 1$ .

**COROLLARY 9.** Let  $p$  be an odd prime power. Suppose there exists a  $\text{BIBD}(p^i + 1, qp^j(p^i + 1), qp^i, p^{i-j}, q(p^{i-j} - 1))$  where  $i \geq j$ , and  $q$  are integers. Then there exists a

$\text{BIBD}(p^{i+j} + p^j + 1, \alpha qp^j(p^{i+j} + p^j + 1), \alpha qp^i(p^i + 1), p^i + 1, \alpha qp^i)$   
 where  $2\alpha q(p^i - p^{i-j} + 1) = \beta p^{i-j}(p^i - 1)$ , there is a  
 $\text{BIBD}(p^{i+j} + p^j + 1, p^i(p^i - 1)(p^{i+j} + p^j + 1), p^i(p^{2i} - 1), p^i + 1, p^{2i-j}(p^i - 1))$ .

*Proof:* Use the  $\text{GH}(p^i, \text{EA}(p^j))$ ,  $i > j$  given by Drake (1979) or Butson (1963).  $\square$

**Remark.** This would be a multiple of the  $\text{SBIBD}(p^{i+j} + p^j + 1, p^i + 1, p^{i-j})$ . Since  $p^{i+j} + p^j + 1$  is odd, in order for this to exist, the diophantine equation

$$z^2 = (p^i - p^{i-j} + 1)x^2 + (-1)^{j(p^i+1)} p^{i-j} y^2$$

must have a solution in the integers for  $x, y, z$  not all zero.

**Example 7.** Let  $i = 2, j = 1, q = 1$  and  $p = 5$ . A  $\text{BIBD}(26, 130, 25, 5, 4)$  exists. Hence a  $\text{BIBD}(131, 600 \cdot 131, 600 \cdot 26, 26, 600 \cdot 5)$  exists.

#### 4. USING GENERALIZED WEIGHING MATRICES IN THE CONSTRUCTION

As noted in Seberry (1979), and Geramita and Seberry (1979), infinite families of GW matrices are known.

**THEOREM 10.** Let  $p^r$  be a prime power and  $q \mid p^r - 1, q$  odd. Then there exists a

$$\text{GBRD}(p^r + 1, \frac{1}{2}p^r(p^r + 1), \frac{1}{2}p^r(p^r - 1), p^r - 1, \frac{1}{2}(p^r - 1)(p^r - 2); \mathbb{Z}_q)$$

and  $B$ , a GDD with parameters

$$(q(p^r + 1), \frac{1}{2}qp^r(p^r + 1), \frac{1}{2}p^r(p^r - 1), p^r - 1, \lambda_1 = 0, \lambda_2 = \frac{1}{2}(p^r - 1)(p^r - 2)/q, m = q, n = p^r + 1).$$

Hence if,  $q = p^r - 1$ , there exists a

$$\text{BIBD}(p^{2r} - 1, \frac{1}{2}(p^{2r} - 2)(p^r + 1), \frac{1}{2}(p^{2r} - 2), p^r - 1, \frac{1}{2}(p^r - 2)).$$

If  $q \mid p^r - 1, q \neq p^r - 1$  and there exists a  $\text{BIBD}(p^r + 1, b, \rho, (p^r - 1)/q, \lambda)$ ,  $A$ , where



$\lambda q p^r = \rho(p^r - q - 1)$ . Using  $A$  to form a  
 $GDD(q(p^r + 1), qb, \rho, p^r - 1, \lambda_1 = r, \lambda_2 = \lambda)$

then

[ $\alpha$  copies of  $A$  :  $\beta$  copies of  $B$  ]

where  $2q\alpha(\rho - \lambda) = \beta(p^r - 1)(p^r - 2)$  gives a BIBD  $(q(p^r + 1), B, R, p^r - 1, \alpha\rho)$ .

*Proof:* We note first that a  $GW(p^r+1, p^r, p^r-1; Z_q)$  exists for all  $p^r$ . The proof, then, is identical to the first part of the proof of Theorem 3.  $\square$

**Example 8.** A  $GW(17, 16, 15; Z_q)$ ,  $q = 15, 5$  and  $3$  exists. This gives a BIBD  $(15 \cdot 17, 127 \cdot 17, 127, 15, 7)$ . Also, we have  $GDD(5 \cdot 17, 40 \cdot 17, 120, 15, \lambda_1 = 0, \lambda_2 = 21, m = 5, n = 17)$  and a  $GDD(3 \cdot 17, 24 \cdot 17, 120, 15, \lambda_1 = 0, \lambda_2 = 35, m = 3, n = 17)$ . Since BIBD  $(17, 8 \cdot 17, 24, 3, 3)$  and BIBD  $(17, 4 \cdot 17, 20, 5, 5)$  exist, we have a BIBD  $(85, 34 \cdot 24, 6 \cdot 24, 15, 24)$  with  $q = 5, r = 24, \lambda = 3, \alpha = \beta = 1$  and a BIBD  $(51, 1700, 500, 15, 140)$  with  $q = 3, r = 20, \lambda = 5, \alpha = 7, \beta = 3$ .

**Example 9.**

We note that there exists a

$$GW((p^{n+1} - 1)/(p - 1), p^n; Z_q)$$

for all  $q | p^n(p - 1)$ . So we can choose  $q$  odd and proceed as in the previous theorem. We do not give full results but note some examples: the  $GW(21, 16; Z_3)$  gives a GBRD  $(21, 12, 66; Z_3)$  and a BIBD  $(63, 12, 22 \cdot 60)$ , the  $GW(31, 25; Z_5)$  gives a GBRD  $(31, 20, 190; Z_5)$  and a BIBD  $(155, 20, 19 \cdot 20)$ , and the  $GW(85, 64; Z_3)$  gives a GBRD  $(85, 48, 24 \cdot 47; Z_3)$  and a BIBD  $(255, 24, 94 \cdot 336)$ .

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