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# A Construction for Orthogonal Designs with Three Variables

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*TO ALEX ROSA ON HIS FIFTIETH BIRTHDAY*

*ABSTRACT*

We show how orthogonal designs $OD(48p^2t;16p^2t, 16p^2t,16p^2t)$ can be constructed from an Hadamard matrix of order $4p$ and an $OD(4t;t,t,t,t)$. This allows us to assert that $OD(48p^2t; 16p^2t,16p^2t,16p^2t)$ exist for all $t,p \leq 102$ except possibly for $t \in \{67,71,73,77,79,83,86,89,91,97\}$. These designs are new.

## 1. Introduction

Let $H = (h_{ij})$ be a matrix of order $h$ with $h_{ij} \in \{1,-1\}$. $H$ is called an *Hadamard matrix* of order $n$, if $HH^T = hI_h$ where $I_h$ denotes the identity matrix order of $h$.

An *orthogonal design* $A$, of order $n$, type $(p_1,p_2, \ldots, p_u)$, denoted $OD(n;p_1,p_2, \ldots, p_u)$, on the commuting variables $(\pm x_1, \pm x_2, \ldots, \pm x_u, 0)$ is a square matrix of order $n$ with entries $\pm x_k$ where each $x_k$ occur $p_k$ times in each row and column such that the rows are pairwise orthogonal.

In other words

$$AA^T = (p_1 x_1^2 + \cdots + p_u x_u^2) I_n .$$

It is known that the maximum number of variables is an orthogonal design is $\rho(n)$, the Radon number, where for $n = 2^a b$, $b$ odd, set $a = 4c + d$, $0 \leq d < 4$, then $\rho(n) = 8c + 2^d$.

$OD(4t;t,t,t,t)$, otherwise called Baumert-Hall arrays, and $OD(2^p;a,b,2^p-a-b)$ have been extensively used to construct Hadamard matrices and weighing matrices. For details see Geramita and Seberry (1979).

Geramita, Geramita and Wallis (=Seberry) observed (see Geramita and Seberry [1979, §4.3]) that if $A,B,C,D$ are four circulant or type 1 matrices of order $n$ satisfying

$$AA^T + BB^T + CC^T + DD^T = (\sum_{i=1}^{u} p_i x_i^2)\, I_n$$

then $A,B,C,D$ can be used in the Goethals-Seidel array or ( J. Seberry Wallis-Whiteman array)

$$\begin{bmatrix} A & BR & CR & -DR \\ -BR & A & D^TR & -C^TR \\ CR & -D^TR & A & B^TR \\ DR & C^TR & -B^TR & A \end{bmatrix} \qquad (1).$$

to form an orthogonal design $OD(4n;p_1,p_2,\ldots,p_u)$.

## 2. Background

Kharaghani (1985) defined $C_k = [h_{ij}.h_{kj}]$ and applying that to Hadamard matrices of order $4p$, obtained matrices satisfying

$$C_i C_j = 0,\ i \neq j \qquad (2).$$
$$\sum_{i=1}^{4p} C_i^2 = (4p)^2 I_{4p}.$$

He then used this to show there are Bush-type (blocks $J_{4p}$ down the diagonal) and Szekeres-type ($h_{ij} = -1 \Rightarrow h_{ji} = 1$ and *not* necessarily vice versa) Hadamard matrices. By using a symmetric Latin square he could also have shown that regular symmetric Hadamard matrices with constant diagonal of order $(4p)^2$ could be constructed by his method.

Hammer, Sarvate and Seberry applied Kharaghani's method to $OD(n;s_1,\ldots,s_u)$ and in particular $OD(4t;t,t,t,t)$ and $OD4s;s,s,s,s)$ obtaining existence of $OD(48s^2t$ $12s^2t,\ 12s^2t,\ 12s^2t,\ 12s^2t)$ and $OD(80s^2t;\ 20s^2t,\ 20s^2t,\ 20s^2t,\ 20s^2t)$ from $OD(4s;s,s,s,s)$ and $0D(4t;t,t,t,t)$. Seberry (to appear) extended this further to obtain $OD(16ks^2t;4ks^2t,4ks^2t,4ks^2t,4ks^2t)$ for $\{1,3,5,\ldots\}$.

We modify their techniques to obtain new orthogonal designs.

## 3. Construction

Let $C_1,C_2,\ldots,C_{4p}$ be the Kharaghani matrices of order $4p$ obtained from an orthogonal design $OD(4p;p,p,p,p)$.

Let $a,b,c$ be commuting variables and write

$$[aC_1 : aC_2 :\ldots: aC_p : bC_{p+1} :\ldots: bC_{2p} : cC_{2p+1} :\ldots: cC_{3p}]$$
$$[aC_{3p+1} :\ldots: aC_{4p} : cC_{p+1} :\ldots: cC_{2p} : bC_{2p+1} :\ldots: bC_{3p}]$$

for the first rows of two block circulant matrices $W_1$ and $W_2$. It can be checked that $W_1W_2^T = W_2W_1^T$. Write

$$[aC_{2p+1}:....:aC_{3p} : bC_1:...:bC_p : cC_{3p+1}:...:cC_{4p}]$$
$$[aC_{p+1}:....:aC_{2p} : bC_1:...bC_p : bC_{3p+1}:....bC_{4p}]$$

for the first rows of two block back-circulant matrices $W_3$ and $W_4$. It can be checked that $W_3W_4^T = W_4W_3^T$.

Now by virtue of being circulant and back-circulant

$$W_iW_j^T = W_jW_i^T \qquad i \in \{1,2\}, \quad j \in \{3,4\}.$$

Example. Let $p = 3$ so there are 12 matrices of order 12, $C_1, \ldots, C_{12}$. Then

$$W_1 = \begin{bmatrix} aC_1 & aC_2 & aC_3 & bC_4 & bC_5 & bC_6 & cC_7 & cC_8 & cC_9 \\ aC_3 & aC_1 & aC_2 & bC_6 & bC_4 & bC_5 & cC_9 & cC_7 & cC_8 \\ aC_2 & aC_3 & aC_1 & bC_5 & bC_6 & bC_4 & cC_8 & cC_9 & cC_7 \\ cC_7 & cC_8 & cC_9 & aC_1 & aC_2 & aC_3 & bC_4 & bC_5 & bC_6 \\ cC_9 & cC_7 & cC_8 & aC_3 & aC_1 & aC_2 & bC_6 & bC_4 & bC_5 \\ cC_8 & cC_9 & cC_7 & aC_2 & aC_3 & aC_1 & bC_5 & bC_6 & bC_4 \\ bC_4 & bC_5 & bC_6 & cC_7 & cC_8 & cC_9 & aC_1 & aC_2 & aC_3 \\ bC_6 & bC_4 & bC_5 & cC_9 & cC_7 & cC_8 & aC_3 & aC_1 & aC_2 \\ bC_5 & bC_6 & bC_4 & cC_8 & cC_9 & cC_7 & aC_2 & aC_3 & aC_1 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} aC_{10} & aC_{11} & aC_{12} & cC_4 & cC_5 & cC_6 & bC_7 & bC_8 & bC_9 \\ aC_{12} & aC_{10} & aC_{11} & cC_6 & cC_4 & cC_5 & bC_9 & bC_7 & bC_8 \\ aC_{11} & aC_{12} & aC_{10} & cC_5 & cC_6 & cC_4 & bC_8 & bC_9 & bC_7 \\ bC_7 & bC_8 & bC_9 & aC_{10} & aC_{11} & aC_{12} & cC_4 & cC_5 & cC_6 \\ bC_9 & bC_7 & bC_8 & aC_{12} & aC_{10} & aC_{11} & cC_6 & cC_4 & cC_5 \\ bC_8 & bC_9 & bC_7 & aC_{11} & aC_{12} & aC_{10} & cC_5 & cC_6 & cC_4 \\ cC_4 & cC_5 & cC_6 & bC_7 & bC_8 & bC_9 & aC_{10} & aC_{11} & aC_{12} \\ cC_6 & cC_4 & cC_5 & bC_9 & bC_7 & bC_8 & aC_{12} & aC_{10} & aC_{11} \\ cC_5 & cC_6 & cC_4 & bC_8 & bC_9 & bc_7 & aC_{11} & aC_{12} & aC_{10} \end{bmatrix}$$

$$W_1W_2^T = I_q \times bc \sum_{i=4}^{q} C_i c_i^T$$

but $C_iC_i^T$ is symmetric and so $W_1W_2^T = W_2W_1^T$.

$W_3$ and $W_4$ are formed similarly but are back-circulant in blocks (i.e., type 2) in the language of Seberry-Wallis and Whiteman).

Thus $W_1, W_2, W_3, W_4$ are Williamson-type matrices of order $12p^2$ they can be used to replace the variables of an $OD(4;1,1,1,1)$ to get an $OD(48p^2;\ 16p^2,\ 16p^2,\ 16p^2,\ 16p^2)$. In general they can be used to replace the variables of an $OD(4t;t,t,t,t)$ so we have

**Theorem:** If there is an Hadamard matrix of order $4p$ and an $OD(4t;t,t,t,t)$ then there is an $OD(48p^2t;16p^2t,16p^2t,16p^2t,16p^2t)$.

Since Hadamard matrices of order $4p$ exist for all $p \leq 102$ and $OD(4t;t,t,t,t)$ exist for all $t \leq 102$ except possibly for $t \in S$, $S=\{67,71,73,77,79,83,86,89,91,97\}$ (see Seberry (1986a)) we have

**Corollary:** *$OD(48p^2t;16p^2t,16p^2t,16p^2t,16p^2t)$ exist for all $t,p \leq 102$ except possibly for $t \in S$.*

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