A Note on Orthogonal Designs

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ABSTRACT

We extend a method of Kharaghani and obtain some new constructions for weighing matrices and orthogonal designs. In particular we show that if there exists an $OD(s_1,...,s_r)$, where $w = \sum s_i$, of order n, then there exists an $OD(s_1w, s_2w, ..., s_rw)$ of order $n(n+k)$ for $k \geq 0$ an integer. If there is an $OD(t,t,t,t)$ in order n, then there exists an $OD(12t,12t,12t,12t)$ in order $12n$. If there exists an $OD(s,s,s,s)$ in order $4s$ and an $OD(t,t,t,t)$ in order $4t$ there exists an $OD(12s^2t,12s^2t,12s^2t,12s^2t)$ in order $48s^2t$ and an $OD(20s^2t,20s^2t,20s^2t20s^2)$ in order $80s^2t$.

1. Introduction.

Let $W = [w_{ij}]$ be a matrix of order n with $w_{ij} \in \{0,1,-1\}$. W is called a *weighing matrix* of weight p and order n, if $WW^T = W^TW = pI_n$, where I_n denotes the identity matrix of order n. Such a matrix is denoted by $W(n,p)$. If squaring all its entries gives an incidence matrix of a SBIBD then W is called a *balanced* weighing matrix.

An *orthogonal design* (OD), A, say, of order n and type $(s_1,s_2,...,s_t)$ on the commuting variables $(\pm x_1, \ldots, \pm x_t)$ and 0, is a square matrix of order n with entries from $(\pm x_1, \ldots, \pm x_t)$ and 0. Each row and column of A contains s_k entries equal to x_k in absolute value, the remaining entries in each row and column being equal to 0. Any two distinct rows of A are orthogonal.

In other words

$$AA^T = (x_1x_1^2 + \cdots + s_tx_t^2)I_n.$$

An Hadamard matrix $W = [w_{ij}]$ is a $W(n,n)$ i.e. it is a square matrix

*Supported in part by an ACRB grant.

of order n with entries $w_{ij} \in \{1,-1\}$ which satisfies

$$WW^T = W^TW = nI_n$$

OD's have been used to construct new Hadamard matrices. For details see Geramita and Seberry (1979).

Kharaghani (1985) defined $C_k = [w_{ki}, w_{kj}]$ and with that obtained skew symmetric and symmetric $W(n^2+2n, p^2)$ from $W(n,p)$, where s is any positive integer such that $n + s$ is even. Each C_k is a symmetric $\{0,1,-1\}$ matrix of order n. We define C_k by the Kronecker product and by extending Kharaghani's method we obtain some new constructions of weighing matrices and orthogonal designs.

2. Some properties of C_k's.

The C_k's can be defined as a Kronecker product of the kth row of W with its transpose, in other words, if R_k denotes the kth row of W, then $C_k = R_k \times R_k^T$. Similarly, we define C_k's corresponding to the OD, A, as follows:

Let U be a weighing matrix obtained from A by replacing all the variables of A by 1. Let A_k and U_k denote the kth rows of A and U respectively. Then $C_k = A_k \times U_k^T$.

Lemma 2.1. *Let V_i be the ith row of an SBIBD(v,p,λ). Consider*

$$X = [V_1 \times V_1^T, \ldots, V_n \times V_n^T]$$

then $XX^T = p((p-\lambda)I + \lambda J)$.

Proof.

$$\begin{aligned} XX^T &= V_1V_1^T \times V_1^TV_1, \ldots, V_nV_n^T \times V_n^TV_n \\ &= p\textstyle\sum_i V_i^TV_i \\ &= p((p-\lambda)I+\lambda J). \quad \square \end{aligned}$$

Corollary 2.2. *Given a balanced $W(n,p)$, based on an SBIBD(n,p,λ), consider*

$$X = [C_1' : C_2' : \cdots : C_n']$$

where C_1' is obtained from C_1 by squaring all its entries. Then the inner product of any two distinct rows of X is λp.

Proof. Observe that $C_i' = V_1 \times V_1^T$. □

3. A new construction of orthogonal designs.

Many constructions in orthogonal design theory have been expressed in terms of Kronecker products of matrices, for example see Gastineau-Hills (1983) and Gastineau-Hills and Hammer (1983). The Kronecker product of two or more designs is not in general a design since products of variables appear, for example:

$$\begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix} \times \begin{bmatrix} y_1 & y_2 \\ y_2 & -y_1 \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_2y_1 & x_1y_2 & x_2y_2 \\ -x_2y_1 & x_1y_1 & -x_2y_2 & x_1y_2 \\ x_1y_2 & x_2y_2 & -x_1y_1 & -x_2y_1 \\ -x_2y_2 & x_1y_2 & x_2y_1 & -x_1y_1 \end{bmatrix} = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ -z_2 & z_1 & -z_4 & z_3 \\ z_3 & z_4 & -z_1 & -z_2 \\ -z_4 & z_3 & z_2 & -z_1 \end{bmatrix}$$

(where $z_1 = x_1y_1$, $z_2 = x_2y_1$, $z_3 = x_1y_2$, $z_4 = x_2y_2$) is not orthogonal if we take z_1,z_2,z_3 and z_4 as independent. However it is a different matter if we take a Kronecker product of an OD with a weighing matrix.

A construction of Kharaghani can be extended to give the following result:

Theorem 3.1. *If there exists an OD, A, of type* $(s_1,s_2,\ldots,s_r)$, *where*

$$w = \sum_{k=1}^{r} s_k,$$

and order n on the variables $(\pm x_1, \ldots, \pm x_r, 0)$ *then there exist n matrices* $C_1,\ldots,C_n$ *of order n satisfying*

$$\sum_{i=1}^{n} C_iC_i^T = \sum_{k=1}^{n} s_k$$

$$C_kC_j^T = 0, \quad k \neq j.$$

Proof. Let $A = (a_{ij})$ be the OD. Replace all the variables of A by 1 making it a (0,1,−1) weighing matrix $U = (u_{ij})$ of order n and weight w. Write A_k and U_k for the kth rows of A and U respectively. Form

$$C_k = A_k \times U_k^T.$$

Then

$$\begin{aligned} C_kC_j^T &= (A_k \times U_k^T)(A_j \times U_j^T)^T \\ &= (A_kA_j^T \times U_k^TU_j) \end{aligned}$$

$= 0$ if $k \neq j$ because A is an orthogonal design.

Now

$$\sum_{k=1}^{n} C_k C_k^T = \sum_{k=1}^{n} (A_k \times U_k^T)(A_k^T \times U_k)$$
$$= \sum A_k A_k^T \times U_k^T U_k$$
$$= \sum s_j x_j^2 (\sum U_k^T U_k)$$
$$= \sum s_j x_j^2 (w I_n) \text{ by the properties of } U. \quad \square$$

Example 3.2. Let

$$A = \begin{bmatrix} -a & b & c & -d \\ b & a & d & c \\ c & -d & a & -b \\ -d & -c & b & a \end{bmatrix}; \quad U = \begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

Then

$$C_1 = \begin{bmatrix} a & -a & -a & a \\ -b & b & b & -b \\ -c & c & c & -c \\ d & -d & -d & d \end{bmatrix}^T, \quad C_2 = \begin{bmatrix} b & b & b & b \\ a & a & a & a \\ d & d & d & d \\ c & c & c & c \end{bmatrix}^T$$

$$C_3 = \begin{bmatrix} c & -c & c & -c \\ -d & d & -d & d \\ a & -a & a & -a \\ -b & b & -b & b \end{bmatrix}^T, \quad C_4 = \begin{bmatrix} d & d & -d & -d \\ c & c & -c & -c \\ -b & -b & b & b \\ -a & -a & a & a \end{bmatrix}^T.$$

Thus we have:

Theorem 3.3. *Suppose there exists an $OD(s_1,...,s_r)$, where $w = \sum s_i$, of order n. Then there exists an $OD(s_1 w, s_2 w, ..., s_r w)$ of order $n(n+k)$ for $k \geq 0$ an integer.*

Proof. Form $C_1,...,C_n$ as in the previous theorem. Form a latin square of order $n + k$ and replace n of its elements by $C_1,...,C_n$ and the other elements by the $n \times n$ zero matrix. $\square$

For instance, using Theorem 3.3 we can construct an $OD(4,4,4,4)$ of order $4n$, for $n \geq 4$. Using inequivalent Latin squares in Theorem 3.3 will yield inequivalent ODs.

Corollary 3.4. *If there is an $OD(t,t,t,t)$ in order $4t$, then there is an $OD(4t^2,4t^2,4t^2,4t^2)$ in every order $4t(4t+k)$, $k \geq 0$ an integer.*

But this construction can be used in other ways.

Example 3.5. Write 1,2,3,4 for $C_1,...,C_4$. Define

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 4 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 4 & 3 \\ 4 & 3 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 4 & 2 \\ 4 & 2 & 1 \end{bmatrix}.$$

Then $A_k A_j^T = A_j A_k^T$. Thus A_1, A_2, A_3, A_4 can be used to replace the variables of any $OD(t,t,t,t)$. □

Hence we have

Theorem 3.6. *Suppose there is an $OD(t,t,t,t)$ in order n. Then there exists an $OD(12t,12t,12t,12t)$ in order $12n$.*

Proof. Use the $OD(1,1,1,1)$ in order 4 to form $C_1,...,C_4$ of order 4. Substitute these in $A_1,...,A_r$ of Example 3.5 to obtain Williamson-type matrices of order 12, on 4 variables each repeated 12 times. Use these to replace the variables of the $OD(t,t,t,t)$ to get the result. □

Now if we had started to construct $C_1,...,C_{4s}$ of order $4s$ from an $OD(s,s,s,s)$ in order $4s$ we would have each of 4 variables occurring $4s^2$ times in each row of $[C_1\colon\ C_2\colon\ \cdots\ \colon C_{4s}]$. But we can use these to form Williamson type matrices in a number of ways:

Let A_i, be a circulant matrix with first row $(i+1,i+2,...,i+s)$, $i = 0,s,2s$, and $3s$. These four matrices can be substituted in an $OD(t,t,t,t)$. Hence we have:

Theorem 3.7. *If there exists an $OD(s,s,s,s)$ in order $4s$ and an $OD(t,t,t,t)$ in order $4t$, then there exists an $OD(4s^2t,4s^2t,4s^2t,4s^{t})$ in order $16s^2t$.*

Now if we write i for B_i we can proceed exactly as in Example 3.5 so we have:

Theorem 3.8. *If there exists an $OD(s,s,s,s)$ in order $4s$ and an $OD(t,t,t,t)$ in order $4t$, then there exists an $OD(12s^2t,12s^2t,12s^2t,12s^2t)$ in order $48s^2t$.* □

Consider the $OD(5,5,5,5)$ in order 20. The construction gives $C_1,C_2,...,C_{20}$ of order 20 and hence an $OD(300,300,300,300)$ in order 1200.

Example 3.10. We suppose as before that 1,2,3,4 are matrices of order n such that $ij^T = 0$ and $\sum ii^T = \sum nx_i^2 I_n$.

Define

$$A_1 = \begin{bmatrix} 3 & 1 & 2 & -2 & 1 \\ 1 & 3 & 1 & 2 & -2 \\ -2 & 1 & 3 & 1 & 2 \\ 2 & -2 & 1 & 3 & 1 \\ 1 & 2 & -2 & 1 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 3 & 4 & -4 & 3 \\ 3 & 1 & 3 & 4 & -4 \\ -4 & 3 & 1 & 3 & 4 \\ 4 & -4 & 3 & 1 & 3 \\ 3 & 4 & -4 & 3 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 4 & 1 & 2 & 2 & -1 \\ 1 & 2 & 2 & -1 & 4 \\ 2 & 2 & -1 & 4 & 1 \\ 2 & -1 & 4 & 1 & 2 \\ -1 & 4 & 1 & 2 & 2 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 2 & 3 & 4 & 4 & -3 \\ 3 & 4 & 4 & -3 & 2 \\ 4 & 4 & -3 & 2 & 3 \\ 4 & -3 & 2 & 3 & 4 \\ -3 & 2 & 3 & 4 & 4 \end{bmatrix}.$$

Then $A_iA_j^T = A_jA_i^T$ and $\sum A_iA_i^T = \sum 5x_i^2 I_{5n}$.

Thus if B_i are as described after Theorem 3.7 we have

Theorem 3.11. *Suppose there is an $OD(s,s,s,s)$ in order $4s$ and an $OD(t,t,t,t)$ in order $4t$. Then there is an $OD(20s^2t,20s^2t,20s^2t,20s^2t)$ in order $80s^2t$.*

4. Method used to form inequivalent Hadamard matrices.

Construction 4.1. Let H be Hadamard of order n. Form C_i, $i = 1,2,...,n$, from H as before. Let L and M be Hadamard matrices of order t. Then

$$(L \times C_i)(M \times C_j) = 0, \qquad i \neq j.$$

So if $H_1,...,H_n$ are Hadamard matrices of order t (inequivalent or just different equivalence operations applied to one) then the matrices

$$H_{i_1} \times C_1,\ H_i^2 \times C_2, \ldots, H_{i_n} \times C_n,\ i_j \in \{1,2,...,n\}$$

can be put into a latin square of order n to form Hadamard matrices of order n^2t. The method will possibly give many inequivalent Hadamard matrices. The method can be generalized to give weighing matrices and orthogonal designs which are also possibly inequivalent.

5. Method used with coloured designs to form rectangular weighing matrices.

In a recent paper Rodger, Sarvate and Seberry (1987) have studied coloured BIBDs showing every BIBD can be coloured. By definition a coloured BIBD is the incidence matrix of the $BIBD(v,b,r,k,\lambda)$ whose nonzero entries are replaced by r fixed symbols such that each row and column has no repeated symbol. Consider a coloured symmetric $BIBD(v,k,\lambda)$ and a $W(k,p)$. If we replace the ith symbol by C_i for $i = 1,2,\ldots,k$ and the 0 entries by the k by k zero matrix, we get $W(vk,p^2)$. In general, if we consider a coloured $BIBD(v,b,r,k,\lambda)$ and there exists a weighing matrix $W(r,p)$ then we form the C_i, $i = 1,\ldots,r$ and replace the ith colour by C_i and zeros by the zero matrix of order r. This matrix, B, has size $vr \times vr$, rp nonzero elements in each row and pk non-zero elements in each column. Hence we have:

Theorem 5.1. *Suppose there is a $BIBD(v,b,r,k,\lambda)$ and a $W(r,p)$. Then there is a $(0,1,-1)$ matrix B with rp nonzero elements in each row and pk nonzero elements in each column such that*

$$BB^T = rpI.$$

In particular, if the BIBD is symmetric then we have a $W(vk,p^2)$. □

Remark. If we replace entries of an n-dimensional latin cube by suitable C_i's then we will get n-dimensional orthogonal designs.

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